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Condensation phenomena of a semilinear elliptic equation on non-smooth domains

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1 Introduction

In this paper we consider a singular perturbation problem to a semilinear Neumann problem

$$\begin{cases} \epsilon^2\Delta u - u + u^p = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a boundedn domain, $p > 1$ and $(N-2)p < N+2$ are satisfied, $\epsilon > 0$ is a constant, and ν is the outer unit normal vector on $\partial\Omega$. We are interested in the asymptotic behavior of solutions to (1.1).

Such a singular perturbation problem is originally considered in the stationary Keller-Segel model or the shadow system of the Gierer-Meinhardt model, and (1.1) is obtained as a reduced problem of those problems. In the case that Ω has a smooth boundary, various researchers investigated (1.1). Lin, Ni and Takagi [16] are pioneers researchers of the problem, and they proved the existence and some properties of least-energy solutions to (1.1) for sufficiently small $\epsilon > 0$. Here a least-energy solution is a solution which attains the least positive critical value of the associated energy functional with (1.1):

$$J_\epsilon(u) = \frac{1}{2}\int_\Omega (\epsilon^2|\nabla u|^2 + u^2)\,dx - \frac{1}{p+1}\int_\Omega u^{p+1}\,dx \quad \text{for } u \in H^1(\Omega).$$

Next Ni and Takagi [18, 19] investigated the asymptotic behavior of a least-energy solution as $\epsilon \to 0$. They proved that, by singular perturbation, the point condensation phenomena of a least-energy solution occurs, that is, the solution concentrates at its maximum point $P_\epsilon \in \partial\Omega$. Furthermore there holds

$$H(P_\epsilon) \to \max_{P\in\partial\Omega} H(P) \quad \text{as } \epsilon \to 0,$$

where $H(P)$ is the mean curvature of $\partial\Omega$ at $P \in \partial\Omega$. Other many mathematicians also investigated condensation phenomena; e.g., Byeon [3], del Pino and Felmer [4] (condensation at a single point), Gui [10], Gui and Wei [11], Gui, Wei and Winter [12] (condensation at many points). Kabeya and Ni [13] investigated the condensation phenomena of semilinear elliptic problems with exponential nonlinearity instead of power nonlinearity. On the other hand, under the Dirichlet boundary condition, condensation phenomena was also investigated by, e.g., [4], the author [14], Ni and Wei [20].

In precedents results above, we assumed that $\partial\Omega$ is smooth. On the other hand, we are interested in the case that $\partial\Omega$ has *piecewise smoothness*, and we know less results on that case than results on the smooth boundary case. Dipierro [5, 6] invesigated (1.1) with the piecewise smooth $\partial\Omega$ and $(N-2)$ dimensional subset $\partial\Omega_0 \subset \partial\Omega$ of non-smooth points.

Namely they assumed that $\partial\Omega$ has a smooth edge $\partial\Omega_0$, and $H_0(P)$ denotes the opening angle at $P \in \partial\Omega_0$. Then they constructed a solution which concentrates at $P \in \partial\Omega_0$ where $H_0(P)$ attains its strict local maximum or minimum.

Dipierro's result implies that condensation phenomena occurs on edges, that is, $(N-2)$ dimensional subset, and $H_0(P)$ plays a similar role to the mean curvature $H(P)$. On the other hand, our aim in this paper is to prove that condensation phenomena occurs at the *vertex* as well as the edge. Especially we focus our attention on a least-energy solution u_ϵ to (1.1), and we show that u_ϵ concentrates at the point having the least *angle* (the least *solid angle* if $N = 3$) less than π $(N = 2)$ or 2π $(N = 3)$. Moreover we obain the asymptotic profile of u_ϵ as $\epsilon \to 0$.

In order to investigate the behavior of least-energy solutions by our method, we are required to use the regularity properties of the solutions. Hence we only assume the case $N = 2, 3$. If we assume higher dimensional case $N \geq 4$, then it is difficult to show the regularity of solutions.

In arguments below we consider more general problems. Namely let $\Omega \subset \mathbb{R}^N$ $(N = 2, 3)$ be a Lipschitz domain. In arguments below we consider the following problem

$$\begin{cases} \epsilon^2 \Delta u - u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\epsilon > 0$, and ν is the outer unit normal vector on $\partial\Omega$. Here $f(t)$ satisfies the following conditions:

$(f\mathrm{i})$ $f(t) \in C^{1,l}(\mathbb{R})$ with $0 < l < 1$, $f(t) > 0$ $(t > 0)$ and $f(t) \equiv 0$ $(t \leq 0)$;

$(f\mathrm{ii})$ $\lim_{t\to 0} f(t)/t = 0$ and $f(t)/t$ is increasing for $t > 0$;

$(f\mathrm{iii})$ $f(t) = O(t^p)$ as $t \to \infty$ with $p > 1$ and $(N-2)p < N + 2$;

$(f\mathrm{iv})$ there exists a constant $\theta \in (0, 1/2)$ such that $F(t) \leq \theta t f(t)$ for $t \geq 0$.

Since Ω is not smooth in our problem, we discuss about a weak solution $u \in H^1(\Omega)$ to (1.2), which satisfies

$$\int_\Omega (\nabla u \nabla \varphi + u\varphi - f(u)\varphi)\, dx = 0 \quad \text{for any } \varphi \in H^1(\Omega).$$

Moreover we focus our attention on a least-energy solution to (1.2), which attains the least positive critical value of

$$J_\epsilon(u) = \frac{1}{2}\int_\Omega (\epsilon^2 |\nabla u|^2 + u^2)\, dx - \int_\Omega F(u)\, dx \quad \text{for } u \in H^1(\Omega) \tag{1.3}$$

with $F(t) = \int_0^t f(s)\, ds$.

In our arguments we will show the asymptotic profile of a solution to (1.2), and then some kind of solution $w = w_0$ to the following entire space problem plays an important role, that is,

$$\begin{cases} \Delta w - w + f(w) = 0 & \text{in } \mathbb{R}^N, \\ w > 0 & \text{in } \mathbb{R}^N, \\ w(z) \to 0 & \text{as } |z| \to \infty, \\ w(0) = \max\limits_{z\in\mathbb{R}^N} w(z). \end{cases} \tag{1.4}$$

The solution w_0 is characterized by the energy functional

$$I(w) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla w|^2 + w^2)\ dz - \int_{\mathbb{R}^N} F(w)\ dz \quad \text{for } w \in H^1(\mathbb{R}^N).$$

and the following proposition:

Proposition 1.1 *Under conditions* (fi)–(fiv), *there exists a solution* w_0 *to* (1.4) *such that*

(a) $w_0 > 0$ *in* $\mathbb{R}^N$ *and* $w_0 \in C^2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$;

(b) *for any solutions* $w \in H^1(\mathbb{R}^N)$ *to* (1.4), *there holds* $0 < I(w_0) \leq I(w)$;

(c) $w_0(z) = w_0(r)$ *with* $r = |z|$ *and* $w_0'(r) < 0$ *for* $r > 0$;

(d) $w_0(r), w_0'(r) \leq Cr^{-\frac{N-1}{2}}e^{-r}$ *for* $r > 0$ *with some constant* $C > 0$.

The existence of w_0 to (1.4) satisfying (a) and (b) is proved by Berestycki, Gallouët, Kavian [1] ($N = 2$) or Berestycki, Lions [2] ($N = 3$). Moreover, by Gidas, Ni and Nirenberg's result (cf., Theorem 2 in [8]), it is known that w_0 satisfies (c) and (d). The solution w_0 found in Proposition 1.1 is said to be a *ground state solution* to (1.4). Through arguments in this paper, we also assume the next condition:

(fv) a ground state solution w_0 to (1.4) is unique.

Since w_0 is radially symmetric, (fv) is equivalent to the uniqueness of the following ODE

$$\begin{cases} w'' + \dfrac{N-1}{r}w' + f(w) = 0 & \text{for } r > 0, \\ w > 0 & \text{for } r > 0, \\ w(r) \to 0 & \text{as } r \to \infty. \end{cases} \tag{1.5}$$

For example a solution to (1.5) is unique if $f(t) = t^p$ holds with $p > 1$ and $(N-2)p < N+2$ (e.g., Kwong [15]). Concerning more general cases, e.g., Pucci and Serrin [22].

Next we introduce some notations and assume the geometrical condition of the Lipschitz domain Ω. Let $x_0 \in \partial\Omega$. In some neighborhood of x_0, the boundary $\partial\Omega$ is expressed by a graph. If the graph is of class $C^{2,\gamma}$, then x_0 is said to be a *smooth point.* On the other hand, if not, then x_0 is said to be a *non-smooth point.* For two kinds of points, we assume the following geometrical condition:

> If $x_0 \in \partial\Omega$ is a smooth point, then some neighborhood of x_0 only consists of smooth points. On the other hand, if $x_0 \in \partial\Omega$ is a non-smooth point, then there exists $R_* > 0$ such that the intersection $B_{R_*}(x_0) \cap \Omega$ is expressed by
>
> $$B_{R_*}(x_0) \cap \Omega = \{x \in \mathbb{R}^N \mid x - x_0 = r\sigma, 0 < r < R_*, \sigma \in \mathcal{S}(x_0)\}, \tag{BD}$$
>
> where $\mathcal{S}(x_0) \subset \mathbb{S}^{N-1}$ is a Lipschitz domain. Moreover, for any $x_1, x_2 \in B_{R_*}(x_0) \cap \Omega$, it holds that
>
> $$ax_1 + bx_2 \in \{x \in \mathbb{R}^N \mid x - x_0 = r\sigma, 0 < r < \infty, \sigma \in \mathcal{S}(x_0)\}$$
>
> for any $a, b > 0$

Condition (BD) implies that a neighborhood of any non-smooth point is *convex cone.* For example polyhedra satisfy the condition (BD).

Under (fi)–(fv) and (BD), we obtain the following theorem. Here, for a set $\mathcal{U} \subset \mathbb{R}^k$ ($k \leq N$), the Hausdorff measure of $\mathcal{U}$ is denoted by $\mathcal{H}_k(\mathcal{U})$.

Theorem 1.1 *Let Ω be a bounded Lipschitz domain. Assume $N = 2, 3$, $(f\mathrm{i})$–$(f\mathrm{v})$, (BD) and the existence of a non-smooth point $P_0 \in \partial\Omega$ such that*

$$\mathcal{H}_{N-1}(\mathcal{S}(P_0)) = \min_{P\in\partial\Omega} \mathcal{H}_{N-1}(\mathcal{S}(P)) < \frac{1}{2}\mathcal{H}_{N-1}(\mathbb{S}^{N-1}). \tag{1.6}$$

Then the following statements hold:

(i) *For sufficiently small $\epsilon > 0$, there exists a least-energy solution $u_\epsilon \in H^2(\Omega)$. Moreover $u_\epsilon \in C^2(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$ to (1.1) with some $0 < \alpha < 1$. Moreover $u_\epsilon > 0$ in Ω and it holds that*

$$J_\epsilon(u_\epsilon) = \epsilon^N \left\{ \frac{\mathcal{H}_{N-1}(\mathcal{S}(P_0))}{\mathcal{H}_{N-1}(\mathbb{S}^{N-1})} I(w_0) + o(1) \right\} \quad \epsilon \to 0. \tag{1.7}$$

(ii) *Take sufficiently small $R > 0$, and let x_ϵ be a maximum point of u_ϵ. Then, for any $\epsilon_j \to 0$ $(j \to \infty)$, there exists a subsequence $\{\epsilon_{j_k}\}$ and a point $P_1 \in \partial\Omega$ such that $\mathcal{H}_{N-1}(\mathcal{S}(P_1)) = \mathcal{H}_{N-1}(\mathcal{S}(P_0))$ and $x_{\epsilon_{j_k}} \to P_1$ as $k \to \infty$. Moreover it holds that*

$$\begin{aligned} &\frac{1}{2}\int_{B_R(P_1)\cap\Omega} (\epsilon_{j_k}^2 |\nabla u_{\epsilon_{j_k}}|^2 + u_{\epsilon_{j_k}}^2)\, dx - \int_{B_R(P_1)\cap\Omega} F(u_{\epsilon_{j_k}})\, dx \\ &= J_{\epsilon_{j_k}}(u_{\epsilon_{j_k}}) + o\left(\epsilon_{j_k}^N\right) \quad (k \to \infty) \end{aligned} \tag{1.8}$$

and, for any $R_0 > 0$,

$$\left\| u_{\epsilon_{j_k}}(x) - w_0\left(\frac{x - P_1}{\epsilon_{j_k} R_0}\right) \right\|_{C^{0,\alpha}\left(B_{\epsilon_{j_k}R_0}(P_1)\cap\Omega\right)} \to 0 \quad (k \to \infty). \tag{1.9}$$

For a least-energy solution v_ϵ concentrating at a smooth point, it is known (cf. [18]) that v_ϵ satisfies

$$J_\epsilon(v_\epsilon) = \epsilon^2 \left\{ \frac{1}{2} I(w_0) + o(1) \right\} \quad \text{as } \epsilon \to 0. \tag{1.10}$$

Hence the condition (1.6) guarantees that u_ϵ condenses at non-smooth point (compare (1.7) with (1.10)). On the other hand, if $\mathcal{H}_{N-1}(\mathcal{S}(P_0)) > 2^{-1}\mathcal{H}_{N-1}(\mathbb{S}^{N-1})$ for any non-smooth point P_0, then u_ϵ does not concentrates at a non-smooth point (the condensation phenomena of u_ϵ occurs at a smooth point on $\partial\Omega$). Moreover (1.8) implies that the energy condensation occurs, and (1.9) implies that the asymptotic profile of u_ϵ is characterized by the ground state solution w_0 to (1.4).

2 Properties of positive solutions on infinite cones

Let $N \geq 2$. Before investigating properties of solutions to (1.2), we consider some Neumann problem defined in an *infinite cone.* We define an infinite cone by

$$\mathcal{F}_\infty := \{z \in \mathbb{R}^N \mid z = r\sigma, r > 0, \sigma \in \mathcal{S}\},$$

where $\mathcal{S} \subset \mathbb{S}^{N-1}$ is a Lipschitz domain. In Section 3 we usually fix the origin at 0. If w defined in $\mathcal{F}_\infty$ satisfies $w(z) = w(|z|)$ in $\mathcal{F}_\infty$, then, by the analogy from $\mathbb{R}^N$, w is also said to be *radially symmetric.*

Let $N \geq 2$, and we state some results on the following problem

$$\begin{cases} \Delta w - w + f(w) = 0 & \text{in } \mathcal{F}_\infty, \\ w > 0 & \text{in } \mathcal{F}_\infty, \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial \mathcal{F}_\infty, \\ w(z) \to 0 & \text{as } |z| \to \infty. \end{cases} \tag{2.1}$$

For $w \in H^1(\mathcal{F}_\infty)$, we also define the following functionals

$$T_{\mathcal{F}}(w) = \frac{1}{2} \int_{\mathcal{F}_\infty} |\nabla w|^2 \, dz,$$
$$V_{\mathcal{F}}(w) = \int_{\mathcal{F}_\infty} \left\{ -\frac{1}{2} w^2 + F(w) \right\} \, dz,$$

and then the energy functional of (2.1) is defined by

$$I_{\mathcal{F}}(w) = T_{\mathcal{F}}(w) - V_{\mathcal{F}}(w). \tag{2.2}$$

A critical point $w \in H^1(\mathcal{F}_\infty)$ is also a weak solution to (2.1). If $w \in H^1(\mathcal{F}_\infty)$ satisfies

$$I_{\mathcal{F}}(w) \leq I_{\mathcal{F}}(v)$$

for any solution v to (2.2), then w is said to be a ground state solution to (2.1). Moreover let

$$\mathcal{F}_R := \{ z \in \mathbb{R}^N \mid z = r\sigma, 0 < r < R, \sigma \in \mathcal{S} \} \quad \text{for } R > 0,$$

and we define the additional condition:

a solution $w \in H^1(\mathcal{F}_\infty)$ to (2.1) satisfies $w \in W^{2,q}(\mathcal{F}_R) \cap L^\infty(\mathcal{F}_\infty)$ for $q > \min\{N/2, 2\}$, and $w(z_0) = \max_{z \in \overline{\mathcal{F}_\infty}} w(z)$ for some $z_0 \in \overline{\mathcal{F}_\infty}$. (R)

Recall Proposition 1.1 and $(f\mathrm{v})$. Our aim in this section is to prove the following lemma:

Lemma 2.1 *Assume* $(f\mathrm{i})$–$(f\mathrm{v})$, (BD), (R) *and* $\mathcal{H}_{N-1}(\mathcal{S}) < 2^{-1}\mathcal{H}_{N-1}(\mathbb{S}^{N-1})$. *If* $w \in H^1(\mathcal{F}_\infty)$ *is a ground state solution to* (2.1) *satisfying* (R), *then* $w(z) \equiv w_0(z)|_{\mathcal{F}_\infty}$ *in* $\mathcal{F}_\infty$, *and it holds that*

$$I_{\mathcal{F}}(w) = \frac{\mathcal{H}_{N-1}(\mathcal{S})}{\mathcal{H}_{N-1}(\mathbb{S}^{N-1})} I(w_0). \tag{2.3}$$

Lemma 2.1 is required to show Theorem 1.1. If we only consider a solution $w \in H^1(\mathcal{F}_\infty)$ which is radially symmetric, then the uniqueness of w easily follows:

Lemma 2.2 *Assume* $(f\mathrm{i})$–$(f\mathrm{v})$. *If a ground state solution* $w \in H^1(\mathcal{F}_\infty)$ *is radially symmetric, then* $w(r) \equiv w_0(r)$ *for* $|z| = r > 0$ *and* w *satisfies* (2.3).

Proof. Since $w \in H^1(\mathcal{F}_\infty)$ is radially symmetric, w is naturally extended as a radial function W defined on $\mathbb{R}^N$, that is, $W(z) := w(|z|)$ for $z \in \mathbb{R}^N$. Then $W \in H^1(\mathbb{R}^N)$ is a weak solution to (1.4), and it is known that $W \in C^2(\mathbb{R}^N)$ (cf. Lemma 1 of [2]). Hence W satisfies Proposition 1.1 (a). Moreover it is also known that a solution $W \in C^2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ satisfies Proposition 1.1 (c) and (d) (cf. Theorem 2 in [8]).

Finally we show $I(W) = I(w_0)$. If $I(W) > I(w_0)$, then, since $w_0|_{\mathcal{F}_\infty}$ is a solution to (2.1), it holds that $I_{\mathcal{F}}(w) > I_{\mathcal{F}}(w_0|_{\mathcal{F}_\infty})$. Since w is a ground state solution, it is a contradiction.

Hence $I(W) = I(w_0)$. Therefore, from $(f\mathrm{v})$, it follows that $w(r) \equiv w_0(r)$ for $|z| = r > 0$, and w satisfies (2.3). ■

Hence it suffices to show that any ground state solution is radially symmetric under assumptions in Lemma 2.1. We requires the following minimizing problem

$$\inf\{T_{\mathcal{F}}(w) \mid w \in \mathcal{M}\}, \tag{2.4}$$

where

$$\mathcal{M} := \{w \in H^1(\mathcal{F}_\infty) \mid V_{\mathcal{F}}(w) = 1 \ (N \geq 3) \text{ or } V_{\mathcal{F}}(w) = 0 \ (N = 2)\}$$

For the minimizing problem (2.4), the next lemma holds:

Lemma 2.3 *If w be a ground state solution to* (2.1) *satisfying* (R), *then $w_\rho(z) = w(z/\rho)$ is a minimizer to* (2.4), *where ρ is defined by*

$$\rho(w) := (V_{\mathcal{F}}(w))^{-\frac{1}{N}}.$$

Conversely a minimizer satisfying (R) *is also a ground state solution up to scale transformation.*

From Lemma 2.3 the following corollary immediately follows:

Corollary 2.1 *A ground state solution to* (2.1) *satisfying* (R) *is unique if and only if a minimizer to* (2.4) *satisfying* (R) *is unique.*

Moreover, from Lemmas 2.2 and 2.3, the next statements immediately follows:

Corollary 2.2 *If any minimizer to* (2.4) *satisfying* (R) *is radially symmetric, then a minimizer to* (2.4) *is unique.*

From Corollaries 2.1 and 2.2, it suffices to show that any minimizer to (2.4) is radially symmetric. For this purpose we introduce some kind of rearrangement of $w \in H^1(\mathcal{F}_\infty)$ on $\mathcal{F}_\infty$ in arguments below. Our rearrangement is similar to the α-symmetrization, which is studied by Lions et al. [17] and Pacella et al. [21] in detail.

Let $w \in H^1(\mathcal{F}_\infty)$. Then a distribution function $\mu(t)$ of $w(x)$ is defined by

$$\mu(t) = \mathcal{H}_N\left(\{x \in \mathcal{F}_\infty \mid |w(z)| > t\}\right) \quad \text{for } t \geq 0.$$

By $\mu(t)$ we define the decreasing rearrangement

$$w^*(s) = \inf\{t \geq 0 \mid \mu(t) < s\}.$$

We define the rearrangement $Cw(z)$ of $w(z)$ by

$$Cw(x) = w^*\left(\frac{\mathcal{H}_{N-1}(\mathcal{S})}{N}|z|^N\right) \quad \text{for } z \in \mathcal{F}_\infty.$$

If we assume (BD) and $\mathcal{H}_{N-1}(\mathcal{S}) < 2^{-1}\mathcal{H}_{N-1}(\mathbb{S}^{N-1})$, then it holds that

$$\int_{\mathcal{F}_\infty} |w|^q \, dx = \int_{\mathcal{F}_\infty} |Cw|^q \, dz \quad \text{for any } q > 0.$$

and

$$\int_{\mathcal{F}_\infty} |\nabla w|^2 \, dx \geq \int_{\mathcal{F}_\infty} |\nabla Cw|^2 \, dz.$$

Moreover we can prove the following lemma:

Lemma 2.4 *Assume* (fi)–(fv), (BD), (R), *and* $\mathcal{H}_{N-1}(\mathcal{S}) < 2^{-1}\mathcal{H}_{N-1}(\mathbb{S}^{N-1})$ *If* $w \in H^1(\mathcal{F}_\infty)$ *is a minimizer to* (2.4), *then* w *is radially symmetric.*

The idea of the proof of Lemma 2.4 follows Friedman and McLeod [7]. By using preliminaries above, we can prove Lemma 2.1.

3 Proof of Theorem 1.1

In this section we state Theorem 1.1 and some lemmas in order to prove Theorem 1.1. In our argument we require the result on the unique solvability of Neumann problems defined on $\mathcal{F}_R$. Name we require the following result:

Lemma 3.1 *We fix* $R_0 > 0$. *Let the Neumann problem*

$$\begin{cases} \Delta u - \lambda u = g & \text{in } \mathcal{F}_R, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathcal{F}_R \end{cases} \tag{3.1}$$

where $g \in L^2(\mathcal{F}_R)$ *and* $\lambda > 0$. *Then, for* $R \geq R_0$, *the Neumann problem* (3.1) *has a unique solution* $u \in H^2(\mathcal{F}_R)$ *satisfying*

$$\|u\|_{H^2(\mathcal{F}_R)} \leq C\|g\|_{L^2(\mathcal{F}_R)},$$

where $C > 0$ *only depends on* λ *and* R_0.

This result is proved by the result on the unique solvability of the Neumann problem defined in non-smooth domains (Grisvard [9]). By Lemma 3.1 the following lemma holds:

Lemma 3.2 *Suppose the same assumptions as in Theorem* 1.1. *Then, for sufficiently small* $\epsilon > 0$, *there exists a non-trivial critical point* $u_\epsilon \in H^1(\Omega)$ *to* (1.3). *Moreover,* $u_\epsilon \in H^2(\Omega)$, $u_\epsilon \in C^2(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$ *with some* $0 < \alpha < 1$, *and* $u_\epsilon > 0$ *in* Ω.

The idea of Lemma 3.2 follows Lin, Ni Takagi [16] and Ni, Takagi [18]. Moreover the following inequalities are also obtained:

Lemma 3.3 *For* u_ϵ *found in Lemma* 3.2, *there holds*

$$\|u_\epsilon\|_{L^\infty(\Omega)} < C_1$$

and

$$\int_\Omega u_\epsilon^q \, dx \leq C_2(q)\epsilon^N \quad \text{for } q \geq 1,$$

where C_1 *and* $C_2(q)$ *are positive constant such that those are independent of* ϵ.

For the least-energy solution u_ϵ found in Lemma 3.2, the next lemma holds:

Lemma 3.4 *Assume the same assumptions as in Lemma* 3.2. *Then it holds that*

$$J_\epsilon(u_\epsilon) = \epsilon^N \left\{ \frac{\mathcal{H}_{N-1}(\mathcal{S}(P_0))}{\mathcal{H}_{N-1}(\mathbb{S}^{N-1})} I(w_0) + o(1) \right\} \quad \text{as } \epsilon \to 0.$$

The idea of the proof of Lemma 3.4 follows the arguments in Ni and Takagi [18]. By using Lemma 3.4 and some results shown in the proof of Lemma 3.4, we can prove Theorem 1.1.

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