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Remarks on λ-commuting operators

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Abstract

In this paper, we study properties of λ-commuting operators. We give spectral and local spectral relations between λ-commuting operators. Moreover, we show that the operators λ-commuting with a unilateral shift are representable as weighted composition operators. We also provide the polar decomposition of the product of (λ, μ)-commuting operators where λ and μ are real numbers with $\lambda\mu > 0$. Finally, we find the restriction of μ for the product of (λ, μ)-commuting quasihyponormal operators to be quasihyponormal.

1 Introduction

This paper is part of a paper submitted for possible publication in some journal.

Let $\mathcal{H}$ be a separable complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For $T \in \mathcal{L}(\mathcal{H})$, we write $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$, $\sigma_{le}(T)$, and $r(T)$ for the spectrum, the point spectrum, the approximate point spectrum, the left essential spectrum, and the spectral radius of T, respectively.

We say that operators S and T in $\mathcal{L}(\mathcal{H})$ are *λ-commuting* if $ST = \lambda TS$, where λ is a complex number. In [3], S. Brown showed that every operator λ-commuting with a nonzero compact operator has a nontrivial hyperinvariant subspace, as one of the generalizations of the famous Lomonosov's theorem (see [10]). Since then, many mathematicians have been interested in λ-commuting operators.

Different classes of operators can be specified depending on the restriction on λ (see [11]). An operator $T \in \mathcal{L}(\mathcal{H})$ is called *normal* if $T^*T = TT^*$. We say that $T \in \mathcal{L}(\mathcal{H})$ is *hyponormal* if $T^*T \geq TT^*$. In [12], J. Yang and H. Du showed that if S and T are λ-commuting normal operators with $ST \neq 0$, then $|\lambda| = 1$. Moreover, M. Cho, J. Lee, and T. Yamazaki proved in [4] that if S and T are λ-commuting operators such that both S^* and T are hyponormal and $ST \neq 0$, then $|\lambda| \leq 1$.

For $\lambda, \mu \in \mathbb{C}$, operators $S, T \in \mathcal{L}(\mathcal{H})$ are said to be *(λ, μ)-commuting* if $ST = \lambda TS$ and $S^*T = \mu TS^*$. By Fuglede-Putnam Theorem, if $A, B \in \mathcal{L}(\mathcal{H})$ are normal and $AX = XB$ for some $X \in \mathcal{L}(\mathcal{H})$, then $A^*X = XB^*$ (see [7]). Hence, if S is

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normal and λ-commuting with T, then S and T are $(\lambda, \overline{\lambda})$-commuting. For a simple example, given any fixed complex constant λ with $|\lambda| \leq 1$, suppose D is a diagonal operator given by $De_n = \lambda^n e_n$ for $n \geq 0$, where $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis for $\mathcal{H}$. Then, every weighted shift W on $\mathcal{H}$ given by $We_n = \alpha_n e_{n+1}$ for $n \geq 0$ satisfies $DW = \lambda WD$. Since D is normal, the operators D and W are $(\lambda, \overline{\lambda})$-commuting by Fuglede-Putnam Theorem; we also observe that W and D are (λ^{-1}, λ)-commuting. For another example, the 2×2 matrices $S = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ are $(\frac{1}{3}, 3)$-commuting.

In this paper, we study properties of λ-commuting operators. We give spectral and local spectral relations between λ-commuting operators. Moreover, we show that the operators λ-commuting with a unilateral shift are representable as weighted composition operators. We also provide the polar decomposition of the product of (λ, μ)-commuting operators where λ and μ are real numbers with $\lambda\mu > 0$. Finally, we find the restriction of μ for the product of (λ, μ)-commuting quasihyponormal operators to be quasihyponormal.

2 Preliminaries

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *the single-valued extension property* (or SVEP) if for every open set G in $\mathbb{C}$ and every analytic function $f : G \to \mathcal{H}$ with $(T - z)f(z) \equiv 0$ on G, we have $f(z) \equiv 0$ on G. For an operator $T \in \mathcal{L}(\mathcal{H})$ and a vector $x \in \mathcal{H}$, the set $\rho_T(x)$, called the *local resolvent* of T at x, consists of elements z_0 in $\mathbb{C}$ such that there exists an $\mathcal{H}$-valued analytic function $f(z)$ defined in a neighborhood of z_0 which verifies $(T - z)f(z) \equiv x$. The *local spectrum* of T at x is given by $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$. Moreover, we define the *local spectral subspace* of T as $H_T(F) := \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$, where F is a subset of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Dunford's property* (C) if $H_T(F)$ is closed for each closed subset F of $\mathbb{C}$. We say that $T \in \mathcal{L}(\mathcal{H})$ is said to have *Bishop's property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to \mathcal{H}$ of $\mathcal{H}$-valued analytic functions such that $(T - z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, then $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. The following implications are well known (see [2], [5], or [9] for more details):

$$\text{Bishop's property } (\beta) \Rightarrow \text{Dunford's property } (C) \Rightarrow \text{SVEP}.$$

3 Main results

In this section, we give several properties of λ-commuting operators. We first consider the product of λ-commuting operators. We say that $T \in \mathcal{L}(\mathcal{H})$ is *quasinilpotent* if $\sigma(T) = \{0\}$.

Theorem 3.1. Let S and T be operators in $\mathcal{L}(\mathcal{H})$ such that $ST = \lambda TS$ for some $\lambda \in \mathbb{C}$. Then the following statements hold:
(i) $r(ST) \leq r(S)r(T)$ and $r(TS) \leq r(S)r(T)$.
(ii) If $|\lambda| \neq 1$, then ST and TS are quasinilpotent.

Recall that an operator T in $\mathcal{L}(\mathcal{H})$ is called *normaloid* if $\|T\| = r(T)$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to belong to *class* A if $|T^2| \geq |T|^2$. Every operator which belongs to class A is normaloid, and hyponormal operators belong to class A (see [6]).

Corollary 3.2. Let S and T be operators in $\mathcal{L}(\mathcal{H})$ such that $ST = \lambda TS$ for some $\lambda \in \mathbb{C}$ and ST belongs to class A. If S or T is quasinilpotent, then $ST = TS = 0$.

We next provide spectral properties of λ-commuting operators.

Theorem 3.3. Suppose that $S, T \in \mathcal{L}(\mathcal{H})$ satisfy $ST = \lambda TS$ for some $\lambda \in \mathbb{C}$. For $\sigma_\Delta \in \{\sigma_p, \sigma_{ap}, \sigma_{le}\}$, the following assertions hold:
(i) either $0 \in \sigma_\Delta(T)$ or else $\lambda\sigma_\Delta(S) \subset \sigma_\Delta(S)$;
(ii) either $0 \in \sigma_\Delta(S)$ or else $\sigma_\Delta(T) \subset \lambda\sigma_\Delta(T)$.

Remark. One can derive that $T\ker(S - \mu) \subset \ker(S - \lambda\mu)$ and $S\ker(T - \mu) \subset \ker(\lambda T - \mu)$ for each $\mu \in \mathbb{C}$. Hence, $\ker(S)$ and $\ker(T)$ are common invariant subspaces for S and T.

Corollary 3.4. Let S and T be operators in $\mathcal{L}(\mathcal{H})$ such that $ST = \lambda TS$ for some $\lambda \in \mathbb{C}$. Then the following assertions hold:
(i) If $0 \notin \sigma_{ap}(T)$, then $\sigma_{ap}(S) = \{0\}$ or $|\lambda| \leq 1$.
(ii) If $0 \notin \sigma_{ap}(S)$, then $\sigma_{ap}(T) = \{0\}$ or $|\lambda| \geq 1$.
Hence, if $0 \notin \sigma_{ap}(S) \cup \sigma_{ap}(T)$, then $|\lambda| = 1$.

When λ is a root of unity, the inclusions in Theorem 3.3 become equalities, as follows:

Corollary 3.5. Let $S, T \in \mathcal{L}(\mathcal{H})$ satisfy that $ST = \lambda TS$ where λ is a root of unity. Then the following statements hold for $\sigma_\Delta \in \{\sigma_p, \sigma_{ap}, \sigma_{le}\}$:
(i) If $0 \notin \sigma_\Delta(T)$, then $\sigma_\Delta(S) = \lambda\sigma_\Delta(S)$;
(ii) If $0 \notin \sigma_\Delta(S)$, then $\sigma_\Delta(T) = \lambda\sigma_\Delta(T)$.

Recall that $T \in \mathcal{L}(\mathcal{H})$ is said to be an *m-isometry* if $\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*j}T^j = 0$, where m is a positive integer. In [1], it turned out that every m-isometry has approximate point spectrum contained in the unit circle.

Corollary 3.6. Suppose that S and T are operators in $\mathcal{L}(\mathcal{H})$ such that $ST = \lambda TS$ for some $\lambda \in \mathbb{C}$. If $|\lambda| \neq 1$ and S is an m-isometry for some positive integer m, then $0 \in \sigma_p(T)$.

We now consider local spectral properties of λ-commuting operators.

Proposition 3.7. Let $S, T \in \mathcal{L}(\mathcal{H})$. If $ST = \lambda TS$ for some $\lambda \in \mathbb{C}$, then the following statements hold:
(i) $\sigma_S(Tx) \subset \lambda\sigma_S(x)$ and $\lambda\sigma_T(Sx) \subset \sigma_T(x)$ for all $x \in \mathcal{H}$.
(ii) $TH_S(F) \subset H_S(\lambda F)$ for any subset F of $\mathbb{C}$.
(iii) If $\lambda \neq 0$, then $SH_T(\lambda F) \subset H_T(F)$ for any subset F of $\mathbb{C}$.

Corollary 3.8. Suppose that $S, T \in \mathcal{L}(\mathcal{H})$ are λ-commuting where λ is a root of unity with order k. If λ is a root of unity with order k and S has Dunford's property (C), then $H_S(F)$ is a common invariant subspace of S and T^k, where F is any closed subset of $\mathbb{C}$.

Combining Corollary 3.8 with [12], we obtain the following corollary.

Corollary 3.9. Assume that $S, T \in \mathcal{L}(\mathcal{H})$ are λ-commuting. If $S \in \mathcal{L}(\mathcal{H})$ is hyponormal and $\sigma(ST)$ consists of k distinct nonzero elements, then $H_S(F)$ is a common invariant subspace of S and T^k.

For an operator $T \in \mathcal{L}(\mathcal{H})$, we define the *quasinilpotent part* of T, denoted by $H_0(T)$, as $H_0(T) := \{x \in \mathcal{H} : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}$ (see [2] and [9] for more details).

Proposition 3.10. Let $S, T \in \mathcal{L}(\mathcal{H})$. If $ST = \lambda TS$ for some $\lambda \in \mathbb{C} \setminus \{0\}$, then $H_0(S)$ is invariant for T.

Let $H^2 = H^2(\mathbb{D})$ be the canonical Hardy space of the open unit disk $\mathbb{D}$, and let H^∞ be the space of bounded functions in H^2. For an analytic map φ from $\mathbb{D}$ into itself and $u \in \mathbb{D}$, the *weighted composition operator* $W_{f,\varphi} : H^2 \to H^2$ is defined by $W_{u,\varphi}h = u \cdot (h \circ \varphi)$. In particular, $C_\varphi := W_{1,\varphi}$ is said to be a *composition operator.* In the following theorem, we assert that if $|\lambda| = 1$, then the operators λ-commuting with the unilateral shift U on H^2 given by $(Uf)(z) = zf(z)$ are representable as weighted composition operators.

Theorem 3.11. Let U be the unilateral shift on H^2 given by $(Uf)(z) = zf(z)$. Assume that $S \in \mathcal{L}(H^2)$ and $\lambda \in \partial\mathbb{D}$. Then $SU = \lambda US$ if and only if $S = W_{u,\lambda z}$ for some $u \in H^\infty$.

For a bounded sequence $\{\alpha_n\}_{n=0}^\infty$ in $\mathbb{C}$, a *weighted shift* on $\mathcal{H}$ with weights $\{\alpha_n\}$ is an operator T such that $Te_n = \alpha_n e_{n+1}$ for $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ denotes an orthonormal basis for $\mathcal{H}$.

Proposition 3.12. Let S and T be weighted shifts in $\mathcal{L}(\mathcal{H})$ with weights $\{\alpha_n\}$ and $\{\beta_n\}$, respectively, and let $\lambda \in \mathbb{C}$. Then $ST = \lambda TS$ if and only if $\alpha_{n+1}\beta_n = \lambda\beta_{n+1}\alpha_n$ for all n.

In the following example, we consider the case when S is the *Bergman shift* determined by the weights $\{\sqrt{\frac{n+1}{n+2}}\}_{n=0}^\infty$.

Example 3.13. If S is the Bergman shift, then its weights form an increasing sequence. Then S is hyponormal. Suppose that T is any weighted shift with positive weights $\{\beta_n\}$ and $\lambda \in \mathbb{C} \setminus \{0\}$. By Proposition 3.12, it follows that $ST = \lambda TS$ if and only if $\beta_{n+1} = \frac{n+2}{\lambda\sqrt{(n+1)(n+3)}}\beta_n$ for $n \geq 0$, that is, $\beta_n = \frac{1}{\lambda^n}\sqrt{\frac{2(n+1)}{n+2}}\beta_0$ for $n \geq 0$.

For a positive integer $n > 1$, define J_r and J_l on $\bigoplus_1^n \mathcal{H}$ by

$$J_r = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{pmatrix} \quad \text{and } J_l = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Proposition 3.14. Let $T \in \mathcal{L}(\bigoplus_1^n \mathcal{H})$. For a complex number λ, the following statements hold:
(i) $TJ_r = \lambda J_r T$ if and only if

$$T = \begin{pmatrix} T_1 & 0 & \cdots & \cdots & 0 & 0 \\ T_2 & \lambda T_1 & \ddots & \ddots & 0 & 0 \\ T_3 & \lambda T_2 & \lambda^2 T_1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ T_{n-1} & \lambda T_{n-2} & \ddots & \ddots & \lambda^{n-2}T_1 & 0 \\ T_n & \lambda T_{n-1} & \cdots & \cdots & \lambda^{n-2}T_2 & \lambda^{n-1}T_1 \end{pmatrix}$$

where $\{T_j\}_{j=1}^n \subset \mathcal{L}(\mathcal{H})$.
(ii) $TJ_l = \lambda J_l T$ if and only if

$$T = \begin{pmatrix} \lambda^{n-1}T_n & \lambda^{n-2}T_{n-1} & \cdots & \cdots & \lambda T_2 & T_1 \\ 0 & \lambda^{n-2}T_n & \cdots & \cdots & \lambda T_3 & T_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & \ddots & \lambda T_{n-1} & T_{n-2} \\ 0 & 0 & \cdots & \cdots & \lambda T_n & T_{n-1} \\ 0 & 0 & \cdots & \cdots & 0 & T_n \end{pmatrix}$$

where $\{T_j\}_{j=1}^n \subset \mathcal{L}(\mathcal{H})$.

We next consider (λ, μ)-commuting operators. To obtain the polar decomposition of the product of (λ, μ)-commuting operators, we show that their partial isometric parts and positive parts satisfy the following extended commuting relationships.

Lemma 3.15. Let $S, T \in \mathcal{L}(\mathcal{H})$ be (λ, μ)-commuting where λ and μ are real numbers with $\lambda\mu > 0$. If $S = U_S|S|$ and $T = U_T|T|$ denote the polar decompositions, then the following statements hold:
(i) $|T|S = (\lambda^{-1}\mu)^{\frac{1}{2}} S|T|$ and $|S|T = (\lambda\mu)^{\frac{1}{2}} T|S|$;
(ii) $|S|U_T = (\lambda\mu)^{\frac{1}{2}} U_T|S|$ and $|T|U_S = (\lambda^{-1}\mu)^{\frac{1}{2}} U_S|T|$;
(iii) $|S||T| = |T||S|$, $|S^*||T| = |T||S^*|$, and $|S||T^*| = |T^*||S|$;
(iv) $U_S U_T = U_T U_S$ and $U_S^* U_T = U_T U_S^*$ if λ and μ are positive, and $U_S U_T = -U_T U_S$ and $U_S^* U_T = -U_T U_S^*$ if λ and μ are negative.

Theorem 3.16. Assume that $S, T \in \mathcal{L}(\mathcal{H})$ are (λ, μ)-commuting where λ and μ are real numbers with $\lambda\mu > 0$. If $ST = U_{ST}|ST|$ is the polar decomposition, then

$$U_{ST} = U_S U_T \text{ and } |ST| = (\lambda\mu)^{\frac{1}{2}} |S||T|.$$

In addition, if $TS = U_{TS}|TS|$ is the polar decomposition, then

$$U_{TS} = U_T U_S \text{ and } |TS| = (\lambda^{-1}\mu)^{\frac{1}{2}} |S||T|.$$

For an operator $T \in \mathcal{L}(\mathcal{H})$ with polar decomposition $T = U|T|$, the *Aluthge transform* $\widetilde{T}$ of T is defined by $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$. In [8], the authors showed several connections between operators and their Aluthge transforms.

Corollary 3.17. If $S, T \in \mathcal{L}(\mathcal{H})$ are (λ, μ)-commuting operators where λ and μ are real numbers with $\lambda\mu > 0$, then the following statements hold:
(i) $\widetilde{S}$ and $\widetilde{T}$ are (λ, μ)-commuting and $\widetilde{ST} = |\mu|^{\frac{1}{2}} \widetilde{S}\widetilde{T} = \lambda|\mu|^{\frac{1}{2}} \widetilde{T}\widetilde{S}$.
(ii) $\widetilde{S}$ and T are (λ, μ)-commuting.
(iii) S and $\widetilde{T}$ are (λ, μ)-commuting.

Corollary 3.18. Let $S, T \in \mathcal{L}(\mathcal{H})$ be λ-commuting for some nonzero real number λ. If $\widetilde{S}$ is hyponormal and T is normal, then the following statements are equivalent:
(i) $\widetilde{ST}$ is hyponormal.
(ii) $\sigma(ST) \neq \{0\}$.
(iii) $\lambda = \pm 1$.

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *quasinormal* if T^*T commutes with T.

Corollary 3.19. Let $S, T \in \mathcal{L}(\mathcal{H})$ be (λ, μ)-commuting quasinormal operators such that $ST \neq 0$, where λ and μ are real numbers with $\lambda\mu > 0$. Then ST is quasinormal if and only if $\mu = \pm 1$. In particular, if ST is quasinormal and one of S and T is normal, then $\lambda = \mu = \pm 1$.

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *quasihyponormal* if $T^*(T^*T - TT^*)T \geq 0$, or $\|T^2x\| \geq \|T^*Tx\|$ for all $x \in \mathcal{H}$. In the following theorem, we show that if $|\mu| \leq 1$, then the product of two (λ, μ)-commuting quasihyponormal operators is again quasihyponormal.

Theorem 3.20. Let S and T be quasihyponormal operators in $\mathcal{L}(\mathcal{H})$ that are (λ, μ)-commuting. If $|\mu| \geq 1$, then ST is quasihyponormal. Furthermore, if $\lambda \neq 0$ and $|\mu| \geq 1$, then TS is quasihyponormal.

An operator T in $\mathcal{L}(\mathcal{H})$ is said to be *nilpotent* if $T^n = 0$ for some positive integer n; in this case, the smallest positive integer n with $T^n = 0$ is referred to as the order of T.

Corollary 3.21. Let S and T be quasihyponormal operators in $\mathcal{L}(\mathcal{H})$ that are (λ, μ)-commuting and $ST \neq 0$. If $|\lambda| \neq 1$ and $|\mu| \geq 1$, then ST is nilpotent of order 2 and one of S and T has a nontrivial invariant subspace.

Corollary 3.22. Let $S \in \mathcal{L}(\mathcal{H})$ be normal and $T \in \mathcal{L}(\mathcal{H})$ be quasihyponormal with $ST \neq 0$. If $ST = \lambda TS$ for some $|\lambda| \geq 1$, then both ST and TS are quasihyponormal; in particular, if $|\lambda| > 1$, then ST and TS are nilpotent of order 2.

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