数理解析研究所講究録
第2033巻 2017年 57-69



# Operator monotonicity of functions related to the Stolarsky mean and $\exp\{f(x)\}$

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**Abstract**

The weighted power mean is one of the most famous 2-parameter operator mean, and its representing function is $P_{s,\alpha}[(1-\alpha)+\alpha x^{s}]^{\frac{1}{s}}$ $(s \in [-1,1],\ \alpha \in [0,1])$. In [6] we constructed a 2-parameter family of operator monotone function $F_{r,s}(x)$ $(r, s \in [-1,1])$ by integration of the function $P_{s.\alpha}(x)$ of $\alpha \in [0,1]$. We shall extend its range of parameters $r$ and $s$. We also consider operator monotonicity of $\exp\{f(x)\}$ for a non-constant continuous function $f(x)$ defined on $(0, \infty)$.

## 1 Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. We assume that a function is not a constant throughout this paper. A continuous function $f(x)$ defined on an interval $I$ is called an operator monotone function, if $A \leq B$ implies $f(A) \leq f(B)$ for every pair $A, B \in \mathcal{B}(\mathcal{H})$ whose spectra $\sigma(A)$ and $\sigma(B)$ lie in $I$. We call $f(x)$ a Pick function if $f(x)$ has an analytic continuation to the upper half-plane $\mathbb{C}^{+}=\{z \in \mathbb{C}\ |\Im z>0\}$ and $f(z)$ maps from $\mathbb{C}^{+}$ into itself, where $\Im z$ means the imaginary part of $z$. It is well known that a Pick function is an operator monotone function and conversely an operator monotone function is a Pick function (Löwner's theorem, cf. [2]).

A map $\mathfrak{M}(\cdot, \cdot)$: $\mathcal{B}(\mathcal{H})_{+}^{2} \rightarrow \mathcal{B}(\mathcal{H})_{+}$ is called an *operator mean* [3] if the operator $\mathfrak{M}(A, B)$ satisfies the following four conditions for $A, B \in \mathcal{B}(\mathcal{H})_{+}$;

(1) $A \leq C$ and $B \leq D$ imply $\mathfrak{M}(A, B) \leq \mathfrak{M}(C, D)$,

(2) $C(\mathfrak{M}(A, B))C \leq \mathfrak{M}(CAC, CBC)$ for all self-adjoint $C \in \mathcal{B}(\mathcal{H})$,

(3) $A_{n} \searrow A$ and $B_{n} \searrow B$ imply $\mathfrak{M}(A_{n}, B_{n}) \searrow \mathfrak{M}(A, B)$,

(4) $\mathfrak{M}(I, I) = I$.

Next theorem is so important to study operator means;

**Theorem K-A** (Kubo-Ando [3]). *For any operator mean $\mathfrak{M}(\cdot,\cdot)$, there uniquely exists an operator monotone function $f \geq 0$ on $[0,\infty)$ with $f(1) = 1$ such that*

$$f(x)I = \mathfrak{M}(I, xI), \quad x \geq 0.$$

*Then the following hold:*

(1) *The map $\mathfrak{M}(\cdot,\cdot) \mapsto f$ is a one-to-one onto affine mapping from the set of all operator means to the set of all non-negative operator monotone functions on $[0,\infty)$ with $f(1) = 1$. Moreover, $\mathfrak{M}(\cdot,\cdot) \mapsto f$ preserves the order, i.e., for $\mathfrak{M}(\cdot,\cdot) \mapsto f$, $\mathfrak{N}(\cdot,\cdot) \mapsto g$,*

$$\mathfrak{M}(A, B) \leq \mathfrak{N}(A, B) \quad (A, B \in \mathcal{B}(\mathcal{H})_+) \Longleftrightarrow f(x) \leq g(x) \quad (x \geq 0).$$

(2) *When $A > 0$,* $\quad \mathfrak{M}(A, B) = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}}$.

The function $f(x)$ is called the *representing function* of $\mathfrak{M}(\cdot,\cdot)$. When we study operator means, we usually consider their representing functions.

The 2-parameter family of operator monotone functions $\{F_{r,s}(x)\}_{r,s\in[-1,1]}$;

$$F_{r,s}(x) := \left(\frac{r(x^{r+s} - 1)}{(r+s)(x^r - 1)}\right)^{\frac{1}{s}}$$

is constructed in [6] by integration the function $[(1-\alpha) + \alpha x^p]^{\frac{1}{p}}$, which representing the weighted power mean, of the parameter $\alpha \in [0,1]$. This family interpolates many well-known operator monotone functions and has monotonicity of $r$ and $s$, namely, $-1 \leq r_1 \leq r_2 \leq 1$, $-1 \leq s_1 \leq s_2 \leq 1$ imply $F_{r_1,s_1}(x) \leq F_{r_2,s_2}(x)$. From this fact, we can easily get the following inequalities;

$$\frac{2x}{x+1} \leq \frac{x \log x}{x-1} \leq x^{\frac{1}{2}} \leq \frac{x-1}{\log x} \leq \exp\left(\frac{x \log x}{x-1} - 1\right) \leq \frac{x+1}{2}.$$

Moreover, $\{F_{r,s}(x)\}_{r,s\in[-1,1]}$ interpolates some famous 1-parameter family of operator monotone functions. By connecting ranges of parameter for the cases $s = 1$ and $s = -1$, we obtain a 1-parameter family $\{PD_r(x)\}_{r\in[-1,2]}$ of operator monotone functions such that

$$PD_r(x) = \frac{(r-1)(x^r - 1)}{r(x^{r-1} - 1)} \quad (-1 \leq r \leq 2).$$

This family is called the power difference mean and the optimality of its range of the parameter $-1 \leq r \leq 2$ is well known. $F_{s,s}(x) := P_s(x)$;

$$P_s(x) = \left(\frac{x^s + 1}{2}\right)^{\frac{1}{s}} \quad (-1 \leq s \leq 1)$$

is the representing function of the power mean, and its range of parameter $-1 \leq s \leq 1$ is optimal. If $r = 1$ and $s = p - 1$, then $F_{1,p-1}(x) := S_p(x)$;

$$S_p(x) = \left(\frac{p(x-1)}{x^p - 1}\right)^{\frac{1}{1-p}} \quad (0 \leq p \leq 2).$$

$S_p(x)$ is well known as the representing function of the Stolarsky mean, and is operator monotone if and only if $-2 \leq p \leq 2$ ([5]). But we cannot prove operator monotonicity of $S_p(x)$ for $-2 \leq p < 0$ by the same way, because $s = p - 1 \in [-1, 1]$. So we think that the range of parameter of $\{F_{r,s}(x)\}_{r,s \in [-1,1]}$ such that $F_{r,s}(x)$ is operator monotone is not optimal. In Section 2, we consider the range of parameter of $\{F_{r,s}(x)\}$ in which the function is operator monotone, and try to extend it by using operator monotonicity of $S_p(x)$ and $F_{r,s}(x)$ for $p \in [-2, 2]$ and $r, s \in [-1, 1]$, respectively.

On the other hand, we have operator monotonicity of the following function from $\{S_p(x)\}_{p \in [-2,2]}$;

$$S_1(x) := \lim_{p \to 1} S_p(x) = \exp\left(\frac{x \log x}{x - 1} - 1\right).$$

(This function is known as the representing function of the identric mean.) The exponential function $\exp(x)$ is well known as a function which is not operator monotone, in contrast with its inverse function $\log x$ is so. But there exists a function $f(x)$ such that $\exp\{f(x)\}$ is an operator monotone function besides constant, like $S_1(x)$. In general, it is so difficult to check operator monotonicity of $\exp\{f(x)\}$ because $\exp\{f(x)\}$ is a composite function of the non-operator monotone function $\exp(x)$ with $f(x)$. In Section 3, we give a characterization of $f(x)$ such that $\exp\{f(x)\}$ is operator monotone. Thanks to this result, it has become easy to check operator monotonicity of $\exp\{f(x)\}$ by simple computation, and by applying this result we get some examples of functions $f(x)$ such that $\exp\{f(x)\}$ is operator monotone.

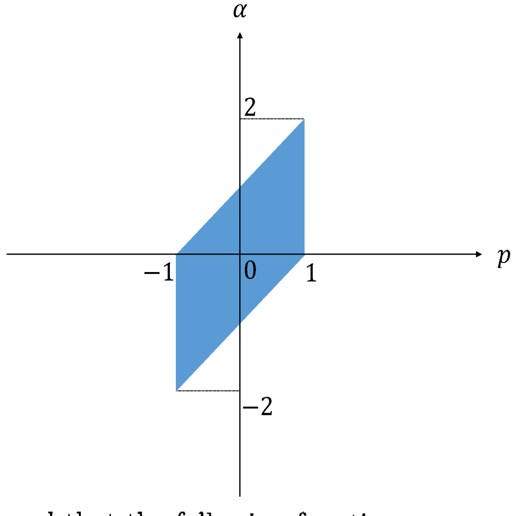
## 2 2-parameter Stolarsky mean

First of this section, we replace symbols $r, s$ with symbols $p, \alpha - p$ as the following;

$$F_{r,s}(x) = \left(\frac{r(x^{r+s}-1)}{(r+s)(x^r-1)}\right)^{\frac{1}{s}} \xrightarrow{r\to p,\ s\to\alpha-p} \left(\frac{p(x^\alpha-1)}{\alpha(x^p-1)}\right)^{\frac{1}{\alpha-p}}.$$

Here we denote by $S_{p,\alpha}(x)$ the above function. From operator monotonicity of $\{F_{r,s}(x)\}_{r,s\in[-1,1]}$, we can find the fact that $S_{p,\alpha}(x)$ is operator monotone if

$$p \in [-1, 1] \text{ and } p - 1 \le \alpha \le p + 1.$$




In [4], they showed that the following function

$$h_{p,\alpha}(x) = \frac{\alpha(x^p - 1)}{p(x^\alpha - 1)}$$

is operator monotone if and only if

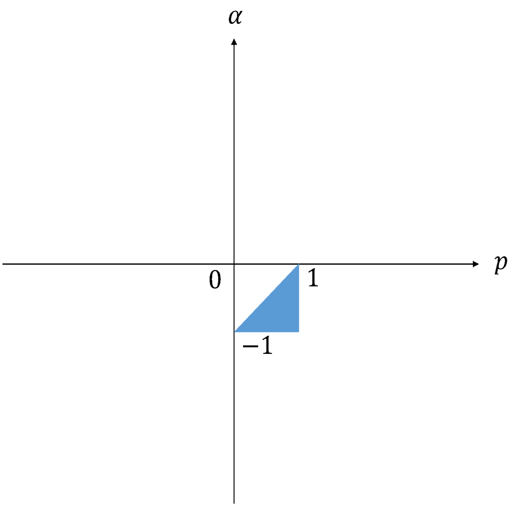
$$(p, \alpha) \in \left\{(p, \alpha) \in \mathbb{R}^2 \;\middle|\; 0 < p-\alpha \le 1, p \ge -1, \text{and } \alpha \le 1\right\} \cup ([0, 1] \times [-1, 0]) \backslash \{(0, 0)\}.$$

Also, if $(p, \alpha) \in \{(p, \alpha) \in \mathbb{R}^2 | 0 \le p \le 1, -1 \le \alpha \le 0 \text{ and } \alpha \le p - 1\}$, then

$$\frac{1}{p - \alpha} \in \left[\frac{1}{2}, 1\right].$$

From these results and Löwner-Heinz inequality, we can find that $S_{p,\alpha}(x) = h_{p,\alpha}(x)^{\frac{1}{p-\alpha}}$ is operator monotone if

$$(p, \alpha) \in \{(p, \alpha) \in \mathbb{R}^2 | 0 \le p \le 1, -1 \le \alpha \le 0 \text{ and } \alpha \le p - 1\}.$$

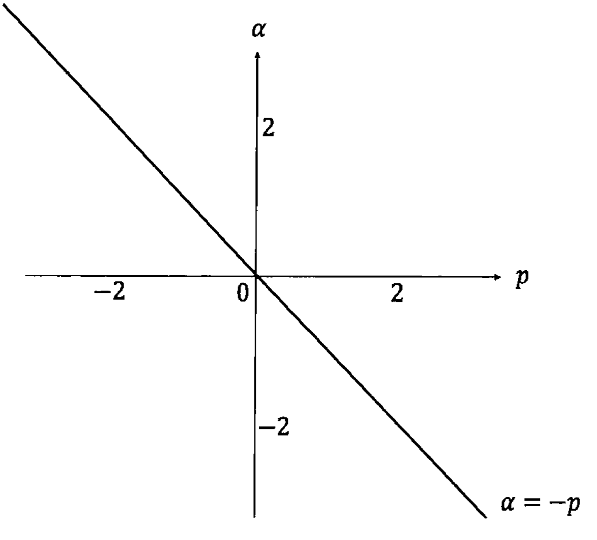



**Trivial part.**

There is a case where $S_{p,\alpha}(x)$ is operator monotone regardless of the value of $p$ or $\alpha$. If $\alpha = -p$, then

$$S_{p,-p}(x) = \left( \frac{p(x^{-p} - 1)}{(-p)(x^p - 1)} \right)^{\frac{1}{-2p}} = \left( \frac{1}{x^p} \right)^{\frac{1}{-2p}} = x^{\frac{1}{2}}.$$

Hence, we find that operator monotonicity of $S_{p,\alpha}(x)$ always holds if $\alpha = -p$.



**Extension from operator monotonicity of $\{S_p(x)\}_{p\in[-2,2]}$.**

From Löwner's theorem and operator monotonicity of the 1-parameter family $\{S_p(x)\}_{p\in[-2,2]}$, $z\in\mathbb{C}^+$ implies $S_p(z)\in\mathbb{C}^+$ for all $p\in[-2,2]$, namely, the argument of $S_p(z)$ has the following property

$$0<\arg\left(\frac{p(z-1)}{z^p-1}\right)^{\frac{1}{1-p}}\left(=\frac{1}{1-p}\arg\left(\frac{p(z-1)}{z^p-1}\right)\right)<\pi$$

$(z\in\mathbb{C}^+,\ -2\le p\le 2)$. So we get

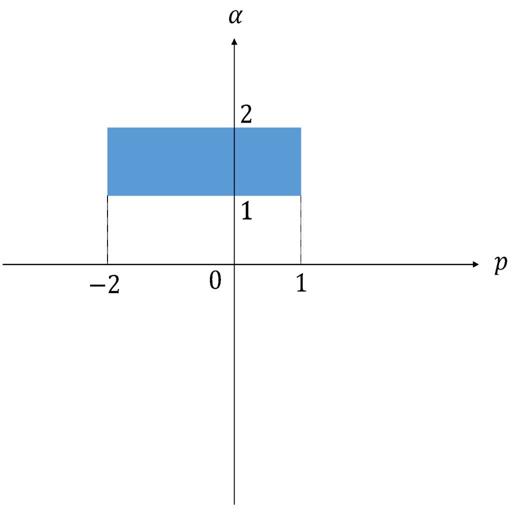
$$0<\arg\left(\frac{p(z-1)}{z^p-1}\right)<(1-p)\pi\quad(-2\le p<1),$$

$$0<\arg\left(\frac{z^p-1}{p(z-1)}\right)<(p-1)\pi\quad(1<p\le 2),$$

respectively. By these inequalities we obtain

$$\begin{aligned}0&<\arg\left(\frac{p(z^\alpha-1)}{\alpha(z^p-1)}\right)^{\frac{1}{\alpha-p}}\\&=\frac{1}{\alpha-p}\left\{\arg\left(\frac{p(z-1)}{z^p-1}\right)+\arg\left(\frac{z^\alpha-1}{\alpha(z-1)}\right)\right\}\\&<\frac{1}{\alpha-p}\{(\alpha-1)\pi+(1-p)\pi\}=\pi\end{aligned}$$

for the case $-2\le p<1$, $1<\alpha\le 2$.



On the other hand,

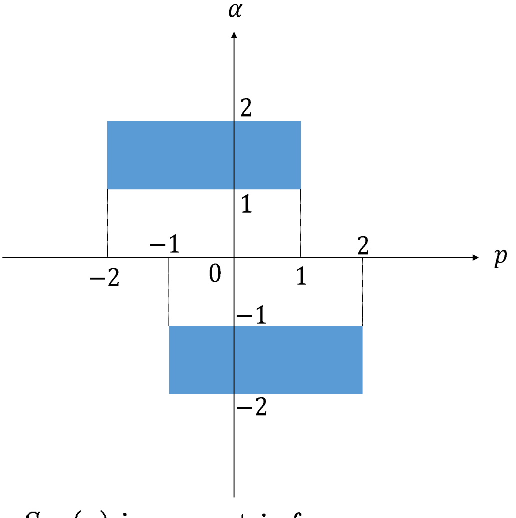
$$S_{-p}(x^{-1})^{-1} = \left(\frac{x(x^p - 1)}{p(x-1)}\right)^{\frac{1}{1+p}}$$

is operator monotone for $-2 \le p \le 2$ too. So we have

$$0 < \frac{1}{1+p} \arg\left(\frac{z(z^p - 1)}{p(z-1)}\right) < \pi \ (z \in \mathbb{C}^+, \ -2 \le p \le 2)$$

and we can show the case $-1 < p \le 2$, $-2 \le \alpha < -1$ similarly, because

$$\arg\left(\frac{p(z^\alpha - 1)}{\alpha(z^p - 1)}\right)^{\frac{1}{\alpha - p}} = \frac{1}{p-\alpha}\left\{\arg\left(\frac{z(z^p-1)}{p(z-1)}\right) + \arg\left(\frac{\alpha(z-1)}{z(z^\alpha - 1)}\right)\right\}.$$

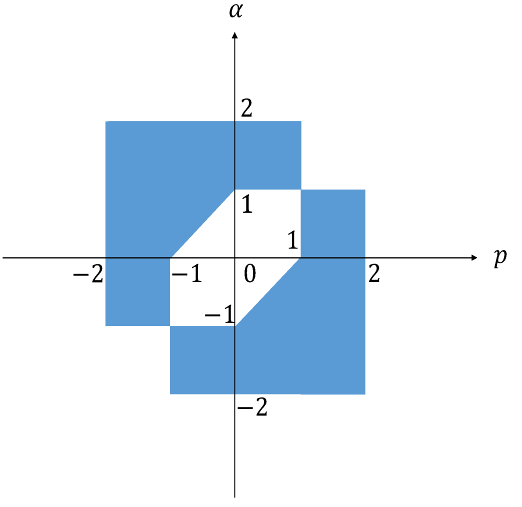



Moreover, since $S_{p,\alpha}(x)$ is symmetric for $p, \alpha$, we can extend the range of parameter symmetrically from the above results. Namely, we have

$$(-2 \le p < 1, \ 1 < \alpha \le 2) \longrightarrow (-2 \le \alpha < 1, \ 1 < p \le 2),$$

$$(-1 < p \le 2, \ -2 \le \alpha < -1) \longrightarrow (-1 < \alpha \le 2, \ -2 \le p < -1),$$

$$(p, \alpha) \in \{(p, \alpha) \in \mathbb{R}^2 | 0 \le p \le 1, -1 \le \alpha \le 0 \text{ and } \alpha \le p - 1\}$$
$$\longrightarrow (p, \alpha) \in \{(p, \alpha) \in \mathbb{R}^2 | 0 \le \alpha \le 1, -1 \le p \le 0 \text{ and } p \le \alpha - 1\}.$$

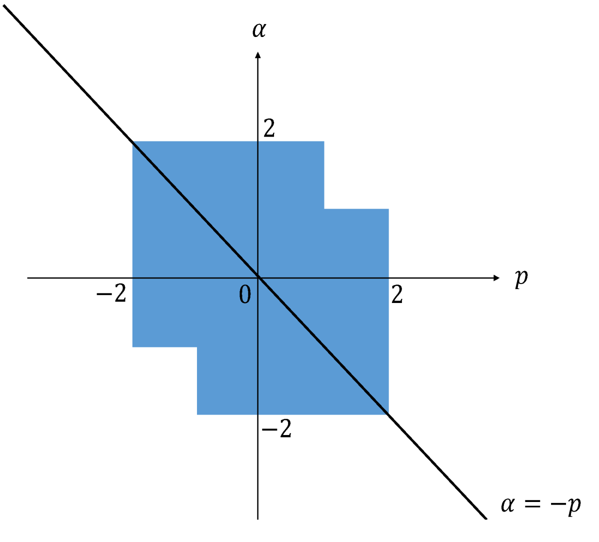



**Theorem 1.** *Let*

$$S_{p,\alpha}(x) = \left(\frac{p(x^\alpha - 1)}{\alpha(x^p - 1)}\right)^{\frac{1}{\alpha - p}} \quad (x > 0).$$

*Then* $S_{p,\alpha}(x)$ *is operator monotone if* $(p,\alpha) \in \mathcal{A} \subset \mathbb{R}^2$*, where*

$$\mathcal{A} = \big([-2,1] \times [-1,2]\big) \cup \big([-1,2] \times [-2,1]\big) \cup \big\{(p,\alpha) \in \mathbb{R}^2 \;\big|\; \alpha = -p\big\}.$$



## 3 Operator monotonicity of $\exp\{f(x)\}$

First of this section we give a characterization of a continuous function $f(x)$ on $(0,\infty)$ such that $\exp\{f(x)\}$ is an operator monotone function. It is clear that $\exp\{\log x\} = x$ is operator monotone. The principal branch of Log$z$ is defined as

$$\mathrm{Log}z := \log r + i\theta \ (z := re^{i\theta}, 0 < \theta < 2\pi).$$

It is an analytic continuation of the real logarithmic function to $\mathbb{C}$. Moreover it is a Pick function, namely an operator monotone function, and satisfies $\Im \mathrm{Log}z = \theta$. In the following we think about the case $f(x)$ is not the logarithmic function:

**Theorem 2.** *Let $f(x)$ be a continuous function on $(0,\infty)$. If $f(x)$ is not a constant or $\log(\alpha x)$ $(\alpha > 0)$, then the following are equivalent:*
*(1) $\exp\{f(x)\}$ is an operator monotone function,*
*(2) $f(x)$ is an operator monotone function, and there exists an analytic continuation satisfying*

$$0 < v(r,\theta) < \theta,$$

*where*

$$f(re^{i\theta}) = u(r,\theta) + iv(r,\theta) \ (0 < r,\ 0 < \theta < \pi).$$

**Remark 1.** *In [1] Hansen proved a necessary and sufficient condition for $\exp\{F(\log x)\}$ to be an operator monotone function, that is, $F$ admits an analytic continuation to $\mathbb{S} = \{z \in \mathbb{C} \mid 0 < \Im z < \pi\}$ and $F(z)$ maps from $\mathbb{S}$ into itself. A condition of Theorem 2 is more rigid than this statement.*

*Proof.* (2) $\Longrightarrow$ (1) Clear.
(1) $\Longrightarrow$ (2).
Since $\exp\{f(x)\}$ is operator monotone, $\log\{\exp\{f(x)\}\} = f(x)$ is operator monotone, too. Also $\exp\{f(x)\}$ is a Pick function, so there exists an analytic continuation to the upper half plane $\mathbb{C}^+$ and $z \in \mathbb{C}^+$ implies $\exp\{f(z)\} \in \mathbb{C}^+$. For $z = s+it \in \mathbb{C}^+$ $(s \in \mathbb{R},\ 0 < t)$, let $f(z) = f(s+it) = p(s,t)+iq(s,t)$. Then $q(s,t) > 0$ since $f(x)$ is a Pick function. Using Euler's formula, we obtain

$$\exp\{f(z)\} = \exp\{p(s,t)\}\big(\cos\{q(s,t)\} + i\sin\{q(s,t)\}\big).$$

So we have $\Im\exp\{f(z)\} = \exp\{p(s,t)\}\sin\{q(s,t)\}$, and hence $0 < \sin\{q(s,t)\}$. Also, $q(s,t)$ belongs to $C^1$, so $q(s,t)$ is continuous on its domain. From these facts, we can find that $2n\pi < q(s,t) < (2n+1)\pi$ holds for the unique

$n \in \mathbb{N} \cup \{0\}$.Moreover $\lim_{t \to 0} f(s+it) = f(s) \in \mathbb{R}$, namely, $q(s,t) \to 0$ $(t \to 0)$ holds. This implies $n = 0$ and

$$0 < q(s,t) < \pi.$$

Here by putting $z = re^{i\theta}$ $(0 < r,\ 0 < \theta < \pi)$, $f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ again, we have

$$0 < v(r,\theta) < \pi.$$

On the other hand, from the operator monotonicity of $\exp\{f(x)\}$ and the assumption of Theorem 2, $x[\exp\{f(x)\}]^{-1}$ is a positive operator monotone function on $(0,\infty)$, too. So we get

$$\begin{aligned} z[\exp\{f(z)\}]^{-1} &= \exp\{\mathrm{Log} z - f(z)\} \\ &= \exp\{(\log r - u(r,\theta)) + i(\theta - v(r,\theta))\} \\ &= \exp\{\log r - u(r,\theta)\}\big(\cos\{\theta - v(r,\theta)\} + i\sin\{\theta - v(r,\theta)\}\big). \end{aligned}$$

From the above,

$$2m\pi < \theta - v(r,\theta) < (2m+1)\pi$$

holds for the unique $m \in \mathbb{Z}$. Moreover, $0 < v(r,\theta) < \pi$ and $0 < \theta < \pi$ are required from the assumption and the above argument, and hence

$$-\pi < -v(r,\theta) < \theta - v(r,\theta) < \theta < \pi.$$

From these facts, $v(r,\theta)$ must satisfy $0 < \theta - v(r,\theta) < \pi (**)$, so we get

$$0 < v(r,\theta) < \theta$$

by the left side inequality of $(**)$. □

By using Theorem 2, we can check numerically that $\exp\{f(x)\}$ is operator monotone or not if the imaginary part of $f(z)$ can be expressed concretely. Now we apply Theorem 2 and give some examples by "only" using simple computation.

**Example 1** (Harmonic, geometric and logarithmic means).

$$H(x) = \frac{2x}{x+1},\ G(x) = x^{\frac{1}{2}} \text{ and } L(x) = \frac{x-1}{\log x}$$

*are operator monotone functions on* $[0,\infty)$, *but* $\exp\{H(x)\}$, $\exp\{G(x)\}$ *and* $\exp\{L(x)\}$ *are not operator monotone. Actually, by putting* $z = re^{i\theta}$ $(0 < r, 0 < \theta < \pi)$, *we have*

$$v_H(r,\theta) := \Im H(z) = \frac{2r\sin\theta}{r^2+1+2r\cos\theta},\ v_G(r,\theta) := \Im G(z) = r^{\frac{1}{2}}\sin\frac{\theta}{2}$$

*and*

$$v_L(r,\theta) := \Im L(z) = \frac{(r\log r)\sin\theta - \theta(r\cos\theta - 1)}{(\log r)^2 + \theta^2}.$$

*When* $r = 1, \theta = \frac{5}{6}\pi$, *we get* $v_H\left(1, \frac{5}{6}\pi\right) = 2 + \sqrt{3} > \frac{5}{6}\pi$, *hence we can find* $\exp\{H(x)\}$ *is not an operator monotone function by Theorem 2. We can also obtain* $v_G\left(2\pi^2, \frac{\pi}{2}\right) = \pi > \frac{\pi}{2}$ *and* $v_L\left(\exp\left\{\frac{\pi}{2}\right\}, \frac{\pi}{2}\right) = \frac{\exp\{\frac{\pi}{2}\}+1}{\pi} > \frac{\pi}{2}$, *so* $\exp\{G(x)\}$ *and* $\exp\{L(x)\}$ *are not operator monotone too.*

**Example 2** (Dual of Logarithmic mean).

$$DL(x) = \frac{x\log x}{x-1}$$

*is an operator monotone function on* $[0,\infty)$ *and* $\exp\{DL(x)\}$ *is operator monotone, too. In the following we verify that* $DL(x)$ *satisfies the condition of Theorem 2:*
*By putting* $z = re^{i\theta}$ $(0 < r, 0 < \theta < \pi)$, *we have*

$$v_{DL}(r,\theta) := \Im DL(z) = \frac{r}{r^2+1-2r\cos\theta}\big\{\theta(r-\cos\theta) - (\log r)\sin\theta\big\}.$$

$0 < v_{DL}(r,\theta)$ *is clear since* $DL(x)$ *is operator monotone. So we only show* $v_{DL}(r,\theta) < \theta$.
*Proof of* $v_{DL}(r,\theta) < \theta$;
$v_{DL}(r,\theta) < \theta$ *is equivalent to* $r\big\{\theta\cos\theta - (\log r)\sin\theta\big\} < \theta$. *By using the following inequalities*

$$\theta\cos\theta \le \sin\theta < \theta \ (0 < \theta < \pi), \ r(1-\log r) \le 1 \ (0 < r),$$

*we obtain*

$$\begin{aligned} r\big\{\theta\cos\theta - (\log r)\sin\theta\big\} &\le r\big\{\sin\theta - (\log r)\sin\theta\big\} \\ &= r(1-\log r)\sin\theta \\ &\le \sin\theta < \theta. \end{aligned}$$

**Example 3.**

$$IL(x) := -L(x)^{-1} = -\frac{\log x}{x-1}$$

*is a negative operator monotone function on* $(0,\infty)$ *and* $\exp\{IL(x)\}$ *is operator monotone, too.*
*By putting* $z = re^{i\theta}$ $(0 < r, 0 < \theta < \pi)$, *we have*

$$v_{IL}(r,\theta) := \Im IL(z) = \frac{(r\log r)\sin\theta - \theta(r\cos\theta - 1)}{r^2+1-2r\cos\theta}.$$

*We can show* $0 < v_{IL}(r,\theta) < \theta$ *as Example 2.*

| $f(x)$ | $-\dfrac{\log x}{x-1}$ | $\dfrac{2x}{x+1}$ | $\dfrac{x\log x}{x-1}$ | $x^{\frac{1}{2}}$ | $\dfrac{x-1}{\log x}$ |
|---|---|---|---|---|---|
| Operator monotonicity of $f(x)$ | ○ | ○ | ○ | ○ | ○ |
| Operator monotonicity of $\exp\{f(x)\}$ | ○ | × | ○ | × | × |

Results of Example 2 and Example 3 are extended as the following;

**Theorem 3.** *Let*

$$DL_p(x) = \frac{x^p \log x}{x^p - 1}.$$

$\exp\{DL_p(x)\}$ *is an operator monotone function if and only if* $p \in [-1,1] \setminus \{0\}$.

*Proof.* Firstly we show that $DL_p(x)$ satisfies the condition of Theorem 2 for the case $p \in (0,1]$:
By putting $z = re^{i\theta}$ $(0 < r, 0 < \theta < \pi)$, we have

$$v(r,\theta) := \Im DL_p(z) = \frac{r^p}{r^{2p}+1-2r^p\cos(p\theta)}\{\theta(r^p - \cos(p\theta)) - (\log r)\sin(p\theta)\}.$$

(1) Proof of $v(r,\theta) < \theta$;

$v(r,\theta) < \theta$ is equivalent to $r^p\theta\cos(p\theta) - (r^p\log r)\sin(p\theta) < \theta$.

$$\begin{aligned} r^p\theta\cos(p\theta) - (r^p\log r)\sin(p\theta) &\leq r^p\left(\frac{1}{p}\right)\sin(p\theta) - (r^p\log r)\sin(p\theta) \\ &= \sin(p\theta)\left(\frac{1}{p}\right)(r^p - r^p\log r^p) \\ &\leq \sin(p\theta)\left(\frac{1}{p}\right) < (p\theta)\left(\frac{1}{p}\right) = \theta. \end{aligned}$$

(2) Proof of $0 < v(r,\theta)$;

$$DL_p(x) = \frac{1}{p}DL(x^p)$$

is operator monotone for $p \in (0,1]$, so $0 < v(r,\theta)$.
From (1) and (2), $\exp\{DL_p(x)\}$ is operator monotone if $p \in (0,1]$

Next, when $p \in [-1,0)$,

$$DL_p(z) = \frac{z^p\mathrm{Log}z}{z^p-1} = \frac{z^{-p}z^p\mathrm{Log}z}{z^{-p}(z^p-1)} = \frac{\mathrm{Log}z}{1-z^{|p|}}$$

and

$$\nu(r,\theta) := \Im DL_p(re^{i\theta}) = \frac{(r^{|p|}\log r)\sin(|p|\theta) - \theta(r^{|p|}\cos(|p|\theta) - 1)}{r^{2|p|} + 1 - 2r^{|p|}\cos(|p|\theta)}.$$

We can show $0 < \nu(r,\theta) < \theta$ by the same technique. So we have that $\exp\{DL_p(x)\}$ is operator monotone if $p \in [-1,1] \setminus \{0\}$.

Next we assume $p > 1$. Then

$$v(r,\theta) < \theta \iff (l(p,r,\theta) =)r^p \left(\cos(p\theta) - (\log r)\frac{\sin(p\theta)}{\theta}\right) < 1.$$

Take $\theta$ as $\frac{\pi}{p} < \theta < \min\left\{\pi, \frac{2\pi}{p}\right\}$, then $\sin(p\theta) < 0$ and

$$\lim_{r\to\infty} l(p,r,\theta) = \infty.$$

Therefore $\exp\{DL_p(x)\}$ is not operator monotone if $1 < p$ from Theorem 2. We can also show the case $p < -1$ similarly. □

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