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DIFFERENTIALS OF COMPLEX INTERPOLATION PROCESSES

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1. The classical theory revisited

Let me explain first what a differential process is, how they appear and what do they mean. Assume one has a space $\mathfrak{F}$ of Banach space valued functions defined on some "parameter" space F; for which we can assume that is a Banach space itself when endowed with a natural norm $\|\cdot\|_{\mathfrak{F}}$ (say, the supremum on F). Evaluation at a point $\theta\in F$ produces a Banach space $X_\theta=\{f(\theta):f\in\mathfrak{F}\}$, which is the quotient space of $\mathfrak{F}$ endowed with the natural quotient norm $\|x\|_\theta=\inf\{\|f\|:f(\theta)=x\}$. If δ_θ denotes the evaluation map then $X_\theta=\delta_\theta(\mathfrak{F})$. Since δ_θ is an open mapping, in fact $\delta_\theta(B_{\mathfrak{F}})=B_{X_\theta}$, given $x\in X_\theta$, an ε-extremal f_x is an element $f_x\in\mathfrak{F}$ so that $f_x(\theta)=x$ and such that $\|f_x\|\leq(1+\varepsilon)\|x\|_\theta$.

A differential process is a correspondence $x\to f_x'(\theta)$. Of course that one has assumed that function of $\mathfrak{F}$ are differentiable. Much trickier is the point of where $f_x'(\theta)$ lies.

To clarify this, let us consider the perfect situation where a differential process appears: complex interpolation. Recall that an interpolation method means to have a space parameter F so that given two (in the simplest version) Banach spaces A, B one can assign to each $\theta\in F$ a Banach space X_θ that is intermediate between A and B in the sense that when an operator sends $A\to A$ and $B\to B$ automatically sends $X_\theta\to X_\theta$. There are many interpolation methods: real methods (several), minimal methods, orbit method, methods specific for Köthe spaces,... We focus now on the complex method.

We describe it in the following way: Assume that both A, B can be embedded into a larger Banach space and consider as $\mathfrak{F}$ the so called Calderon space of holomorphic $A+B$ valued functions defined on the unit complex strip $\mathcal{S}=\{z:0\leq Rez\leq 1\}$ such that $f(it)\in A$ and $f(1+it)\in B$. A few technical properties are usually added. Then, for each inner point $\theta\in\mathcal{S}$, the space $X_\theta=\delta_\theta(\mathfrak{F})$ is an interpolation space between A and B.

Let us present a simple and natural example: Pick $A=\ell_1$ and $B=\ell_\infty$. Interpolation at $z=1/2$ produces the space ℓ_2 by the Riesz-Thorin theorem: i.e., each holomorphic function that takes values in ℓ_1 at 0 and takes values in ℓ_∞ at 1 takes values in ℓ_2 at 1/2. What is the differential process here? Well, given $x\in\ell_2$ an extremal can be given by $f_x(z)=x^{2(1-z)}$ since —we simplify a bit just picking 0 and 1 to represent points it and $1+it$— $f_x(0)=x^2\in\ell_1$ and $f_x(1)=1\in\ell_\infty$. Hence differentiation on the positive cone produces the process:

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$$x \to -2x \log |x|.$$

This has to be understood as the map that associated to the sequence x the sequence $(-2x(n) \log |x(n)|)_n$ What is the meaning of this map? More precisely, What is the meaning of this non-linear, non-continuous, and of course not even well defined from ℓ_2 into ℓ_2 map?

Well, given a Banach space X, non-linear, non-continuous maps defined on X but taking values in some bigger space (from now on simply referred to as "the cloud") have their place in Banach space theory. Indeed, let $\Omega : X \to \bigcirc$ be one of those maps. We will ask it a couple of conditions: to be homogeneous ($\Omega(\lambda x) = \lambda\Omega(x)$) and to verify that for any two points $x, y \in X$ the Cauchy difference $\Omega(x+y) - \Omega(x) - \Omega(y) \in X$. In other words, the map Ω can take values wherever it wants, but the Cauchy differences fall in X. With such a map one can form the derived space

$$d_\Omega X = \{(w, x) \in \bigcirc \times X : x \in X; w - \Omega x \in X\}$$

endowed with the "norm" $\|(w,x)\|_\Omega = \|w - \Omega x\| + \|x\|$. Actually, $\|\cdot\|_\Omega$ is not a norm, only a quasi-norm since it fails the triangle inequality. But in most cases it is equivalent to a norm — the one having as unit ball the closed convex hull of the unit ball of $\|\cdot\|_\Omega$— so we can simplify and just call it "a norm". The space $d_\Omega X$ contains the subspace $\{(x, 0) : x \in X\}$ isometric to X and the quotient space $d_\Omega X/\{(x,0) : x \in X\}$ is again isometric to X. This situation is well known in homological algebra: An exact sequence $0 \to Y \to X \to Z \to 0$, where Y, Z are Banach spaces and the arrows are (bounded) operators is a diagram in which the kernel of each arrow coincides with the image of the preceding one. By the open mapping theorem this means that Y is (isomorphic to) a subspace of X in such a way that the corresponding quotient X/Y is isomorphic to Z. The space X is usually called a twisted sum of Y and X.

Thus, there is an exact sequence of Banach spaces

$$0 \longrightarrow X \longrightarrow d_\Omega X \longrightarrow X \longrightarrow 0$$

and the space $d_\Omega X$ is a twisted sum of X. Twisted sums have been studied long since. The starting problem is probably that of deciding if a twisted sum of Hilbert spaces must itself be a Hilbert space. The first negative answer came from Enflo, Lindenstrauss and Pisier [13] using an ad-hoc construction; but it was Kalton [14] who showed that all twisted sums of two quasi-Banach spaces Y, X are actually induced by a quasi-linear map $\Omega : X \to Y$. Recall that a map $\Omega : X \to Y$ is called *quasi-linear* if it is homogeneous and there is a constant M such that $\|\Omega(u+v) - \Omega(u) - \Omega(v)\| \leq M\|u+v\|$ for all $u, v \in X$. In fact, there is a correspondence (see [11, Theorem 1.5.c, Section 1.6]) between exact sequences $0 \to Y \to \diamondsuit \to X \to 0$ of Banach spaces and a special kind of quasi-linear maps $\omega : X \to Y$, called z-linear maps, which are those quasi-linear maps satisfying $\|\omega(\sum_{i=1}^n u_i) - \sum_{i=1}^n \omega(u_i)\| \leq M \sum_{i=1}^n \|u_i\|$ for all finite sets $u_1, \ldots, u_n \in X$. A quasi-linear map $\Omega : X \to Y$ induces the exact sequence $0 \to Y \xrightarrow{j} Y \oplus_\Omega X \xrightarrow{p} X \to 0$ in which $Y \oplus_F X$ denotes the vector space $Y \times X$ endowed with the quasi-norm $\|(y,x)\|_\Omega = \|y - \Omega(x)\| + \|x\|$. The embedding is $j(y) = (y, 0)$ while the quotient map is $p(y,x) = x$. When Ω is z-linear, this quasi-norm is equivalent to a norm [11, Chapter 1]. On the other hand, the process to obtain a z-linear map out from an exact sequence $0 \to Y \xrightarrow{i} \diamondsuit \xrightarrow{q} X \to 0$ of Banach spaces is the following: get a homogeneous bounded selection $b : X \to \diamondsuit$ for the quotient map q, and then a linear $\ell : X \to \diamondsuit$ selection for the quotient map. Then $\omega = i^{-1}(b - \ell)$ is a z-linear map $X \to Y$. The commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \stackrel{i}{\longrightarrow} & \diamondsuit & \stackrel{q}{\longrightarrow} & X & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle T} & & \| & & \\ 0 & \longrightarrow & Y & \underset{j}{\longrightarrow} & Y \oplus_\Omega X & \underset{p}{\longrightarrow} & X & \longrightarrow & 0 \end{array}$$

obtained by taking as $T : \diamondsuit \to Y \oplus_\Omega X$ the operator $T(x) = (x - \ell qx, qx)$ shows that the upper and lower exact sequences are equivalent. Two quasi-linear maps $F, G : X \to Y$ are said to be equivalent, denoted $F \equiv G$, if the difference $F - G$ can be written as $B + L$, where $B : X \to Y$ is a homogeneous bounded map (not necessarily linear) and $L : X \to Y$ is a linear map (not necessarily bounded).

In [18] Kalton and Peck show that in the particular case of ℓ_p spaces (more generally, Banach spaces with an unconditional basis) it is possible to give an explicit quasi-linear map $\Omega_p : \ell_p \to \ell_p$ by means of

$$\Omega_p(x) = x \log \frac{|x|}{\|x\|_p}$$

when x is a finitely supported sequence, and with the understanding that $\log 0 = 0$. And this means that on the unit sphere $\|x\|_2 = 1$ the Kalton-Peck map Ω_2 is, up to a -2 factor, the differential process corresponding to the natural interpolation scale (ℓ_1, ℓ_2) by the complex method.

To have a better understanding of this connection between interpolation scales and differential processes it will help us to display a few elements of homological algebra in Banach space theory. Two exact sequences $0 \to Y \to X_1 \to Z \to 0$ and $0 \to Y \to X_2 \to Z \to 0$ are *equivalent* if there exists an operator $T : X_1 \to X_2$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \stackrel{i}{\longrightarrow} & X_1 & \stackrel{q}{\longrightarrow} & Z & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle T} & & \| & & \\ 0 & \longrightarrow & Y & \underset{j}{\longrightarrow} & X_2 & \underset{p}{\longrightarrow} & Z & \longrightarrow & 0 \end{array}$$

The classical 3-lemma (see [11, p. 3]) shows that T must be an isomorphism. An exact sequence is trivial if and only if it is equivalent to $0 \to Y \to Y \times Z \to Z \to 0$, where $Y \times Z$ is endowed with the product norm. In this case we say that the exact sequence *splits*.

Of course two exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Y \oplus_\Omega X & \longrightarrow & X & \longrightarrow & 0 \\ & & \| & & & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & Y \oplus_\Psi X & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

induced by two quasi-linear maps Ω, Ψ are equivalent if and only if Ω and Ψ are equivalent.

Given an exact sequence $0 \to Y \to X \to Z \to 0$ with associated quasi-linear map F and an operator $\alpha : Y \to Y'$, there is a commutative diagram

(1)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \stackrel{i}{\longrightarrow} & X & \stackrel{q}{\longrightarrow} & Z & \longrightarrow & 0 \\ & & {\scriptstyle \alpha}\downarrow & & {\scriptstyle T}\downarrow & & \| & & \\ 0 & \longrightarrow & Y' & \stackrel{i'}{\longrightarrow} & PO & \stackrel{q'}{\longrightarrow} & Z & \longrightarrow & 0. \end{array}$$

The lower sequence is called the *push-out sequence*, its associated quasi-linear map is equivalent to $\alpha \circ F$; and the space PO is called the *push-out space*. When F is z-linear, so is $\alpha \circ F$.

The push-out technique will clearly show us the connection between differential processes and quasi-linear maps. Indeed, the complex interpolation method works, as we indicated above, by considering the exact sequence

$$0 \longrightarrow \ker \delta_\theta \longrightarrow \mathfrak{F} \xrightarrow{\delta_\theta} X_\theta \longrightarrow 0$$

Now, a basic principle in [5, Theorem 4.1] establishes that, even if functions of $\mathfrak{F}$ can have their derivatives taking values in the cloud, the functions of $\ker \delta_\theta$ take their value at θ precisely in X_θ; thus, if δ'_θ denotes the operator $f \to f'(\theta)$, the result above says that $\delta'_\theta : \ker \delta_\theta \to X_\theta$ is a linear continuous operator. To formulate this in the homological language, observe that we have already the setting for a push-out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \delta_\theta & \longrightarrow & \mathfrak{F} & \xrightarrow{\delta_\theta} & X_\theta & \longrightarrow & 0 \\ & & \delta'_\theta \big\downarrow & & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & X_\theta & \longrightarrow & PO & \longrightarrow & X_\theta & \longrightarrow & 0. \end{array} \qquad (2)$$

So, it turns out that if one has a complex interpolation scale (X_0, X_1) and interpolates at θ and obtains the space X_θ then the method yields a twisted sum of X_θ. To know exactly which one one just has to follow the diagram: if B_θ, L_θ are homogeneus bounded and linear selctions for δ_θ then the quasi-linear map associated to the upper sequence is $B_\theta - L_\theta$ and consequently the quasi-linear map associated to the lower push-out sequence is

$$\delta'_\theta(B_\theta - L_\theta).$$

In other words, up to a linear map it is the differential process $\delta'_\theta B_\theta$.

Kalton [15, 16] started the study of differential processes and did a tremendously deep work in the context of Köthe function spaces. X over a measure space (Σ, μ). As a particular case of which are the Banach spaces with a 1-unconditional basis. We denote by L_0 the space of all μ-measurable functions and this is going to be our cloud in this context. A centralizer on X is a homogeneous L_0-valued map $\Omega : X \to L_0$ such that $\|\Omega(ax) - a\Omega(x)\|_X \leq C\|x\|_X\|a\|_\infty$ for all $a \in L_\infty$ and $x \in X$. This notion coincides with Kalton's notion of "strong centralizer" introduced in [15]. Centralizers arise naturally in a complex interpolation scheme in which the interpolation scale of spaces share a common L_∞-module structure: in such case, the space $\mathcal{H}$ also enjoys the same L_∞-module structure in the form $(u \cdot f)(z) = u \cdot f(z)$. In this way, the fundamental sequence of the interpolation scheme $0 \to \ker \delta_\theta \to \mathcal{H} \to X_\theta \to 0$ is an exact sequence in the category of L_∞-modules. In an interpolation scheme starting with a couple (X_0, X_1) of Köthe function spaces, the map $\Omega_\theta = \delta_\theta{}' B_\theta$ is a centralizer on X_θ.

Thus, to some extent, it can be said that the centralizer is the core part of the quasi-linear map, or that the quasi-linear map is a linear perturbation of the centralizer. Moreover, centralizers on Köthe function spaces are quasi-linear maps [15, Lemma 4.2]. In addition to this, Kalton proved in [15, Section 4] that every self-extension of a Köthe function space X is (equivalent to) the extension induced by a centralizer on X. Of course that two centralizers Ω_1, Ω_2 on X are *equivalent* if and only if the induced exact sequences are equivalent, which happens if and only if there exists a linear map $L : X \to L_0$ so that $\Omega_1 - \Omega_2 - L$ is bounded. It makes therefore sense to define a centralizer Ω on X to be *bounded* when there exists a constant $C > 0$ so that $\|\Omega(x)\|_X \leq C\|x\|_X$ for all $x \in X$; which in particular means that $\Omega(x) \in X$ for all $x \in X$. Two centralizers Ω_1, Ω_2 are said to be *boundedly equivalent* when $\Omega_1 - \Omega_2$ is bounded.

We have the following outstanding result of Kalton [16, Theorem 7.6]:

Theorem 1.1. *Let X be a separable superreflexive Köthe function space. Then there exists a constant c (depending on the concavity of a q-concave renorming of X) such that if Ω is a real centralizer on X with $\rho(\Omega) \leq c$, then*

(1) *There is a pair of Köthe function spaces X_0, X_1 such that $X = (X_0, X_1)_{1/2}$ and $\Omega - \Omega_{1/2}$ is bounded.*
(2) *The spaces X_0, X_1 are uniquely determined up to equivalent renorming.*

An example is in order: taking the couple (ℓ_1, ℓ_∞), the map $B(x) = x^{2(1-z)}$ is a homogeneous bounded selection for the evaluation map $\delta_{1/2} : \mathcal{H} \to \ell_2$; hence the interpolation procedure yields the centralizer $-2\Omega_2$; while the couple (ℓ_p, ℓ_{p^*}) yields $-2(\frac{1}{p} - \frac{1}{p^*})\Omega_2$. As we see the two centralizers are the same up to the scalar factor. That scalar factor however cannot be overlooked since it actually determines the end points X_0, X_1 in the interpolation scale.

2. Perspectives

The research program in which I'm involved contemplates to advance in the understanding of Kalton's theorem. The basic question can be stated as:

Problem 1. Determine to what extent Kalton's theorem works for general interpolation scales.

A second line of research focuses on the underlying connection between Kalton's theorem and the theory of twisted sums. Observe that it is contained in Kalton's theorem that the associated centralizer is bounded if and only if the two extremes of the scale coincide. In other words, the equality notion that corresponds to differential processes seems to be that of bounded equivalence because one gets that the induced exact sequence is "trivial" if and only if the associated centralizer is bounded. But that is not the notion that appears via homology: the induced sequence is trivial if and only if the associated exact sequence is equivalent to 0; i.e., the sum of a bounded plus a linear map. So the question is

Problem 2. Does there exist a Kalton's theorem valid for standard equivalence of maps (bounded plus linear) instead of bounded equivalence ?

A sentence in [6, p. 364] suggests a positive answer for Banach spaces with unconditional basis: *If (Z_0, Z_1) are two super-reflexive sequence spaces and $Z_\theta = [Z_0, Z_1]_\theta$ for $0 < \theta < 1$ is the usual interpolation space by the Calderon method, one can define a derivative dX_θ which is a twisted sum $X_\theta \oplus_\Omega X_\theta$ which splits if and only if $Z_1 = wZ_0$ for some weight sequence $w = (w(n))$ where $w(n) > 0$ for all n. These remarks follow easily from the methods of* [16]. In fact, an appeal to the methods of [4] allows one to prove it. Here is an sketch of proof. Let (X_0, X_1) be an interpolation couple of superreflexive Banach spaces having a common unconditional basis (e_n) and let $0 < \theta < 1$. If Ω_θ is trivial then $X_1 = wX_0$ for some weight sequence.

Proof. We need to show first that if the associated centralizer Ω_θ is trivial then there is a function $f \in \ell_\infty$ so that $\Omega_\theta(x) - fx$ is bounded. Indeed, if there is a linear map L so that $\Omega_\theta - L$ takes values in X_θ and is bounded there. The techniques in [8] show that after some averaging it is possible to get a linear map Λ such that $\Omega_\theta - \Lambda$ takes values in X_θ and is bounded there and, moreover, $\Lambda(ux) = u\Lambda x$ for every unit of ℓ_∞. From where it follows that $\Lambda(ax) = a\Lambda(x)$ for every $a \in \ell_\infty$. It is then a standard fact that Λ must have the form $\Lambda(x) = fx$ for some function f. And since one can assume $\Omega_\theta(e_n) = 0$, necessarily $f \in \ell_\infty$. The rest is simple, just pick $w = e^f$ and observe that the centralizer associated to the couple (X_0, wX_0) is boundedly equivalent to Ω_θ. Thus, by Kalton's uniqueness theorem $X_1 = wX_0$. $\square$

Turinng to specific properties, the general question is:

Problem 3. Obtain conditions to detect that the differential process Ω_θ has a certain property.

The first property to look at is triviality, and this brings us back to problem 2. The next interesting property is singularity (a quasi-linear map is said to be singular when no restriction to an infinite dimensional subspace is trivial); the Kalton-Peck map Ω_2 is singular. A thorough study on singularity can be found in [8]. Super-singularity has however been considered in [10].

After Problem 3 they come questions involving the stability of the properties of the differential process, a problem for which several partial results have been obtained in [12, 9]. Precisely:

Problem 4. Local stability. Assume that for some θ the differential process Ω_θ is trivial (in some sense). Does there exist a neighborhood V of θ so that Ω_ν is also trivial for all $\nu \in V$?

Problem 5. Global stability. Assume that for some θ the differential process Ω_θ is trivial (in some sense). Does there it follows that Ω_ν is trivial for all ν?

Since interpolation methods there are many, the next set of questions is

Problem 6. Other interpolation methods. Set and solve the problems above for other interpolation methods.

In particular, the paper [3] clearly shows that homological techniques provide a natural frame to iterate differential complex processes. In particular, to obtain twisted sums of twisted sums, ... and so on. It is natural to ask

Problem 6. Iteration. Does a similar iteration process exist for other interpolation methods ?

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