数理解析研究所講究録
第2033巻 2017年 101-105

Geometric constants of Banach spaces and the matrix norms

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1 introduction

In the theory of Banach space geometry, geometric constants which quantify various geometric features of Banach spaces often play fundamental roles. This paper is mainly concerned with, in particular, one of the best-known geometric constants of Banach spaces, that is, James constant.

Let X be a Banach space, and let S_X denote the unit sphere of X. The von Neumann-Jordan constant of a Banach space was first considered by Jordan and von Neumann [4] based on their characterization of inner product spaces. Namely, the von Neumann-Jordan constant $C_{NJ}(X)$ of X is given by

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)} : (x,y) \neq (0,0)\right\}.$$

The following are basic properties of von Neumann-Jordan constant:

(i) $1 \leq C_{NJ}(X) \leq 2$.

(ii) $C_{NJ}(X) = 1$ if and only if X is a Hilbert space.

(iii) $C_{NJ}(X) < 2$ if and only if X is uniformly non-square, that is, there exists a $\delta > 0$ such that $\min\{\|x+y\|, \|x-y\|\} < 2(1-\delta)$ whenever $x, y \in S_X$ ([12]).

Moreover, $C_{NJ}(X)$ has the representation using the norm of certain 2×2 matrix. Indeed, if we put $Y = X \otimes_2 X$, then

$$\frac{\|x+y\|^2+\|x-y\|^2}{\|x\|^2+\|y\|^2} = \left\|\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right\|^2 \left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|^{-2},$$

which implies that

$$C_{NJ}(X) = 2^{-1}\left\|\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} : Y \to Y\right\|^2.$$

This formula has the following applications.

(iv) $C_{NJ}(X^*) = C_{NJ}(X)$.

(v) X is uniformly non-square if and only if X^* is uniformly non-square.

Thus it can be worth considering another formulation of geometric constant, in particular, using the norm of matrices.

On the other hand, the James constant $J(X)$ of X was introduced in 1990 by Gao and Lau [2] as a measure of the squareness of the unit ball. Namely, we define $J(X)$ by

$$J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in S_X\}.$$

It is known that $J(X)$ has the following properties:

(i) $\sqrt{2} \leq J(X) \leq 2$ ([2]).

(ii) If H is a Hilbert space, then $J(H) = \sqrt{2}$.

(iii) If $\dim X \geq 3$, then $J(X) = \sqrt{2}$ implies that X is a Hilbert space ([7]); and hence $J(X) = \sqrt{2}$ if and only if X is a Hilbert space provided that $\dim X \geq 3$.

(iv) There are various non-Hilbert two-dimensional normed spaces X with $J(X) = \sqrt{2}$ ([2, 7, 8, 9]).

(v) $J(X) < 2$ if and only if X is uniformly non-square.

(vi) There exists a two-dimensional normed space X with $J(X^*) \neq J(X)$ ([5]).

Unlike von Neumann-Jordan constant, in general, James constant does not have representations using the norm of matrices. However, for a certain class of norms on $\mathbb{R}^2$, we can consider such a representation.

In this paper, we present a new representation of James constant for $\pi/2$-rotation invariant norms on $\mathbb{R}^2$ by using the norm of a 2×2 matrix. We also give some applications of that representation.

2 A representation of James constant for $\pi/2$-rotation invariant norms

Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$, and let $\theta \in (0, 2\pi)$. Then $\|\cdot\|$ is said to be *θ-rotation invariant* if the θ-rotation matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is an isometry on $(\mathbb{R}^2, \|\cdot\|)$. All norms on $\mathbb{R}^2$ are clearly π-rotation invariant, and the Euclidean norm is θ-rotation invariant for each $\theta \in (0, 2\pi)$.

Recall that the James constant $J(X)$ of a Banach space X has the simple form using the notion of isosceles orthogonality. An element $x \in X$ is said to be *isosceles orthogonal* to $y \in X$, denoted by $x \perp_I y$, if $\|x+y\| = \|x-y\|$. In Gao and Lau [2], it was shown that

$$J(X) = \sup\{\|x+y\| : x, y \in S_X,\ x \perp_I y\}.$$

This formula plays very important role, especially, in the two-dimensional case because of the following result.

Lemma 2.1 (Gao and Lau [2]; Alonso [1]; Ji et al. [3]). *Let X be a two-dimensional normed space. Suppose that $x \in S_X$. Then there exists a unique (up to the sign) element $y \in S_X$ such that $x \perp_I y$.*

The following theorem says that the class of $\pi/2$-rotation invariant norms on $\mathbb{R}^2$ is suitable for the study of James constant.

Theorem 2.2 (Komuro, Saito and Mitani [6]; Komuro, Saito and Tanaka [8]). *Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$. Then the following are equivalent.*

(i) $\|\cdot\|$ *is $\pi/2$-rotation invariant.*

(ii) $x \perp_I y$ *if and only if* $\langle x, y\rangle = 0$ *whenever* $\|x\| = \|y\| = 1$.

In which cases, $x \perp_I R(\pi/2)x$ for each x.

The following is our main result.

Theorem 2.3 (Komuro, Saito and Tanaka [10]). *Let $\|\cdot\|$ be a $\pi/2$-rotation invariant norm on $\mathbb{R}^2$. Then*

$$J((\mathbb{R}^2, \|\cdot\|)) = \sqrt{2}\|R(\pi/4) : (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)\|.$$

Proof. By the preceding theorem, we have

$$\begin{aligned} J((\mathbb{R}^2, \|\cdot\|)) &= \sup\{\|x+y\| : x, y \in \mathbb{R}^2,\ \|x\| = \|y\| = 1,\ x \perp_I y\} \\ &= \sup\{\|x + R(\pi/2)x\| : x \in \mathbb{R}^2,\ \|x\| = 1\} \\ &= \sup\{\|(I + R(\pi/2))x\| : x \in \mathbb{R}^2,\ \|x\| = 1\} \\ &= \|I + R(\pi/2) : (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)\|. \end{aligned}$$

The conclusion follows from $I + R(\pi/2) = \sqrt{2}R(\pi/4)$. □

This simple result has some interesting applications.

3 Applications

In this section, we present two applications of Theorem 2.3. The first one is concerned with the following result of Gao and Lau [2, Proposition 2.8].

Proposition 3.1 (Gao and Lau [2]). *Let $\|\cdot\|$ be a $\pi/4$-rotation invariant norm on $\mathbb{R}^2$. Then $J((\mathbb{R}^2, \|\cdot\|)) = \sqrt{2}$.*

We can make use of our main result for giving a simple proof of partial converse to the preceding proposition.

Theorem 3.2 (Komuro, Saito and Tanaka [8]). *Let $\|\cdot\|$ be a $\pi/2$-rotation invariant norm on $\mathbb{R}^2$. Then $J((\mathbb{R}^2, \|\cdot\|)) = \sqrt{2}$ if and only if $\|\cdot\|$ is $\pi/4$-rotation invariant.*

Proof. The "if" part is the statement of Proposition 3.1. For the converse, suppose that $J((\mathbb{R}^2, \|\cdot\|)) = \sqrt{2}$. Then $\|R(\pi/4)\| = 1$ by Theorem 2.3. Moreover, since $\|R(\pi/2)\| = \|R(-\pi/2)\| = 1$, it follows that

$$\|R(\pi/4)^{-1}\| = \|R(-\pi/4)\| = \|R(-\pi/2)R(\pi/4)\| \leq 1.$$

Thus, for each x, one has

$$\|x\| = \|R(-\pi/4)R(\pi/4)x\| \leq \|R(\pi/4)x\| \leq \|x\|,$$

that is, $\|R(\pi/4)x\| = \|x\|$. This proves that $\|\cdot\|$ is $\pi/4$-rotation invariant. □

Another application is concerned with the sufficient condition that the equality $J(X^*) = J(X)$ holds. Naturally, the dual space X^* of X can be identified with $(\mathbb{R}^2, \|\cdot\|_*)$ under $X^* \ni f \leftrightarrow (f(1,0), f(0,1)) \in \mathbb{R}^2$. In this manner, the adjoint operator (as a Banach space operator) A^* of a matrix A can be represented by the transpose A^T. Moreover, it is well-known that $\|A^*\|_{X^*} = \|A\|_X$.

We need the following results.

Lemma 3.3. *Let $\theta \in \mathbb{R}$. Suppose that $\|\cdot\|$ is a θ-rotation invariant norm on $\mathbb{R}^2$. Then the dual norm $\|\cdot\|_*$ is also θ-rotation invariant.*

Using this and Theorem 2.3, we have the following theorem.

Theorem 3.4 (Komuro, Saito and Tanaka [10]). *Let $\|\cdot\|$ be a $\pi/2$-rotation invariant norm on $\mathbb{R}^2$. Then $J((\mathbb{R}^2, \|\cdot\|)^*) = J((\mathbb{R}^2, \|\cdot\|))$.*

Proof. Let $\|\cdot\|$ be a $\pi/2$-rotation invariant norm on $\mathbb{R}^2$. Put $X = (\mathbb{R}^2, \|\cdot\|)$. Then its dual norm $\|\cdot\|_*$ is also $\pi/2$-rotation invariant by Lemma 3.3. It follows from Theorem 2.3 that $J(X) = \sqrt{2}\|R(\pi/4)\|_X$ and $J(X^*) = \sqrt{2}\|R(\pi/4)\|_{X^*}$. However, since $R(\pi/2)$ is an isometry on X^*, one has that

$$\begin{aligned}\|R(\pi/4)\|_X &= \|R(\pi/4)^*\|_{X^*} = \|R(-\pi/4)\|_{X^*} \\ &= \|R(\pi/2)R(-\pi/4)\|_{X^*} = \|R(\pi/4)\|_{X^*}.\end{aligned}$$

This proves that $J(X^*) = J(X)$, as desired. □

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