数理解析研究所講究録
第2033巻 2017年 32-41



# Complex symmetric operators and their Weyl type theorems

Il Ju An, Eungil Ko, and Ji Eun Lee

**Abstract**

We study a necessary and sufficient condition for complex symmetric operator matrices to satisfy $a$-Weyl's theorem. Moreover, we also give the conditions for such operator matrices to satisfy generalized $a$-Weyl's theorem and generalized $a$-Browder's theorem, respectively. As some applications, we provide various examples of such operator matrices which satisfy Weyl type theorems.

## 1 Introduction

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators acting on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, we write $\sigma(T)$, $\sigma_p(T)$, $\sigma_s(T)$, and $\sigma_a(T)$ for the spectrum, the point spectrum, the surjective spectrum, and the approximate point spectrum of $T$, respectively.

If $T \in \mathcal{L}(\mathcal{H})$, we shall write $N(T)$ and $R(T)$ for the null space and the range of $T$, respectively. Also, let $\alpha(T) := dimN(T)$ and $\beta(T) := dimN(T^*)$, respectively. For $T \in \mathcal{L}(\mathcal{H})$, the smallest nonnegative integer $p$ such that $N(T^p) = N(T^{p+1})$ is called the *ascent* of $T$ and denoted by $p(T)$. If no such integer exists, we set $p(T) = \infty$. The smallest nonnegative integer $q$ such that $R(T^q) = R(T^{q+1})$ is called the *descent* of $T$ and denoted by $q(T)$. If no such integer exists, we set $q(T) = \infty$.

A *conjugation* on $\mathcal{H}$ is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ which satisfies $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$. For any conjugation $C$, there

---

[0]2010 Mathematics Subject Classification; Primary 47A10, 47A53, 47A55.

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(2014R1A1A2056642). This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2013006537). The third author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science, and Technology (2016R1A2B4007035).

is an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for $\mathcal{H}$ such that $Ce_n = e_n$ for all $n$ (see [7] for more details). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *complex symmetric* if there exists a conjugation $C$ on $\mathcal{H}$ such that $T = CT^*C$. In this case, we say that $T$ is complex symmetric with conjugation $C$. This concept is due to the fact that $T$ is a complex symmetric operator if and only if it is unitarily equivalent to a symmetric matrix with complex entries, regarded as an operator acting on an $l^2$-space of the appropriate dimension (see [7]). All normal operators, Hankel matrices, finite Toeplitz matrices, all truncated Toeplitz operators, and some Volterra integration operators are included in the class of complex symmetric operators. We refer the reader to [7]-[9] for more details.

The Weyl type theorems for upper triangular operator matrices have been studied by many authors. In general, even though Weyl type theorems hold for entry operators $T_1$ and $T_2$, neither $\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ nor $\begin{pmatrix} T_1 & T_3 \\ 0 & T_2 \end{pmatrix}$ satisfies Weyl type theorems (see [10], [11], [13], [14], [3], and ect.). So many authors have been studied the relation between a diagonal matrix and an upper triangular operator matrix of Weyl type theorems. Recently, in [17], they provide several forms of complex symmetric operator matrices $\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ and have studied $a$-Weyl's theorem and $a$-Browder's theorem for complex symmetric operator matrices $\begin{pmatrix} A & B \\ 0 & CA^*C \end{pmatrix}$. We now consider how Weyl type theorems hold for upper triangular operator matrices when some entry operators are complex symmetric.

In this paper, we focus on the operator matrix $\begin{pmatrix} A & B \\ 0 & CA^*C \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ when $B$ is complex symmetric with the conjugation $C$. In this case, we are interested in which the operator matrix $\begin{pmatrix} A & B \\ 0 & CA^*C \end{pmatrix}$ satisfies Weyl type theorems under what behavior of the entry operator $A$. In particular, we give a necessary and sufficient condition for this complex symmetric operator matrices to satisfy $a$-Weyl's theorem. Moreover, we also provide the conditions for such operator matrices to satisfy generalized $a$-Weyl's theorem and generalized $a$-Browder's theorem, respectively. As some applications, we give various examples of such operator matrices which satisfy Weyl type theorems.

## 2 Preliminaries

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *upper semi-Fredholm* if it has closed range and finite dimensional null space and is called *lower semi-Fredholm* if it has closed range and its range has finite co-dimension. If $T \in \mathcal{L}(\mathcal{H})$ is either upper or lower semi-Fredholm, then $T$ is called *semi-Fredholm*, and *index of a semi-Fredholm operator*

$T \in \mathcal{L}(\mathcal{H})$ is defined by

$$i(T) := \alpha(T) - \beta(T).$$

If both $\alpha(T)$ and $\beta(T)$ are finite, then $T$ is called *Fredholm*. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *Weyl* if it is Fredholm of index zero and *Browder* if it is Fredholm of finite ascent and descent, respectively. The left essential spectrum $\sigma_{SF+}(T)$, the right essential spectrum $\sigma_{SF-}(T)$, the essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$, and the Browder spectrum $\sigma_b(T)$ of $T \in \mathcal{L}(\mathcal{H})$ are defined as follows;

$$\sigma_{SF+}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not upper semi-Fredholm}\},$$

$$\sigma_{SF-}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not lower semi-Fredholm}\},$$

$$\sigma_e(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\},$$

$$\sigma_w(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$

and

$$\sigma_b(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\},$$

respectively. Evidently

$$\sigma_{SF+}(T) \cup \sigma_{SF-}(T) = \sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \operatorname{acc} \sigma(T),$$

where we write acc $\Delta$ for the accumulation points of $\Delta \subseteq \mathbb{C}$. If we write iso $\Delta = \Delta \setminus \operatorname{acc} \Delta$, then we let

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(T - \lambda) < \infty \},$$

and $p_{00}(T) := \sigma(T) \setminus \sigma_b(T)$. We say that *Weyl's theorem holds for* $T \in \mathcal{L}(\mathcal{H})$ if $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$, and that *Browder's theorem holds for* $T \in \mathcal{L}(\mathcal{H})$ if $\sigma(T) \setminus \sigma_w(T) = p_{00}(T)$. We recall the definitions of some spectra;

$$\sigma_{ea}(T) := \cap\{\sigma_a(T + K) : K \in \mathcal{K}(\mathcal{H})\}$$

is the essential approximate point spectrum, and

$$\sigma_{ab}(T) := \cap\{\sigma_a(T + K) : TK = KT \text{ and } K \in \mathcal{K}(\mathcal{H})\}$$

is the Browder essential approximate point spectrum. We put

$$\pi_{00}^a(T) := \{\lambda \in \operatorname{iso} \sigma_a(T) : 0 < \alpha(T - \lambda) < \infty\}$$

and $p_{00}^a(T) = \sigma_a(T) \setminus \sigma_{ab}(T)$.

Let $T \in \mathcal{L}(\mathcal{H})$. We say that *a-Browder's theorem holds for* $T$ if

$$\sigma_a(T) \setminus \sigma_{ea}(T) = p_{00}^a(T),$$

and *a-Weyl's theorem holds for* $T$ if

$$\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T).$$

It is known that

$$a\text{-Weyl's theorem} \Longrightarrow a\text{-Browder's theorem} \Longrightarrow \text{Browder's theorem},$$

$$a\text{-Weyl's theorem} \Longrightarrow \text{Weyl's theorem} \Longrightarrow \text{Browder's theorem}.$$

Let $T_n = T|_{R(T^n)}$ for each nonnegative integer $n$; in particular, $T_0 = T$. If $T_n$ is upper semi-Fredholm for some nonnegative integer $n$, then $T$ is called a *upper semi-B-Fredholm operator.* In this case, by [4], $T_m$ is a upper semi-Fredholm operator and $ind(T_m) = ind(T_n)$ for each $m \geq n$. Thus, we can consider the *index* of $T$ as the index of the semi-Fredholm operator $T_n$. Similarly, we define *lower semi-B-Fredholm operators.* We say that $T \in \mathcal{L}(\mathcal{H})$ is *B-Fredholm* if it is both upper and lower semi-$B$-Fredholm. Let $SBF_+^-(\mathcal{H})$ be the class of all upper semi-$B$-Fredholm operators such that $ind(T) \leq 0$, and let

$$\sigma_{SBF_+^-}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin SBF_+^-(\mathcal{H})\}.$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *B-Weyl* if it is $B$-Fredholm of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ of $T$ is defined by

$$\sigma_{BW}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a } B\text{-Weyl operator }\}.$$

In addition, we state two spectra as follows;

$$\sigma_{LD}(T) = \{\lambda \in \mathbb{C} |\ T - \lambda \notin LD(\mathcal{H})\},$$
$$\sigma_{RD}(T) = \{\lambda \in \mathbb{C} |\ T - \lambda \notin RD(\mathcal{H})\},$$

where $LD(\mathcal{H}) = \{T \in \mathcal{H} |\ p(T) < \infty \text{ and } R(T^{p(T)+1}) \text{ is closed}\}$, and $RD(\mathcal{H}) = \{T \in \mathcal{H} |\ q(T) < \infty \text{ and } R(T^{q(T)}) \text{ is closed}\}$. The notation $p_0(T)$ (respectively, $p_0^a(T)$) denotes the set of all poles (respectively, left poles) of $T$, while $\pi_0(T)$ (respectively, $\pi_0^a(T)$) is the set of all eigenvalues of $T$ which is an isolated point in $\sigma(T)$ (respectively, $\sigma_a(T)$).

Let $T \in \mathcal{L}(\mathcal{H})$. We say that

(i) $T$ satisfies *generalized Browder's theorem* if $\sigma(T) \setminus \sigma_{BW}(T) = p_0(T)$;

(ii) $T$ satisfies *generalized a-Browder's theorem* if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = p_0^a(T)$;

(iii) $T$ satisfies *generalized Weyl's theorem* if $\sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T)$;

(iv) $T$ satisfies *generalized a-Weyl's theorem* if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \pi_0^a(T)$.

It is known that

$$\text{generalized } a\text{-Weyl's theorem} \Longrightarrow \text{generalized Weyl's theorem}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

generalized $a$-Browder's theorem $\Longrightarrow$ generalized Browder's theorem.

An operator $T \in \mathcal{L}(\mathcal{H})$ has the *single-valued extension property at* $\lambda_0 \in \mathbb{C}$ if for every open neighborhood $U$ of $\lambda_0$ the only analytic function $f : U \longrightarrow \mathcal{H}$ which satisfies the equation $(T-\lambda)f(\lambda) = 0$ is the constant function $f \equiv 0$ on $U$. The operator $T$ is said to have the *single-valued extension property* if $T$ has the single-valued extension property at every $\lambda_0 \in \mathbb{C}$.

# 3 Wyel Type Theorem

In this section, we study Weyl type theorems for complex symmetric operator matrices. In [17], they provide several forms of complex symmetric operator matrices $\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$. Indeed, if $C$ is a conjugation on $\mathcal{H}$, then $\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ is complex symmetric with $\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$ if and only if $T_2$ and $T_3$ are complex symmetric with a conjugation $C$ and $T_4 = CT_1{}^*C$. For example, the complex symmetric operator matrix $\begin{pmatrix} S^* & 0 \\ 0 & S \end{pmatrix}$ does not satisfy Weyl's theorem where $S$ is the unilateral shift on $\mathcal{H}$. They also have studied $a$-Weyl's theorem and $a$-Browder's theorem for complex symmetric operator matrices $\begin{pmatrix} T_1 & T_2 \\ 0 & CT_1{}^*C \end{pmatrix}$. In this paper, we study generalized Weyl theorem and generalized $a$-Weyl theorem for complex symmetric operator matrices $\begin{pmatrix} T_1 & T_2 \\ T_3 & CT_1{}^*C \end{pmatrix}$ where $C$ is a conjugation on $\mathcal{H}$. Put $\Delta^* := \{\overline{z} : z \in \Delta\}$ for any set $\Delta$ in $\mathbb{C}$. For our study, we start with the following lemmas.

**Lemma 3.1** ([17]) *If $C$ is a conjugation on $\mathcal{H}$ and $A \in \mathcal{L}(\mathcal{H})$, then the following identities hold:*
(i) $\sigma(A)^* = \sigma(CAC)$, $\sigma_p(A)^* = \sigma_p(CAC)$, $\sigma_a(A)^* = \sigma_a(CAC)$, *and* $\sigma_s(A) = \sigma_s(CAC)^*$.
(ii) $\sigma_e(A)^* = \sigma_e(CAC)$, *and* $\sigma_w(A)^* = \sigma_w(CAC)$.

Remark that if $S$ is a complex symmetric operator with the conjugation $C$, then it is known from [16, Lemma 3.5] that $S$ has the single-valued extension property if and only if $S^*$ has. With the similar proof of [16], we have the following lemma.

**Lemma 3.2** *Let $C$ be a conjugation on $\mathcal{H}$ and $S \in \mathcal{L}(\mathcal{H})$. Then $S$ has the single-valued extension property if and only if $CSC$ has.*

**Lemma 3.3** *If $C$ is a conjugation on $\mathcal{H}$ and $A \in \mathcal{L}(\mathcal{H})$, then the following identities hold:*
(i) $\sigma_b(A)^* = \sigma_b(CAC)$ *and* $\sigma_D(A)^* = \sigma_D(CAC)$.
(ii) $\sigma_{LD}(A)^* = \sigma_{LD}(CAC)$ *and* $\sigma_{RD}(A) = \sigma_{RD}(CAC)^*$.
(iii) $\sigma_{BF}(A)^* = \sigma_{BF}(CAC)$ *and* $\sigma_{BW}(A)^* = \sigma_{BW}(CAC)$.

*Throughout this paper, for operators $A, B \in \mathcal{L}(\mathcal{H})$ and a conjugation $C$ on $\mathcal{H}$, put* $M(A,B) = \left\{ \begin{pmatrix} A & B \\ 0 & CA^*C \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}) : B \text{ is complex symmetric with the conjugation } C \right\}$. We study $a$-Weyl theorem and generalized $a$-Weyl theorem for complex symmetric operator matrices in $M(A,B)$.

**Theorem 3.4** *Let $T \in M(A,B)$. Suppose that $A$ is complex symmetric which has the single-valued extension property.*
(a) *Then the following statements are equivalent;*
(i) *$A$ satisfies Weyl's theorem.*
(ii) *$A$ satisfies a-Weyl's theorem.*
(iii) *$T$ satisfies Weyl's theorem.*
(iv) *$T$ satisfies a-Weyl's theorem.*
(b) *Then the following statements are equivalent;*
(i) *$A$ satisfies generalized Weyl's theorem.*
(ii) *$A$ satisfies generalized a-Weyl's theorem.*
(iii) *$T$ satisfies generalized Weyl theorem.*
(vi) *$T$ satisfies generalized a-Weyl theorem.*

Let us recall that the Hilbert Hardy space, denoted by $H^2$, consists of all analytic functions $f$ on the open unit disk $\mathbb{D}$ with the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ where } \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

It is clear that $H^2 = \overline{\text{span}\{z^n : n = 0, 1, 2, 3, \cdots\}}$.

For any $\varphi \in L^\infty$, the *Toeplitz operator* $T_\varphi : H^2 \to H^2$ is defined by the formula

$$T_\varphi f = P(\varphi f)$$

for $f \in H^2$ where $P$ denotes the orthogonal projection of $L^2$ onto $H^2$. Let $C_1$ and $C_2$ be the conjugations on $H^2$ given by

$$(C_1 f)(z) = \overline{f(\overline{z})} \text{ and } (C_2 f)(z) = \overline{f(-\overline{z})}$$

for all $f \in H^2$, respectively.

**Corollary 3.5** *Let $C_1$ and $C_2$ be the conjugations on $H^2$ given by $(C_1f)(z) = \overline{f(\overline{z})}$ and $(C_2f)(z) = \overline{f(-\overline{z})}$ for all $f \in H^2$. Suppose that*

$$T = \begin{pmatrix} T_\varphi & T_\psi \\ 0 & C_1T_\varphi{}^*C_1 \end{pmatrix} \text{ or } T = \begin{pmatrix} T_\psi & T_\varphi \\ 0 & C_2T_\psi{}^*C_2 \end{pmatrix}$$

*are in $\mathcal{L}(H^2 \oplus H^2)$ where*

$$\begin{cases} \varphi(z) = \varphi_0 + 2\sum_{k=1}^{\infty} \hat{\varphi}(2k)Re\{z^{2k}\} + 2i\sum_{k=1}^{\infty} \hat{\varphi}(2k-1)Im\{z^{2k-1}\} \\ \psi(z) = \psi_0 + 2\sum_{n=1}^{\infty} \hat{\psi}(n)Re\{z^n\}. \end{cases} \tag{1}$$

*If $T_\varphi$ or $T_\psi$ have the single-valued extension property, then $T$ satisfies a-Weyl's theorem.*

**Example 3.6** Let $C$ be a conjugation on $l^2(\mathbb{Z})$ given by $Cx = \overline{x}$ for all $x$ and let $U_1$ and $U_2$ are bilateral shifts on $l^2(\mathbb{Z})$. Then $\begin{pmatrix} U_1 & U_2 \\ 0 & CU_1^*C \end{pmatrix} \in \mathcal{L}(l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z}))$ satisfies $a$-Weyl's theorem from Theorem 3.4.

**Corollary 3.7** *Let $T \in M(N, B)$ where $N$ is normal and $B = CB^*C$ for a conjugation $C$. Then $T$ satisfies generalized a-Weyl theorem.*

From the similar way with the proof of Theorem 3.4 and [18, Theorem 4.6], we get the following corollary.

**Corollary 3.8** *Let $T \in M(A, B)$. If $A$ is complex symmetric which has the single-valued extension property, then the following statements are equivalent;*
(i) *$A$ satisfies Browder's theorem.*
(ii) *$A$ satisfies a-Browder's theorem.*
(iii) *$A$ satisfies generalized Browder's theorem.*
(iv) *$A$ satisfies generalized a-Browder's theorem.*
(v) *$T$ satisfies Browder's theorem.*
(vi) *$T$ satisfies a-Browder's theorem.*
(vii) *$T$ satisfies generalized Browder's theorem.*
(viii) *$T$ satisfies generalized a-Browder's theorem.*

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *isoloid* if every $\lambda \in \text{iso}\sigma(T)$ is an eigenvalue of $T$. In [17], they proved that if $T \in M(A, B)$ where $A$ and $A^*$ are isoloid operators with the single-valued extension property and if Weyl's theorem holds for both $A$ and $A^*$, then $a$-Weyl's theorem holds for $T$. Finally, we consider complex symmetric operator matrices where main diagonal operators are not complex symmetric.

**Theorem 3.9** *Let $T \in M(A, B)$ where $A$ and $A^*$ have the single-valued extension property. Then the following statements hold:*

(a) *If $A$ satisfies generalized Weyl theorem, then $T$ satisfies generalized a-Weyl theorem.*

(b) *If $A$ is isoloid, then the following statements are equivalent;*
(i) *$A$ satisfies generalized Weyl theorem.*
(ii) *$A$ satisfies generalized a-Weyl theorem.*
(iii) *$T$ satisfies generalized Weyl theorem.*
(iv) *$T$ satisfies generalized a-Weyl theorem.*

(c) *If $A$ is isoloid, then the following statements are equivalent;*
(i) *$A$ and $A^*$ satisfies Weyl's theorem.*
(ii) *$T$ satisfies Weyl's theorem.*
(iii) *$T$ satisfies a-Weyl theorem.*

**Corollary 3.10** *Let $T \in M(A, N)$ where $A$ is decomposable and $N$ is normal or nilpotent of order 2 with $N = CN^*C$. If $A$ satisfies generalized Weyl's theorem, then $T$ satisfies generalized a-Weyl's theorem.*

**Example 3.11** For $x \in \mathbb{C}^n$, define $C^j(\sum_{i=1}^n \alpha_i e_i) = \sum_{i=1}^n \overline{\alpha_i} e_{n-i+1}$. Put $\mathcal{C} = \oplus C^j$. Then $\mathcal{C}$ is a conjugation on $\mathcal{H}$ where $dim\mathcal{H} = \aleph_0$. Suppose that $S$ is written as $S = \oplus_{j=1}^\infty S_j$ where

$$S_j = \begin{pmatrix} 0 & \lambda_1^{(j)} & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2^{(j)} & \cdots & 0 \\ \cdots & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \lambda_{n_j-1}^{(j)} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to an orthonormal basis of $S_j$ with $|\lambda_k^{(j)}| = |\lambda_{n_j-k}^{(j)}|$ for all $1 \leq k \leq n_j - 1$. Then $S$ is complex symmetric with $\mathcal{C}$ from [23, Theorem 3.1]. Let $W$ be a weighted shift on $\mathcal{H}$ defined by

$$W = (x_1, x_2, x_3, \cdots) := (\frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \cdots).$$

If $T = \begin{pmatrix} W^* & S \\ 0 & \mathcal{C}W\mathcal{C} \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$. Then $T$ satisfies generalized $a$-Weyl's theorem. Indeed, since $\sigma(W^*) = \sigma_{BW}(W^*) = \{0\}$ and $\pi_0(W^*) = \emptyset$, it follows that $W^*$ satisfies generalized Weyl's theorem. Moreover, in this case, $W$ and $W^*$ have the single-valued extension property. Hence $T$ satisfies the generalized $a$-Weyl's theorem from Theorem 3.9.

# References


[1] P. Aiena, *Fredholm and local spectral theory with applications to multipliers*, Kluwer Academic Pub. 2004.

[2] P. Aiena, M.T. Biondi. and C. Carpintero, *On Drazin invertibility*, Proc. Amer. Math. Soc. **136** (2008), no. 8, 2839–2848.

[3] I. J. An, *Weyl type theorems for* $2 \times 2$ *operator matrices*, Kyung Hee Univ., Ph.D. Thesis. 2013.

[4] M. Berkani, *On a class of quasi-Fredholm operators*, Int. Eq. Op. Th. **34**(1999), 244-249.

[5] S. R. Garcia, *Aluthge transforms of complex symmetric operators*, Int. Eq. Op. Th. **60**(2008), 357-367.

[6] ________, *Means of unitaries, conjugations, and the Friedrichs operator*, J. Math. Anal. Appl. **335**(2007), 941-947.

[7] S. R. Garcia and M. Putinar, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358**(2006), 1285-1315.

[8] ________, *Complex symmetric operators and applications* II, Trans. Amer. Math. Soc. **359**(2007), 3913-3931.

[9] ________, *Some new classes of complex symmetric operators*, Trans. Amer. Math. Soc. **362**(2010), 6065-6077.

[10] S. V. Djordjevic and Y. M. Han, *A note on Weyl's theorem for operator matrices*, Proc. Amer. Math. Soc. **130**(2003), 2543-2547.

[11] J. K. Han, H.Y. Lee and W.Y. Lee, *Invertible completions of* $2 \times 2$ *upper triangular operator matrices,* Proc. Amer. Math. Soc. **128** (1999), 119-123.

[12] I. S. Hwang and W.Y. Lee, *The boundedness below of* $2 \times 2$ *upper triangular operator matrices,* Int. Eq. Op. Th. **39** (2001), 267-276.

[13] W.Y. Lee, *Weyl spectra of operator matrices*, Proc. Amer. Math. Soc. **129**(2001), 131-138.

[14] ________, *Weyl's theorem for operator matrices*, Int. Eq. Op. Th. **32** (1998), 319-331.

[15] S. Jung, E. Ko, M. Lee, and J. Lee, *On local spectral properties of complex symmetric operators*, J. Math. Anal. Appl. **379**(2011), 325-333.

[16] S. Jung, E. Ko, and J. Lee, *On scalar extensions and spectral decompositions of complex symmetric operators*, J. Math. Anal. Appl. **382**(2011), 252-260.

[17] ________, *On complex symmetric operator matrices*, J. Math. Anal. Appl. **406**(2013), 373-385.

[18] ________, *Properties of complex symmetric operator matrices*, Operators and matrices. **8**(4)(2014), 957-974.

[19] E. Ko and J. Lee, *On complex symmetric Toeplitz operators*, J. Math. Anal. Appl. **434**(2016), 20-34.

[20] K. Laursen and M. Neumann, *An introduction to local spectral theory*, Clarendon Press, Oxford, 2000.

[21] D. Sarason, *Algebraic properties of truncated Toeplitz operators*, Oper. Matrices, **1**(2007), 419-526.

[22] S. Zhang, H. Zhang, and J. Wu, *Spectra of upper-triangular operator matrix*, preprint.

[23] S. Zhu and C. G. Li, *Complex symmetric weigthed shifts*, Trans. Amer. Math. Soc., **365**(1)(2013), 511-530.

Il Ju An
Department of Mathematics
Hankuk University of Foreign Studies
Yongin-si Gyeonggi-do 17035
Korea
e-mail: 66431004@naver.com

Eungil Ko
Department of Mathematics
Ewha Womans University
Seoul 03760
Korea
e-mail: eiko@ewha.ac.kr

Ji Eun Lee
Department of Mathematics-Applied Statistics
Sejong University
Seoul 05006
Korea
e-mail: jieun7@ewhain.net: jieunlee7@sejong.ac.kr