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On a conjugation and a linear operator

by

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1 Abstract

In this note, we introduce the study of some classes of operators concerning with conjugations on a complex Hilbert space.

2 Definition

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{L}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. For $T \in \mathcal{L}(H)$, let $\sigma(T)$, $\sigma_p(T)$, $\sigma_a(T)$, $\sigma_s(T)$, $\sigma_e(T)$, $\sigma_w(T)$ be the spectrum, the point spectrum, the approximate point spectrum, the surjective spectrum, the essential spectrum and the Weyl spectrum, respectively.

Definition 1. For $T \in \mathcal{L}(\mathcal{H})$, we define $\alpha_m(T)$ and $\beta_m(T)$ as follows;

(1) $\alpha_m(T) = \sum_{j=0}^{m}(-1)^j \binom{m}{j} T^{*m-j}T^j$,

(2) $\beta_m(T) = \sum_{j=0}^{m}(-1)^j \binom{m}{j} T^{*m-j}T^{m-j}$.

(1) T is said to be *m-symmetric* if $\alpha_m(T) = 0$. Then $(-i)^{m-1}\alpha_{m-1}(T) \geq 0$ and $\sigma(T) \subset \mathbb{R}$.

(2) T is said to be *m-isometric* if $\beta_m(T) = 0$. Then $\beta_{m-1}(T) \geq 0$ and $\sigma_a(T) \subset \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

It holds that

(1) $T^* \alpha_m(T) - \alpha_m(T)\, T = \alpha_{m+1}(T)$, (2) $T^* \beta_m(T)\, T - \beta_m(T) = \beta_{m+1}(T)$.

Proposition 1 (Proposition 1.23, [1]) *Let T be m-isometric. If m is even and T is invertible, then T is $(m-1)$-isometric.*

When m is odd, we have the following:

Proposition 2 (Theorem 1, [7]) *If m is any odd number, then there exists an invertible m-isometric which is not $(m-1)$-isometric.*

Proposition 3 (Theorem 3.4, [13]) *If T is m-symmetric and m is even, then T is $(m-1)$-symmetric.*

(1) Let T be 1-symmetric. Then $T^* - T = 0$. So T is Hermitian clearly.
(2) Let T be 2-symmetric. By Proposition 3, T is 1-symmetric. Hence T is Hermitian.
(3) Let T be m-symmetric. For sequences of unit vectors $\{x_n\}, \{y_n\}$, if $(T-a)x_n \rightarrow 0$ and $(T-b)y_n \rightarrow 0$ $(a \neq b)$, then $\langle x_n, y_n \rangle \rightarrow 0$. Hence if $Tx = ax, Ty = by$ $(a \neq b)$, then $\langle x, y \rangle = 0$.

- If Q is 2-nilpotent, then Q is 3-symmetric.

In [11], J. W. Helton introduced m-symmetric for the study of Jordan operators.

- If T is 1-isometric, then $T^*T - I = 0$ and T is an isometry.

In [1], J. Agler and M. Stankus studied m-isometric for the research of Dirichlet Differential operators.

We have many results of m-isometric operators. Researchers are Agler, Stankus, Gu, Bermúdes, Martinón and etc.

3 Conjugation

Definition 2 $C : \mathcal{H} \longrightarrow \mathcal{H}$ is said to be *antilinear* if

$$C(ax + by) = \bar{a}\,Cx + \bar{b}\,Cy, \quad \text{for all } a, b \in \mathbb{C}, x, y \in \mathcal{H}.$$

An antilinear operator C is said to be a *conjugation* if

$$C^2 = I, \quad \langle Cx, Cy \rangle = \langle y, x \rangle \quad \text{for all } x, y \in \mathcal{H}.$$

- If C is a conjugation, then $\|Cx\| = \|x\|$ for all $x \in \mathcal{H}$.

4 Example

Example 1

Typical Example of Conjugation: Let $\mathcal{H} = \mathbb{C}^n$.

(1) $J(z_1, z_2, ..., z_n) = (\overline{z_1}, \overline{z_2}, ..., \overline{z_n})$, (2) $C(z_1, z_2, ..., z_n) = (\overline{z_n}, \overline{z_{n-1}}, ..., \overline{z_1})$.

Then J, C are conjugations.
Example 2

T is said to be *complex symmetric* if there exists a conjugation C such that $CTC = T^*$. Typical Example of a complex symmetric operator T: Let $\mathcal{H} = \mathbb{C}^n$ and T be

$$T = \begin{pmatrix} a_0 & a_{-1} & \cdots & a_{-(n-1)} \\ a_1 & a_0 & \cdots & a_{-(n-2)} \\ \vdots & \ddots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 \end{pmatrix} \quad \text{(Toeplitz matrix)}.$$

Then $CTC = T^*$. Hence every Toeplitz matrix is complex symmetric (C-symmetric).

T. Takagi first showed this. He studied antilinear eigen-value problem. There is the following result.

Takagi Factorization Theorem. *Let T be a symmetric and C-symmetric matrix. Then there exist a unitary U and normal and symmetric N such that $T = UN\,{}^tU$.*

5 Symmetric operators

In [12] S. Jung, E. Ko and J. E. Lee showed several results about complex symmetric operators. We only set the following theorem.

Theorem 1. *Let C be a conjugation and $T \in \mathcal{L}(\mathcal{H})$. Then*

$$\sigma(CTC) = \overline{\sigma(T)}, \quad \sigma_p(CTC) = \overline{\sigma_p(T)}, \quad \sigma_a(CTC) = \overline{\sigma_a(T)},$$

$$\sigma_s(CTC) = \overline{\sigma_s(T)}, \quad \sigma_e(CTC) = \overline{\sigma_e(T)}, \quad \sigma_w(CTC) = \overline{\sigma_w(T)}.$$

• It is not need $CTC = T^*$. It is the relation between spectra of T and CTC.

6 (m, C)-symmetric operator

Definition 3. Let C be a conjugation and $T \in \mathcal{L}(\mathcal{H})$. Then

$$\Delta_m(T;C) = \sum_{j=0}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} T^{*j} C T^{m-j} C.$$

T is said to be (m, C)-*symmetric* if $\Delta_m(T;C) = 0$. (In [2] and [3], it is said to be *m-complex symmetric.*)

We have $T^* \cdot \Delta_m(T;C) - \Delta_m(T;C) \cdot (CTC) = \Delta_{m+1}(T;C)$.

Hence if T is (m, C)-symmetric, then T is (n, C)-symmetric for every n $(\geq m)$.

At the last year RIMS Conference, in [5] we already had a talk of this class. (m, C)-*symmetric* means *m-complex symmetric*. Please see [5].

7 $[m, C]$-symmetric operator

Definition 3. Let C be a conjugation and $T \in \mathcal{L}(\mathcal{H})$. Then

$$\alpha_m(T;C) = \sum_{j=0}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} (CT^{m-j}C)\, T^j.$$

T is said to be $[m, C]$-*symmetric* if $\alpha_m(T;C) = 0$.

We have $CTC \cdot \alpha_m(T;C) - \alpha_m(T;C) \cdot T = \alpha_{m+1}(T;C)$.

Hence if T is $[m, C]$-symmetric, then T is $[n, C]$-symmetric for every n $(\geq m)$.

Theorem 2. *Let C be a conjugation and $T \in \mathcal{L}(\mathcal{H})$.*
(a) *T is $[m, C]$-symmetric if and only if so is T^*.*
(b) *If T is $[m, C]$-symmetric, then so is T^n for every $n \in \mathbb{N}$.*
(c) *If T is $[m, C]$-symmetric and invertible, then T^{-1} is $[m, C]$-symmetric.*

Theorem 3. *Let T be $[m, C]$-symmetric. Then*

$$\sigma(T) = \overline{\sigma(T)}, \quad \sigma_p(T) = \overline{\sigma_p(T)}, \quad \sigma_a(T) = \overline{\sigma_a(T)}, \sigma_s(T) = \overline{\sigma_s(T)}.$$

• A pair (T,S) is said to be *C-doubly commuting* if $TS = ST$ and $CSC \cdot T = T \cdot CSC$.

Lemma 1. *Let (T,S) be C-doubly commuting. Then it holds*

$$\alpha_m(T+S;C) = \sum_{j=0}^{m} \binom{m}{j} \alpha_j(T;C) \cdot \alpha_{m-j}(S;C).$$

Theorem 4. *Let T be $[m,C]$-symmetric and S be $[n,C]$-symmetric. If (T,S) is C-doubly commuting, then $T+S$ is $[m+n-1,C]$-symmetric.*

Theorem 4. *Let Q be n-nilpotent. Then Q is $[2n-1,C]$-symmetric for every conjugation C.*

Theorem 5. *Let T be $[m,C]$-symmetric and Q be n-nilpotent. If (T,Q) is C-doubly commuting, then $T+Q$ is $[m+2n-2,C]$-symmetric.*

Lemma 2. *Let (T,S) be C-doubly commuting. Then it holds*

$$\alpha_m(TS;C) = \sum_{j=0}^{m} \binom{m}{j} \alpha_j(T;C) \cdot T^{m-j} \cdot CS^jC \cdot \alpha_{m-j}(S;C).$$

Theorem 6. *Let T be $[m,C]$-symmetric and S be $[n,C]$-symmetric. If (T,S) is C-doubly commuting, then TS is $[m+n-1,C]$-symmetric.*

Theorem 7. *Let T be $[m,C]$-symmetric and S be $[n,D]$-symmetric. Then $T \otimes S$ is $[m+n-1, C \otimes D]$-symmetric.*

Proof. It is clear that $C \otimes D$ is a conjugation on $\mathcal{H} \otimes \mathcal{H}$. And it is easy to see that $T \otimes I$ is $[m, C \otimes D]$-symmetric and $I \otimes S$ is $[n, C \otimes D]$-symmetric. Also it is clear that $(T \otimes I, I \otimes S)$ is $C \otimes D$-doubly commuting. Since $T \otimes S = (T \otimes I)(I \otimes S)$, by the previous theorem we have $T \otimes S$ is $[m+n-1, C \otimes D]$-symmetric. Q.E.D.

8 (m,C)-isometric operator

Definition 4. Let C be a conjugation and $T \in \mathcal{L}(\mathcal{H})$. Then

$$\Lambda_m(T;C) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*m-j}(CT^{m-j}C).$$

T is said to be *(m, C)-isometric* if $\Lambda_m(T;C) = 0$.

We have $T^* \cdot \Lambda_m(T;C) \cdot (CTC) - \Lambda_m(T;C) = \Lambda_{m+1}(T;C)$.

Hence if T is (m, C)-isometric, then T is (n, C)-isometric for every n ($\geq m$).

Theorem 8. *Let T be (m, C)-isometric. Then;*
(a) *T is bounded below,*
(b) $0 \notin \sigma_a(T)$,
(c) *T is injective and $R(T)$ is closed,*
(d) *if $z \in \sigma_a(T)$, then $\frac{1}{\overline{z}} \in \sigma_a(T^*)$,*
(e) *if there exists T^{-1}, then T^{-1} is (m, C)-isometric.*

Theorem 9. *Let T be (m, C)-isometric. If T^* has* SVEP, *then*

$$\sigma(T) = \sigma_a(T) = \sigma_s(T).$$

Theorem 10. *Let T be (m, C)-isometric. If T is power bounded and $T^*CTC - I$ is normaloid, then T is $(1, C)$-isometric, i.e., $T^*CTC = I$.*

• Of course, if T is m-isometric and power bounded, then T is isometric.

• A pair (T, S) is said to be *C-$*$doubly commuting* if $TS = ST$ and $S^* \cdot CTC = CTC \cdot S^*$.

Lemma 3. *Let (T, S) be C-$*$doubly commuting. Then it holds*

$$\Lambda_m(T+S;C) = \sum_{m_1+m_2+m_3=m} \binom{m}{m_1, m_2, m_3}$$
$$\cdot (T^* + S^*)^{m_1} S^{*m_2} \Lambda_{m_3}(T;C) \cdot (CT^{m_2}C) \cdot (CS^{m_1}C).$$

It follows from the following equation:

$$\big((a+b)(c+d) - 1\big)^m = \big((ac-1) + (a+b)d + bc\big)^m$$
$$= \sum_{m_1+m_2+m_3=m} \binom{m}{m_1, m_2, m_3} \cdot (a+b)^{m_1} b^{m_2} (ac-1)^{m_3} c^{m_2} d^{m_1}.$$

Hence we have the following result.

Theorem 11. *Let T be (m, C)-isometric, Q be n-nilpotent and (T, Q) be a commuting pair. Then $T + Q$ is $(m + 2n - 2, C)$-isometric.*

Lemma 4. *Let (T, S) be C-$*$doubly commuting. Then it holds*

$$\Lambda_m(TS; C) = \sum_{j=0}^{m} \binom{m}{j} T^{*j} \cdot \Lambda_{m-j}(T; C)(CT^jC) \cdot \Lambda_j(S; C).$$

It follows from the following equation:

$$\big(abcd - 1\big)^m = \big((ab - 1) + a(cd - 1)b\big)^m$$

$$= \sum_{j=0}^{m} \binom{m}{j} \cdot a^j (ab - 1)^{m-j} b^j (cd - 1)^j.$$

Hence we have the following result.

Theorem 12. *Let T be (m, C)-isometric and S be (n, C)-isometric. If (T, S) is C-$*$doubly commuting, then TS is $(m + n - 1, C)$-isometric.*

Theorem 13. *Let T be (m, C)-isometric and S be (n, D)-isometric. Then $T \otimes S$ is $(m + n - 1, C \otimes D)$-isometric.*

Proof. It is easy to see that $T \otimes I$ is $(m, C \otimes D)$-isometric and $I \otimes S$ is $(n, C \otimes D)$-isometric. Also it is clear that $(T \otimes I, I \otimes S)$ is $C \otimes D$-$*$doubly commuting. Since $T \otimes S = (T \otimes I)(I \otimes S)$, by the previous theorem we have $T \otimes S$ is $(m + n - 1, C \otimes D)$-isometric. Q.E.D.

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