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Product range of 3-by-3 normal matrices

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Abstract. The k-product range of a complex matrix A is defined as the set of the products of any k diagonal entry of A under unitary similarities. For $k=2$, the k-product range of a normal matrix A is convex if the eigenvalues of A form an acute-angled or right-angled triangle.

1. Introduction

Let A be an $n\times n$ complex matrix. For any integer k with $1\leq k\leq n$, the *k-product range of* A is defined as

$$W_k^{\Pi}(A)=\left\{\prod_{i=1}^{k}(UAU^*)_{ii}:U \text{ is a unitary matrix}\right\}. \tag{1.1}$$

When k equals 1, the product range is the *classical numerical range* of A

$$W(A)=\{x^*Ax:x\in\mathbb{C}^n,\ x^*x=1\}.$$

Hence $W_k^{\Pi}(A)$ is one of a generalized numerical range. We refer the reader to [3] for fundamental properties of the numerical ranges.

Especially, when A is a *normal matrix* which means A commutes with its conjugate transpose, we have $W(A)=\mathrm{Conv}(\sigma(A))$ where Conv and $\sigma(A)$ are the convex hull and the spectrum of A, respectively. In this article, we will concern the convexity of the k-product range of any 3×3 normal matrix. The details of our results, see [5].

The concept of product ranges was introduced firstly by Marvin Marcus [6] in 1973. Bebiano, Li and Providência [1] investigated some geometrical properties, such as convexity, star-shapedness and simply connectedness of $W_k^{\Pi}(A)$ in 1993. In particular, they have shown that $W_k^{\Pi}(A)$ is not generally simply connected when A is an $n \times n$ normal matrix with $n \geq 4$. In addition, they also gave an example which shows that $W_3^{\Pi}(A)$ is not convex when A is a 3×3 normal matrix. Hence $W_k^{\Pi}(A)$ is not necessary convex in general even when A is normal. However, for the 2-by-2 case, Hu and Tam [4] have shown that $W_2^{\Pi}(A)$ is a line segment if and only if A is normal. In the case $n = 2$, every doubly stochastic matrix is unistochastic. For $N = \mathrm{diag}(\lambda_1, \lambda_2)$, we get the line segment

$$W_2^{\Pi}(N) = \left[\lambda_1\lambda_2, \left(\frac{\lambda_1 + \lambda_2}{2}\right)^2\right].$$

In addition, the convexity of the range $W_k^{\Pi}(N)$ is characterized in [2] in the case the eigenvalues of N are on a straight line.

2. Preliminaries

By the definition of $W_k^{\Pi}(N)$, when N is normal we may assume $N = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ in (1.1) where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of N. Moreover, having in mind that

$$\prod_{i=1}^{k} (UNU^*)_{ii} = \prod_{i=1}^{k} \sum_{j=1}^{n} |u_{ij}|^2 \lambda_j,$$

we can define $W_k^{\Pi}(N)$ equivalently by

$$W_k^{\Pi}(N) = \left\{ \prod_{i=1}^{k} \sum_{j=1}^{n} a_{ij}\lambda_j : (a_{ij}) \text{ is a unitary matrix} \right\}. \tag{2.1}$$

In the case N is a 3×3 normal matrix and $k = 2$, the 3 eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are on a straight line ℓ, we can characterize the range $W_2^{\Pi}(N)$ according to the different two situation: (i) $0 \in \ell$, (ii) $0 \notin \ell$.

In the case (i), we may assume that $\lambda_1, \lambda_2, \lambda_3$ are real numbers and (i-1) $0 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1$ or (i-2) $\lambda_3 \leq 0 \leq \lambda_2 \leq \lambda_1$. The equations

$$W_2^{\Pi}(N) = [\lambda_2\lambda_3, (\lambda_1 + \lambda_2)^2/4], \tag{2.2}$$

and

$$W_2^{\Pi}(N) = [\lambda_1\lambda_3, (\lambda_1 + \lambda_2)^2/4], \tag{2.3}$$

hold respectively in the case (i-1) and (i-2).

In the case (ii), the range $W_2^{\Pi}(N)$ is not convex [2, Theorem 3.1]. The exact figure of $W_2^{\Pi}(N)$ in this situation is given in [7] under the assumption

$$\mathbf{Im}(\lambda_1) = \mathbf{Im}(\lambda_2) = \mathbf{Im}(\lambda_3) = 1.$$

The range $W_2^{\Pi}(N)$ is characterized as the image of $D(3)$ under a quadratic map. So we may assume that $(\lambda_2-\lambda_3)/(\lambda_1-\lambda_3)$ is a well defined imaginary number. We assume that $\lambda_1, \lambda_2, \lambda_3$ lie on a circle counterclockwisely. The relations (2.2), (2.3) and some numerical experiments suggest the equation

$$W_2^{\Pi}(\operatorname{diag}(\lambda_1,\lambda_2,\lambda_3)) = \operatorname{Conv}(\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3), \tag{2.4}$$

holds only when the the interior Γ of the circumscribed circle of $\triangle\lambda_1\lambda_2\lambda_3$ satisfies

$$0 \in \Gamma \tag{2.5}$$

and

$$0 < \arg((\lambda_3-\lambda_1)/(\lambda_2-\lambda_1)), \arg((\lambda_1-\lambda_2)/(\lambda_3-\lambda_2)), \arg((\lambda_2-\lambda_3)/(\lambda_1-\lambda_3)) \leq \pi/2. \tag{2.6}$$

However, so far we are not sure if these two conditions are necessary for the equation (2.4) to hold, we show its necessity of the condition (2.6) under some special two situations (Corollary 3.2 and Corollary 3.3). We assume these conditions to consider the subject.

The next lemma provides the main motivation to assume the conditions (2.5). For a moment, we assume that the three points $\lambda_1\lambda_2$, $\lambda_1\lambda_3$, $\lambda_2\lambda_3$ are not colinear. For any $1 \leq i < j \leq 3$, let $k = \{1,2,3\}\backslash\{i,j\}$. Denote by H_{ij}, the closed half-plane with the boundary passing through $\lambda_i\lambda_k$, $\lambda_j\lambda_k$ satisfying $\lambda_i\lambda_j \in H_{ij}$. We have the following lemma.

Lemma 2.1. *Let $\lambda_1, \lambda_2, \lambda_3$ be mutually distinct non-zero complex numbers satisfying the condition* (2.5). *Then the three points $\lambda_1\lambda_2$, $\lambda_1\lambda_3$, $\lambda_2\lambda_3$ are not colinear. Moreover the interior of the respective three half-planes H_{12}, H_{13}, H_{23} contains the respective points $\lambda_3^2, \lambda_2^2, \lambda_1^2$.*

3. Main theorems

The following theorem is our main result.

Theorem 3.1. *Suppose that μ_1, μ_2 and μ_3 are three complex numbers with modulus one given by*

$$\mu_2 = 1 = e^{2i(\eta_1+\eta_2+\eta_3)}, \mu_3 = e^{2i\eta_1}, \mu_1 = e^{2i(\eta_1+\eta_2)}$$

with $0 < \eta_1 \leq \eta_2 \leq \eta_3 \leq \pi/2$, $\eta_1+\eta_2+\eta_3 = \pi$. Let

$$\tilde{\Gamma_j} = \{z \in \mathbb{C} : |z| \leq \cos\eta_j\}$$

and $\Gamma_j = \operatorname{Conv}(\tilde{\Gamma_j}, \mu_j)$ for $j = 1,2,3$. Then the triple

$$(\lambda_1, \lambda_2, \lambda_3) = (\mu_1-\mu_0, \mu_2-\mu_0, \mu_3-\mu_0)$$

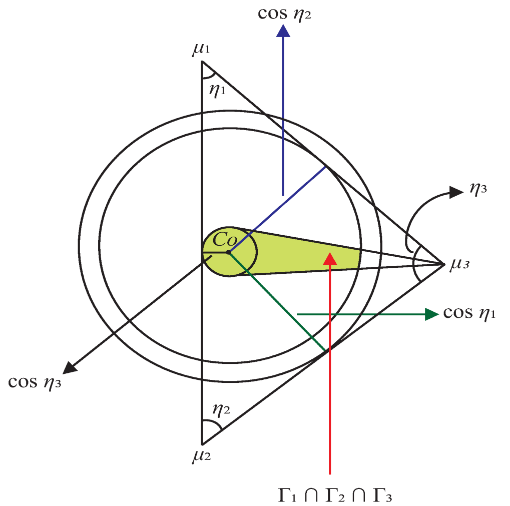

Figure 1. C_O is the circumcenter of $\triangle\mu_1\mu_2\mu_3$.

for a point $\mu_0 \in \mathrm{Conv}(\mu_1, \mu_2, \mu_3)$ *satisfies*

$$W_2^{\Pi}(\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)) = \mathrm{Conv}(\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3)$$

if and only if $\mu_0 \in \Gamma_1 \cap \Gamma_2 \cap \Gamma_3$. *The region* $\Gamma_1 \cap \Gamma_2 \cap \Gamma_3$ *is shown in* Figure 1.

As simple consequences of Theorem 3.1 we provide the following two corollaries.

Corollary 3.2. *Suppose that* $\lambda_1, \lambda_2, \lambda_3$ *are complex numbers with modulus* 1 *and satisfies the condition* (2.6). *Then an acute-angled or right-angled triangle* $\lambda_1, \lambda_2, \lambda_3$ *satisfies*

$$W_2^{\Pi}(\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)) = \mathrm{Conv}(\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3).$$

Corollary 3.3. *Suppose that* $\lambda_1, \lambda_2, \lambda_3$ *are vertices of a acute-angled or a right-angled triangle in the Gaussian plane and satisfy the condition* $\lambda_1 + \lambda_2 + \lambda_3 = 0$. *Then the relation*

$$W_2^{\Pi}(\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)) = \mathrm{Conv}(\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3)$$

holds.

Remark 3.4. One would suppose that the assertion of Corollary 3.2 remains valid if we replace the circumcenter by the incenter of a triangle. However such an assertion is false, see Example 4.2.

Next we will consider some generalizations of Corollary 3.2 and Corollary 3.3.

Proposition 3.5. Let λ_1, λ_2 and λ_3 be three distinct points on the unit circle $|z| = 1$. Then the equation

$$W_2^{\Pi}(\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)) = \mathrm{Conv}(\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3), \tag{3.1}$$

holds only when $\triangle\lambda_1\lambda_2\lambda_3$ is an acute-angled or right-angled triangle.

Proposition 3.6. Let λ_1, λ_2 and λ_3 be three distinct complex numbers satisfying

$$\lambda_1 + \lambda_2 + \lambda_3 = 0.$$

Then the equation (3.1) holds only when $\triangle\lambda_1\lambda_2\lambda_3$ is an acute-angled or right-angled triangle.

For the proof of Theorem 3.1, we firstly prove the following inclusion

$$W_2^{\Pi}(N) \subseteq \mathrm{Conv}(\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3), \tag{3.2}$$

where $N = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$.

By using the half-planes H_{ij} introduced in Lemma 2.1, the inclusion (3.2) is rewritten as

$$W_2^{\Pi}(N) \subseteq H_{12} \cap H_{23} \cap H_{13}.$$

We shall prove the inclusion

$$W_2^{\Pi}(N) \subseteq H_{12}, \tag{3.3}$$

under the assumption $\mu_0 \in \Gamma_3$. Similar inclusions for H_{23} and H_{13} can be proved in the same manner. To prove the inclusion (3.3), we use a functional on the set Uni(3). Let $\xi = \arg(\lambda_1\lambda_3 - \lambda_2\lambda_3) - \pi/2$. Then this angle satisfies

$$\mathbf{Re}(e^{-i\xi}\lambda_1\lambda_3) = \mathbf{Re}(e^{-i\xi}\lambda_2\lambda_3) < \mathbf{Re}(e^{-i\xi}\lambda_1\lambda_2).$$

For any $(a_{ij}) \in \mathrm{Uni}(3)$, the element z of $W_2^{\Pi}(N)$ can be expressed as

$$\begin{aligned} z &= \lambda_1\lambda_2(a_{11}a_{22} + a_{12}a_{21}) + \lambda_1\lambda_3(a_{11}a_{23} + a_{13}a_{21}) + \lambda_2\lambda_3(a_{12}a_{23} + a_{13}a_{22}) \\ &\quad + \lambda_1^2 a_{11}a_{21} + \lambda_2^2 a_{12}a_{22} + \lambda_3^2 a_{13}a_{23}. \end{aligned}$$

The inclusion (3.3) is equivalent to the inequality

$$\mathbf{Re}(e^{-i\xi}z) \geq \mathbf{Re}(e^{-i\xi}\lambda_1\lambda_3)$$

for any $(a_{ij}) \in \mathrm{Uni}(3)$. So we shall prove the positivity of $\mathbf{Re}(e^{-i\xi}(z - \lambda_1\lambda_3))$ for any $(a_{ij}) \in \mathrm{Uni}(3)$. This functional depends on the relative positions of $\lambda_i = \mu_i - \mu_0$ for $i = 1, 2, 3$.

We shall consider the following normalized functional on Uni(3)

$$L((a_{ij})) = \frac{\mathbf{Re}(e^{-i\xi}(z-\lambda_1\lambda_3))}{\mathbf{Re}(e^{-i\xi}(\lambda_1\lambda_2-\lambda_1\lambda_3))}.$$

It is rewritten as

$$L((a_{ij})) = a_{11}a_{22} + a_{12}a_{21} + pa_{13}a_{23} + qa_{11}a_{21} + ra_{12}a_{22}, \tag{3.4}$$

where

$$p = \frac{\mathbf{Re}(e^{-i\xi}(\lambda_3^2-\lambda_1\lambda_3))}{\mathbf{Re}(e^{-i\xi}(\lambda_1\lambda_2-\lambda_1\lambda_3))},$$
$$q = \frac{\mathbf{Re}(e^{-i\xi}(\lambda_1^2-\lambda_1\lambda_3))}{\mathbf{Re}(e^{-i\xi}(\lambda_1\lambda_2-\lambda_1\lambda_3))},$$
$$r = \frac{\mathbf{Re}(e^{-i\xi}(\lambda_2^2-\lambda_1\lambda_3))}{\mathbf{Re}(e^{-i\xi}(\lambda_1\lambda_2-\lambda_1\lambda_3))},$$

and we may assume $p > 0$ by Lemma 2.1 provided that the point μ_0 belongs to the open disc bounded by the circle circumscribed to $\triangle\mu_1\mu_2\mu_3$.

The positive semi-definiteness of $L((a_{ij}))$ is characterized by the following.

Theorem 3.7. *Let $L((a_{ij}))$ be a functional on* Uni(3) *defined by* (3.4). *Then $L((a_{ij})) \geq 0$ for all $(a_{ij}) \in$* Uni(3) *if and only if one of the following conditions holds:*

(i) $0 < p \leq 1$, $p + q \geq 0$ *and* $p + r \geq 0$.

(ii) $p > 1$, $p + q \geq 0$, $p + r \geq 0$ *and* $-1 + 2p + pq + pr + qr \geq 0$.

The corresponding regions of (i) *and* (ii) *in the (q, r)-plane are shown in* Figure 2 *and* Figure 3, *respectively.*

4. Examples

In this section, we provide some examples of product ranges.

Example 4.1. Let $N = \mathrm{diag}(1, \omega, \omega^2)$ where $\omega^3 = 1$. Figure 4 and Figure 5 are the graphs of $W_2^{\Pi}(N)$ and $W_3^{\Pi}(N)$, respectively. Note that the diagonal entries of N satisfy Theorem 2.1 and Theorem 2.2, so $W_2^{\Pi}(N)$ is a triangle $\mathrm{Conv}(1, \omega, \omega^2)$.

Example 4.2. Let $N = \mathrm{diag}(\frac{75}{4}i, -7 - \frac{21}{4}i, 7 - \frac{21}{4}i)$. Note that the origin is the inner center of the triangle consisting of the diagonal entries of N. But $W_2^{\Pi}(N)$ is not convex. See Figure 6.

Example 4.3. So far we have considered the eigenvalues form an acute-angled triangle. However, when the eigenvalues of N form an obtuse-angled triangle, the 2-product range of N is not necessary convex, even the origin is the circumcenter (Figure 7), centroid (Figure 8) or incenter (Figure 9).

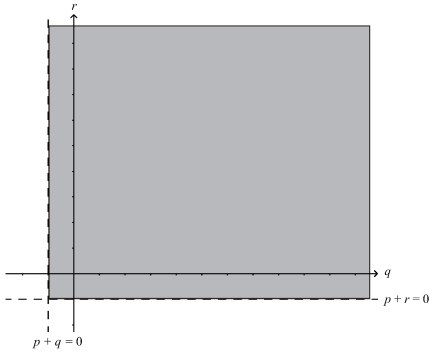

Figure 2

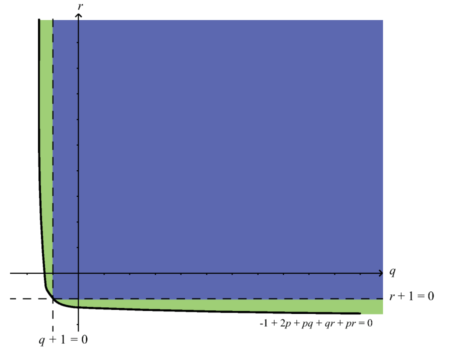

Figure 3

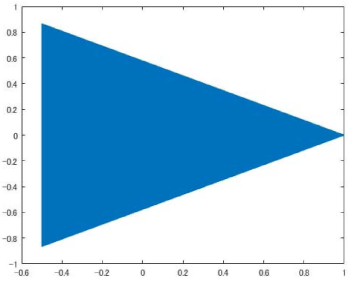

Figure 4: $W_2^{\Pi}(N)$

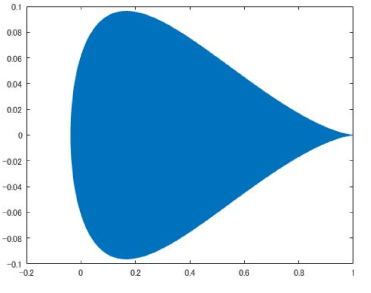

Figure 5: $W_3^{\Pi}(N)$

5. Plotting the product range

In this section, we provide two MATLAB programs for plotting the 2-product range and 3-product range of any 3×3 normal matrix. For convenience, they are written as an m-file for MATLAB. The range $W_k^{\Pi}(N)$ is a compact subset of the Guassian plane $\mathbb{C}$. We approximate it by its finite many representative points. We adopt 200^3 as the number of representative points in the examples in Section 4. The following program is in order to plot the 2-product range.

```
function y = p(p,q,r,m)
a = ones(1,m+1);
t = 0:pi/m:pi;
s = 0:pi/m:pi;
u = 0:pi/m:pi;
b1 = kron(a,a);
```

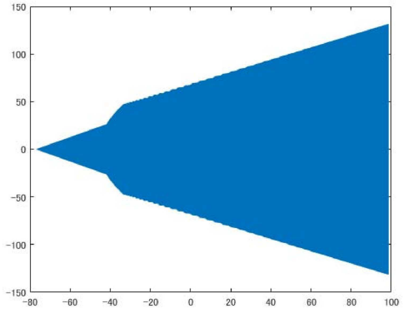

Figure 6: $W_2^{\Pi}(N)$

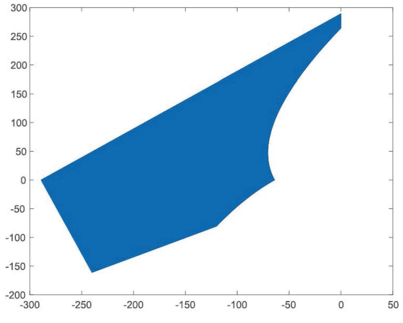

Figure 7: $N = \mathrm{diag}(15+8i, 8+15i, -15+8i)$

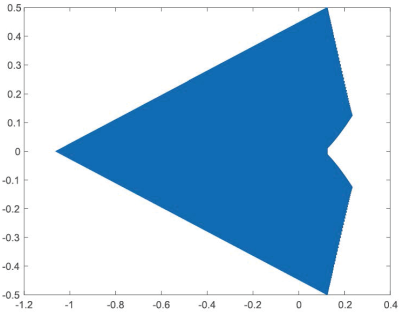

Figure 8: $N = \mathrm{diag}(\frac{1}{2}i, -1-\frac{1}{4}i, 1-\frac{1}{4}i)$

```
b11 = kron(cos(t),b1);
b12 = kron(cos(s),a);
b12 = kron(sin(t),b12);
b21 = kron(a,sin(u));
b21 = kron(sin(t),b21);
b2 = kron(cos(s),sin(u));
bb2 = kron(cos(t),b2);
b3 = kron(sin(s),cos(u));
bb3 = kron(a,b3);
b22 = -bb2 - bb3;
a11 = b11.^2;
a12 = b12.^2;
a21 = b21.^2;
```

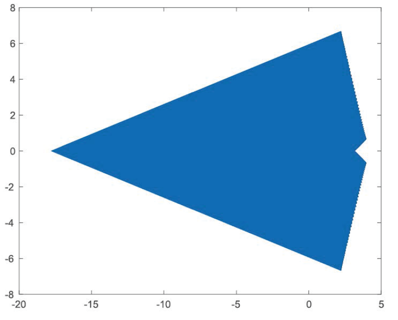

Figure 9: $N = \text{diag}(\frac{5}{3}i, -4 - \frac{4}{3}i, 4 - \frac{4}{3}i)$

```
a22 = b22.^2;
a13 = 1 - a11 - a12;
a23 = 1 - a21 - a22;
a31 = 1 - a11 - a21;
a32 = 1 - a12 - a22;
a33 = 1 - a31 - a32;
X = a11 .* p + a12 .* q + a13 .* r;
Y = a21 .* p + a22 .* q + a23 .* r;
Z = X .* Y;
XX = real(Z);
YY = imag(Z);
grid;
plot(XX,YY)
```

For plotting the 3-product range of a 3×3 normal matrix, we only need to replace 5 codes from the bottom to the above by following:

```
Z = a31 .* p + a32 .* q + a33 .* r;
ZZ = X .* Y .* Z;
XX = real(ZZ);
YY = imag(ZZ);
grid;
plot(XX,YY)
```

In the above two programs, p, q and r are the diagonal entries of the given normal matrix N. For finer approximation, we can replace m by another large number. Of course, it needs more computational times for larger number m.

Although the numerical experiments indicate that the product range of a 3×3 normal matrix is not necessary convex, they seem like star-shaped and simply connected. Hence, we end this note by providing the following problem for further researching investigation.

Question 5.1. Are the product ranges $W_2^{\Pi}(N)$ and $W_3^{\Pi}(N)$ for any 3×3 normal matrix always star-shaped or simply connected?

References

[1] N. Bebiano, C.K. Li and J. da Providência, Product of diagonal elements of matrices, *Linear Algebra Appl.*, **178** (1993), 185-200.

[2] C.-M. Cheng and C. K. Li, On the Hu-Hurley-Tam conjecture concerning the generalized numerical range, Linear Algebra Appl. **305** (2000), 87-97.

[3] T. Furuta, Invitation to Linear Operators, CRC Press, 2002.

[4] S.A. Hu and T.Y. Tam, On the generalized numerical range with principal character, *Linear and Multilinear Algebra*, **30** (1991), 93-107.

[5] P.-R. Huang and H. Nakazato, Product of two diagonal entries of a 3-by-3 normal matrix, submitted.

[6] M. Marcus, Derivations, Plüker relations, and the numerical range, *Indiana Univ. Math. J.*, **22** (1973), 1137-1149.

[7] H. Nakazato, N. Bebiano and J. da Providência, Product of diagonal entries of the unitary orbit of a 3-by-3 normal matrix, *Linear Algebra Appl.*, **429** (2008), 698-715.