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多変数作用素平均の共役とべき平均

The adjoint of multi-variable operator means and power ones

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1 Introduction

Derived from the theory of means of Pusz-Woronowicz [23, 24], Kubo and Ando [19] established the theory of operator means for positive operators on a Hilbert space (see also [3]):

$$A \,\mathrm{m}\, B = A^{\frac{1}{2}} f_{\mathrm{m}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}} \qquad \text{for} \quad f_{\mathrm{m}}(x) = 1 \,\mathrm{m}\, x$$

where f_{m} is an operator monotone function and, by the monotonicity of each terms, $A \,\mathrm{m}\, B = \text{s-lim}_{\varepsilon \to 0} (A + \varepsilon) \mathrm{m} (B + \varepsilon)$ defines an operator mean for all positive operators. Based on this theory, in [14] we introduced the relative operator entropy $\mathsf{S}(A|B)$ putting $f_{\mathrm{m}}(x) = \log x$, which is a relative version of the operator entropy defined by Nakamura-Umegaki [22]. From the viewpoint of Uhlmann, it also defined as the derivative at $t = 0$ of the path of geometric operator means [15] for $f_{\mathrm{m}_t}(x) = x^t$ $(t \in [0, 1])$;

$$A \#_t B = \underset{\varepsilon \downarrow 0}{\text{s-lim}} (A + \varepsilon)^{\frac{1}{2}} \left((A + \varepsilon)^{-\frac{1}{2}} B (A + \varepsilon)^{-\frac{1}{2}} \right)^t (A + \varepsilon)^{\frac{1}{2}}.$$

For invertible A and B, the relative operator entropy has the following variational forms;

$$\mathsf{S}(A|B) = A^{\frac{1}{2}} \log \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}} = \lim_{t \downarrow 0} \frac{A \#_t B - A}{t}.$$

If A and B are not invertible, it is defined under a certain condition like other relative entropies. We also extended the Kubo-Ando theory as solidarities [13].

This view yielded the Finsler space consisting of positive invertible operators, which is called the CPR geometry [5] and it was pointed that the metric, which is now called the Thompson (part) metric, can be defined by $\mathsf{S}(A|B)$:

$$d(A, B) = \| \log \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) \| = \| A^{-\frac{1}{2}} \mathsf{S}(A|B) A^{-\frac{1}{2}} \|.$$

Moreover its Riemannian version was discussed by Bhatia-Holbrook [4] and the multivariate geometric mean for positive definite matrices was introduced. Successively Lim and Pálfia [21] redefined it as the (weighted) matrix Karcher mean defined by the Karcher equation and then Lawson and Lim [20] extended this to the mean for positive invertible operators which is a nice extension of geometric operator means in the Kubo-Ando theory. Here the Karcher equation for positive invertible operators A_j $(j = 1, 2, \ldots, n)$, X and a weight $\{\omega_j\}$ is

$$0 = \sum_{j=1}^{n} \omega_j \log\left(X^{-\frac{1}{2}} A_j X^{-\frac{1}{2}}\right).$$

But their theory depends on the Thompson metric and the power operator mean corresponding to the power function $f_{\mathrm{m}_{r,t}}(x) = (1 - t + tx^r)^{\frac{1}{r}}$. Thus it needs substantially the invertibility of positive operators.

In this note, we extend it to a mean for (non-invertible) positive operators by virtue of the relative operator entropy based on the properties with the existence conditions which are closely related to the kernels and ranges for operators. To study properties of the quantities for non-invertible operators defined by the limit in the strong operator topology, we pay attention to futher approximations, e.g. continuities. To approach to the relative operator entropy, we prepare two tools. One is a bounded double monotone sequence lemma (Lemma 2.7) and another is Izumino's construction (5), see also [8] to express such quantities via commuting operators explained in the last part of this section. Similarly to this section, we often use these tools in this paper. Finally we introduce general operator mean in order to view the negative power means as the adjoint of the positive power means. As a consequence, we can easily observe the relations around Karcher maens and power means.

2 The relative operator entropy

First we review the *relative operator entropy* $\mathsf{S}(A|B)$ for positive (bounded linear) operators A, B on a Hilbert space, see [14, 15, 16, 9, 10, 11, 17]. If B is invertible, then it is defined by $\mathsf{S}(A|B) = B^{\frac{1}{2}} \eta\left(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}\right) B^{\frac{1}{2}}$, where η is the entropy function:

$$\eta(x) = -x \log x \quad \text{if } x > 0, \qquad \eta(0) = 0.$$

In addition, if A is invertible, then $\mathsf{S}(A|B) = A^{\frac{1}{2}} \log\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}$. Since $\mathsf{S}(A|B)$ has the right-term monotone decreasing property of $\mathsf{S}(A|B + \varepsilon)$ as $\varepsilon \downarrow 0$, we define for non-invertible A and B

$$\mathsf{S}(A|B) = \operatorname*{s-lim}_{\varepsilon \downarrow 0} \mathsf{S}(A|B + \varepsilon) \tag{1}$$

if the limit (in the strong operator topology) exists as a bounded operator. But, in general, $\mathsf{S}(A|B)$ does not always exist. On the other hand, based on the fact that $\frac{x^t-1}{t} \searrow \log t$ as $t \downarrow 0$, it follows that $\frac{A\#_t B - A}{t}$ is monotone-decreasing as $t \downarrow 0$, so that another equivalent definition of Uhlmann's type is the derivative one for the path of geometric means $A\#_t B$:

$$\mathsf{S}(A|B) = \operatorname*{s-lim}_{t\downarrow 0} \frac{A\#_t B - A}{t} \tag{2}$$

if the limit exists. If A and B are commuting and $\mathsf{S}(A|B)$ is defined, then

$$\mathsf{S}(A|B) = A\log B - A\log A,$$

in particular, $\mathsf{S}(0|B) = 0$ for all positive operators $B \geq 0$. Though we often use unbounded expressions like $\log A$ from now on, these conventions are surely based on the total boundedness of $A\log A$. Under the existence, we have the following properties of $\mathsf{S}(A|B)$ for positive operators A and B by those for operator means:

Lemma 2.1. *Under the existence, the following properties hold:*

(1) *If* $B \leq B'$, *then* $\mathsf{S}(A|B) \leq \mathsf{S}(A|B')$.

(2) $T^*\mathsf{S}(A|B)T \leq \mathsf{S}(T^*AT|T^*BT)$ *for all* T *(the equality holds for invertible* T*).*

(2′) $\Phi(\mathsf{S}(A|B)) \leq \mathsf{S}(\Phi(A)|\Phi(B))$ *for all normal positive linear maps* Φ.

(3) $\mathsf{S}(A_1|B_1) + \mathsf{S}(A_2|B_2) \leq \mathsf{S}(A_1 + A_2|B_1 + B_2)$.

(3′) $(1-t)\mathsf{S}(A_1|B_1) + t\mathsf{S}(A_2|B_2) \leq \mathsf{S}((1-t)A_1 + tA_2|(1-t)B_1 + tB_2)$ *for all* $t \in [0,1]$.

(4) $\mathsf{S}(A|B) \leq B - A$.

(5) $\ker \mathsf{S}(A|B) \supset \ker A$.

(6) $\mathsf{S}\left(\bigoplus_k A_k \mid \bigoplus_k B_k\right) = \bigoplus_k \mathsf{S}(A_k|B_k)$.

(7) $\mathsf{S}(A|A\#_t B) = t\ \mathsf{S}(A|B)$ *for all* $t \in [0,1]$.

Here we recall the equality condition in the transformer inequality (2) of Lemma 2.1 [8, Theorem 3]: If $\ker T^* \subset \ker A \cap \ker B$ for an operator T, then $T^*(A \,\mathrm{m}\, B)T = (T^*AT)\mathrm{m}(T^*BT)$ holds for all operator means m. Moreover this equality holds for in $\mathsf{S}(A|B)$ since $\mathsf{S}(A|B) = \operatorname{s-lim}_{t\downarrow 0} \frac{A\#_t B - A}{t}$:

Theorem 2.2. *Let A and B be positive operators. If $\mathsf{S}(A|B)$ exists and $\ker T^* \subset \ker A \cap \ker B$ for an operator T, then*

$$T^*\mathsf{S}(A|B)T = \mathsf{S}(T^*AT|T^*BT).$$

Then we have one of the (sufficient) conditions that $\mathsf{S}(A|B)$ exists;

Lemma 2.3. *If A is majorized by B, i.e., $A \leq \alpha B$ for some $\alpha > 0$, then $\mathsf{S}(A|B)$ exists.*

In fact, by Douglas' majorization theorem [6], we have $A^{\frac{1}{2}} = DB^{\frac{1}{2}}$ for some 'derivative' operator D with $\ker D = \ker A \supset \ker B$ and so $\ker B = \ker A \cap \ker B$. Then , for the support projection P_B for B, we have $P_BAP_B = A$ and $P_BD^*DP_B = D^*D$. Hence it follows from Theorem 2.2 that

$$\mathsf{S}(A|B) = \mathsf{S}(B^{\frac{1}{2}}D^*DB^{\frac{1}{2}}|B) = B^{\frac{1}{2}}\mathsf{S}(D^*D|P_B)B^{\frac{1}{2}} = B^{\frac{1}{2}}\eta\,(D^*D)\,B^{\frac{1}{2}}$$

and so $\mathsf{S}(A|B)$ exists.

It is also shown that the majorization $A \leq \alpha B$ is equivalent to the condition for the range inclusion;

$$\operatorname{ran} A^{\frac{1}{2}} \subset \operatorname{ran} B^{\frac{1}{2}}.$$

But it is stronger than the existence condition. In fact, A is not majorized by A^2 if $\sigma(A) = [0,1]$, while we easily see $\mathsf{S}(A|A^2) = A\log A$.

Another candidate is the kernel inclusion

$$\ker A \supset \ker B,$$

which is weaker than the range inclusion. In fact, the kernel condition does not guarantee the existence: For B with $\sigma(B) = [0,1]$ where 0 is not an eigenvalue, it follows that $\mathsf{S}(I|B) = \log B$ diverges while both kernels are trivial.

The third condition between the above ones is B-*absolute continuity* in the sense of Ando's Lebesgue decomposition [2]:

$$A = [B]A \equiv \operatorname*{s-lim}_{n\to\infty} A : nB$$

where $A : B$ defined by

$$\langle A : Bz, z\rangle = \inf_{x+y=z} [\langle Ax, x\rangle + \langle By, y\rangle] \tag{†}$$

is the *parallel addition* [1], which is the half of the *harmonic mean* $A\,\mathrm{h}\,B$ [3]. Kosaki [18] showed that

$$[B]A = A^{\frac{1}{2}}P_MA^{\frac{1}{2}}$$

for the projection P_M on the closed subspace

$$M = \overline{\{y \mid A^{\frac{1}{2}}y \in \operatorname{ran} B\}}.$$

This result implies $A = [B]A = \lim_{t\downarrow 0} A\#_t B$ and hence B-absolute continuity guarantees the continuity of $A\#_t B$ at $t = 0$ and it is a necessary condition for the existence of $\mathsf{S}(A|B)$ as the above derivative [12]. In fact, this continuity is in the norm topology:

Lemma 2.4. *If $\mathsf{S}(A|B)$ exists, then $A\#_t B$ converges uniformly to A for $t \downarrow 0$.*

Since $\ker A\#_t B \supset \ker A \vee \ker B$ for all $t \in (0,1)$ as in [10] (as we will see later, these are equal indeed) and it is related to the ranges, it is a stronger condition than the kernel inclusion. But it is weaker than the existence condition: If A is the range projection P_B for B with $\sigma(B) = [0,1]$, then $\mathsf{S}(P_B|B) = P_B \log B$ is not bounded.

In fact we showed the existence condition expressed by the boundedness of tangent lines in [13]. Let $L_\alpha(A,B) \equiv \dfrac{1}{\alpha}B - A + (\log\alpha)A$ for $\alpha > 0$. Then we see $L_\alpha(A,B) \geq \mathsf{S}(A|B)$:

Lemma 2.5. *The entropy $\mathsf{S}(A|B)$ exists if and only if*

$$L_\alpha(A,B) = \left[\frac{1}{\alpha}B - A + (\log\alpha)A\right] > c \qquad \text{for some } c \text{ for all } \alpha > 0. \tag{3}$$

As we will see in the proof, we have $\mathsf{S}(A|B) \geq c$.

Summing up, we have the following relations around the existence condition:

Theorem 2.6. *The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) hold in the following conditions for a pair of A, $B \geq 0$ and each converse does not always hold.*

(1) **majorization** *or* **range inclusion:** $\exists\alpha > 0;\ A \leq \alpha B$, *i.e.*, $\operatorname{ran} A^{\frac{1}{2}} \subset \operatorname{ran} B^{\frac{1}{2}}$.

(2) **existence condition**: $\mathsf{S}(A|B)$ *exists as a bounded operator, i.e.,*

$$\left[\frac{1}{\alpha}B - A + (\log\alpha)A\right] > \exists c \ (\forall\alpha > 0).$$

(3) B-**absolute continuity:** $A = [B]A \left(= A^{\frac{1}{2}}P_M A^{\frac{1}{2}} = \lim_{t\downarrow 0} A\#_t B\right)$.

(4) **kernel inclusion:** $\ker A \supset \ker B$.

Remark 2.1. If both ranges of A and B are closed, in particular, for the case of matrices, the above conditions in Theorem 2.6 are all equivalent since the relation $\operatorname{ran} A^{\frac{1}{2}} = \overline{\operatorname{ran} A} = (\ker A)^\perp$ holds.

Here we recall the following well-known 'monotone convergence lemma' for monotone double (or multiple) sequences which is our key lemma:

Lemma 2.7. *Let* $\{a_{\delta_1,\delta_2}\}$ *be a bounded double sequence of real numbers for* $\delta_1, \delta_2 \in (0,1]$. *If* $\{a_{\delta_1,\delta_2}\}$ *is monotone decreasing for* $\delta_1, \delta_2 \downarrow 0$, *then there exists the limit with*

$$\lim_{\delta_1,\delta_2 \downarrow 0} a_{\delta_1,\delta_2} = \lim_{\delta_1 \downarrow 0} \lim_{\delta_2 \downarrow 0} a_{\delta_1,\delta_2} = \lim_{\delta_2 \downarrow 0} \lim_{\delta_1 \downarrow 0} a_{\delta_1,\delta_2}.$$

Moreover, it also holds for multi-monotone sequences: If bounded numbers $a_{\delta_1,\ldots,\delta_n}$ *are monotone decreasing for* $\delta_1, \ldots, \delta_n \downarrow 0$, *then there exists the limit* $\lim_{\delta_1,\ldots,\delta_n \downarrow 0} a_{\delta_1,\ldots,\delta_n}$ *and each iterating limit is exchangeable.*

Remark 2.2. In fact, under the existence, the equivalence of two definitions (1) and (2) of $\mathsf{S}(A|B)$ is based on the above fact since $\frac{A\#_t(B+\varepsilon)-A}{t}$ is monotone decreasing for $t, \varepsilon \searrow 0$. See the similar argument in the following theorem and Theorems 3.2 and 4.1.

Here we give a property of an upper semi-continuity type:

Theorem 2.8. *Let* A *and* B *be positive operators. If* $\mathsf{S}(A|B)$ *exists and* $Y_\varepsilon \searrow 0$ *as* $\varepsilon \downarrow 0$ *for a sequence of positive operators* Y_ε, *then* $\mathsf{S}(A+Y_\varepsilon|B+Y_\varepsilon) \searrow \mathsf{S}(A|B)$ *as* $\varepsilon \downarrow 0$.

Finally in this section, we add some new results for $\mathsf{S}(A|B)$. First we see, so to speak, the *interpolational property*. For this, we recall Izumino's construction of operator means [8] which is considered as an operator version for the Pusz-Woronowicz means [23, 24]: Let A and B be positive operators and put $R = (A+B)^{\frac{1}{2}}$. Since $A \leq A+B$ and $B \leq A+B$, it follows from Douglas's majorization theorem that there exists derivatives D, E with $A^{\frac{1}{2}} = DR$, $B^{\frac{1}{2}} = ER$. Then

$$R^2 = A + B = RD^*DR + RE^*ER = R(D^*D + E^*E)R,$$

so that we may assume $E^*E = I - D^*D$ in $\overline{\operatorname{ran} R}$. Thus it follows from $\ker R = \ker A \cap \ker B \subset \ker D^*D \cap \ker E^*E$ that

$$A \,\mathrm{m}\, B = R(D^*D \,\mathrm{m}\, (I - D^*D))R \tag{4}$$

for operator means m and similarly

$$\mathsf{S}(A|B) = R\, \mathsf{S}(D^*D|I - D^*D)\, R = R\,(D^*D \log D^*D - D^*D \log(I - D^*D))\, R \tag{5}$$

if $\mathsf{S}(A|B)$ exists by s-$\lim_{t\downarrow 0} \frac{A\#_t B - A}{t}$. Here we note that the formula (2.5) makes sense as a bounded operator even though $S(D^*D|I - D^*D)$ is not bounded. Moreover, we may use such inner calculations by suitable approximations.

Now we recall that $A\#_t B$ is an interpolational mean;

$$(A\#_p B)\#_r(A\#_q B) = A\#_{(1-r)p+rq}B$$

for $r, p, q \in [0,1]$ under the conventions $A\#_0 B = A$ and $A\#_1 B = B$, see [15, 16]. Then, for $t \in (0,1)$ and $p \in [0,1]$, $\mathsf{S}(A\#_t B|A\#_p B)$ exists and the following properties hold:

Lemma 2.9. *Let A and B be positive operators. For $t \in (0,1)$ and $p, q \in [0,1]$, the entropy $\mathsf{S}(A\#_t B|A\#_p B)$ exists and*

$$\frac{\mathsf{S}(A\#_t B|A\#_p B) + \mathsf{S}(A\#_t B|A\#_q B)}{2} = \mathsf{S}(A\#_t B|A\#_{\frac{p+q}{2}} B).$$

Theorem 2.10. *Let A and B be positive operators where $\mathsf{S}(A|B)$ exists. If $t \in (0,1)$ and $p, q, r \in [0,1]$, the following entropies exist and the interpolational property;*

$$(1-r)\mathsf{S}(A\#_t B|A\#_p B) + r\mathsf{S}(A\#_t B|A\#_q B) = t\mathsf{S}(A\#_t B|A\#_{(1-r)p+rq}B)$$

holds.

For invertible operators A and B, it is easy to see that the positivity (resp. negativity) of $\mathsf{S}(A|B)$ is equivalent to $B \geq A$ (resp. $A \geq B$) and hence $\mathsf{S}(A|B) = 0$ if and only if $A = B$. Second we discuss the non-invertble case:

Theorem 2.11. *Suppose $\mathsf{S}(A|B)$ exists for positive operators A and B. Then $\mathsf{S}(A|B) \geq 0$ (resp. $\mathsf{S}(A|B) \leq 0$) if and only if $A \leq B$ (resp. $A \geq B$). Consequently, $\mathsf{S}(A|B) = 0$ if and only if $A = B$.*

3 Karcher mean for positive operators

Lawson and Lim [20] showed that the *Karcher equation* for positive invertible operators A_j ($j = 1, 2, \dots, n$), X and a weight $\{\omega_j\}$ ($\omega_j \geq 0$ for $j = 1, 2, \dots, n$ and $\sum_{j=1}^n \omega_j = 1$)

(KE)
$$0 = \sum_{j=1}^{n} \omega_j \log\left(X^{-\frac{1}{2}} A_j X^{-\frac{1}{2}}\right)$$

has a unique positive invertible solution

$$X = \mathsf{G}_\mathsf{K}(\omega_j; A_j) = \mathsf{G}_\mathsf{K}(\omega; \mathbb{A}) \text{ for } \omega = (\omega_1, ..., \omega_n) \text{ and } \mathbb{A} = (A_1, ..., A_n).$$

It is called the *(weighted n-variable) Karcher mean.* This definition depends on the invertibility of operators. But, even for non-invertible positive operators A_j, for each

$\varepsilon > 0$ the Karcher mean $X_\varepsilon = \mathsf{G_K}(\omega_j; A_j + \varepsilon) \geq 0$ exists and the monotonicity of $\mathsf{G_K}$ guarantees the strong-operator limit:

$$X_0 = \operatorname*{s\text{-}lim}_{\varepsilon\to 0} X_\varepsilon = \operatorname*{s\text{-}lim}_{\varepsilon\to 0} \mathsf{G_K}(\omega_j; A_j + \varepsilon).$$

Naturally we write $X_0 = \mathsf{G_K}(\omega_j; A_j)$ for non-invertible A_j and call it *the Karcher mean* again.

Here we extend the extremal means with a weight $\{\omega_j\}$ synchronously to $\mathsf{G_K}$: The arithmetic mean A and the harmonic one H for non-invertible A_j are defined by

$$\mathsf{A}(\omega_j; A_j) = \sum_j \omega_j A_j, \quad \mathsf{H}(\omega_j; A_j) = \operatorname*{s\text{-}lim}_{\varepsilon\to 0} \mathsf{H}((\omega_j; A_j+\varepsilon) = \operatorname*{s\text{-}lim}_{\varepsilon\to 0} \left(\sum_j \omega_j (A_j+\varepsilon)^{-1}\right)^{-1}.$$

As for this construction of corresponding mean, we say 'H is the adjoint of A' as in the Kubo-Ando theory [19]. Then we also have the following properties of the Karcher mean for positive operators:

Theorem 3.1. *Let A_j and B_j be positive operators for $j = 1, 2, \ldots, n$ and $\{\omega_j\}$ a weight. Then the following properties hold:*

(1) *If* $A_j \leq B_j$, *then* $\mathsf{G_K}(\omega_j; A_j) \leq \mathsf{G_K}(\omega_j; B_j)$.

(2) $T^*\mathsf{G_K}(\omega_j; A_j)T \leq \mathsf{G_K}(\omega_j; T^*A_jT)$ *for all T (the equality holds for invertible T).*

(2′) $\Phi(\mathsf{G_K}(\omega_j; A_j)) \leq \mathsf{G_K}(\omega_j; \Phi(A_j))$ *for all normal positive linear maps Φ.*

(3) $\mathsf{G_K}(\omega_j; A_j) + \mathsf{G_K}(\omega_j; B_j) \leq \mathsf{G_K}(\omega_j; A_j + B_j)$.

(3′) $(1-t)\mathsf{G_K}(\omega_j; A_j) + t\mathsf{G_K}(\omega_j; B_j) \leq \mathsf{G_K}(\omega_j; (1-t)A_j + tB_j)$ *for all $t \in [0,1]$.*

(4) *If all A_j are commuting, then* $\mathsf{G_K}(\omega_j; A_j) = \prod_{j=1}^n A_j^{\omega_j}$ *with convention $A^0 = I$.*

(5) $\mathsf{G_K}(\omega_j; A_j) = \text{s-lim}_{\varepsilon\downarrow 0}\, \mathsf{G_K}(\omega_j; (A_j+\varepsilon)^{-1})^{-1}$.

(6) $\mathsf{G_K}(\omega_j; c_jA_j) = \prod_{j=1}^n c_j^{\omega_j} \mathsf{G_K}(\omega_j; A_j)$ *for $c_j \geq 0$ $(j = 1, 2, \ldots, n)$.*

(7) $\mathsf{H}(\omega_j; A_j) \leq \mathsf{G_K}(\omega_j; A_j) \leq \mathsf{A}(\omega_j; A_j)$.

(8) $\mathsf{G_K}\left(\omega_j; \bigoplus_m A_{j,m}\right) = \bigoplus_m \mathsf{G_K}(\omega_j; A_{j,m})$.

In fact, the equality in the 'transformer inequality' (2) for the case that all operators are invertible is already shown in [20], so that the equality also holds for non-invertible A_j. In general, (2) follows from (2').

We also have the upper semi-continuity for the Karcher mean:

Theorem 3.2. *Let A_j be positive operators for $j = 1, 2, \ldots, n$ and $\{\omega_j\}$ a weight. If $Y_{\varepsilon_j} \searrow 0$ as $\varepsilon_j \downarrow 0$ for sequences of positive operators Y_{ε_j}, then $\mathsf{G_K}(\omega_j; A_j + Y_{\varepsilon_j}) \searrow \mathsf{G_K}(\omega_j; A_j)$.*

Corollary 3.3. *Let A_j be positive operators for $j = 1, 2, \ldots, n$ and $\{\omega_j\}$ a weight. Then $X = \mathsf{G_K}(\omega_j; A_j)$ implies $X = \mathsf{G_K}(\frac{1}{n}; X \#_{\omega_j} A_j)$.*

The properties in Theorem 3.1 also holds for the arithmetic mean and the harmonic one in non-invertible cases. Moreover, by the sub-additivity (3) in Theorem 3.1, a joint concavity for m and that for its adjoint m*

$$A \ \mathrm{m}^* B = \operatorname*{s\text{-}lim}_{\varepsilon \to 0} \left((A+\varepsilon)^{-1} \, \mathrm{m} (B+\varepsilon)^{-1} \right)^{-1}$$

for operator means m (see [19, Theorems 3.6, 4.8]) hold;

Lemma 3.4. *For positive operators A_j and B_j $(j = 1, 2, \ldots, n)$, and a weight $\{\omega_j\}$,*

$$\mathsf{A}(\omega_j; A_j) \, \mathrm{m} \, \mathsf{A}(\omega_j; B_j) \geq \mathsf{A}(\omega_j; A_j \, \mathrm{m} \, B_j) \text{ and } \mathsf{H}(\omega_j; A_j) \, \mathrm{m} \, \mathsf{H}(\omega_j; B_j) \leq \mathsf{H}(\omega_j; A_j \, \mathrm{m} \, B_j)$$

for any operator means m.

Note that if $\omega_k = 0$ for some k, then n-mean $\mathsf{G_K}(\omega_j; A_j)$ is nothing but $(n-1)$-mean without ω_k, A_k. So we call $\mathsf{G_K}(\omega_j; A_j)$ the *proper* Karcher mean if $\omega_j > 0$ for all j. Then we also call the weight $\{\omega_j\}$ *proper*. Like the 2-variable case Theorem 2.9 (see also [7, 10]), we also have the following properties of ranges:

Lemma 3.5. *For a proper weight $\{\omega_j\}$ and positive operators A_j $(j = 1, 2, \ldots, n)$,*

$$\operatorname{ran} \mathsf{A}(\omega_j; A_j)^{\frac{1}{2}} = \bigvee_j \operatorname{ran} A_j^{\frac{1}{2}} \quad \text{and} \quad \operatorname{ran} \mathsf{H}(\omega_j; A_j)^{\frac{1}{2}} = \bigcap_j \operatorname{ran} A_j^{\frac{1}{2}}.$$

We also extend the vector state expression for the parallel sum, which is obtained inductively:

Lemma 3.6. *For a proper weight $\{\omega_j\}$ and positive operators A_j $(j = 1, 2, \ldots, n)$,*

$$\langle \mathsf{H}(\omega_j; A_j) x, x \rangle = \inf_{x = \sum_j x_j} \sum_j \left\langle \frac{1}{\omega_j} A_j x_j, x_j \right\rangle \quad \text{for every vector } x.$$

Then, similarly to the 2-variable case (Theorem 2.9), we have the following kernel condition for the Karcher mean:

Theorem 3.7. *For a proper Karcher mean,* $\ker \mathsf{G_K}(\omega_j; A_j) = \bigvee_j \ker A_j$.

The above theorem shows that if $A_j = 0$ for some j with $\omega_j > 0$, then $\mathsf{G_K}(\omega_j; A_j) = 0$ since the kernel is the entire space.

The following result is also an extension of 2-variable case:

Corollary 3.8. *For a proper weight* $\{\omega_j\}$, $\mathsf{G_K}(\omega_j; P_j) = \bigwedge_j P_j$ *for projections* P_j $(j = 1, 2, \ldots, n)$.

Remark 3.1. In general, we easily obtain

$$\ker \mathsf{G_K}(\omega_j; A_j) = \bigvee_{\omega_j > 0} \ker A_j \qquad \text{and} \qquad \mathsf{G_K}(\omega_j; P_j) = \bigwedge_{\omega_j > 0} P_j.$$

The Karcher equation (KE) definitely requires the invertibility for A_j and their theory depends on the geometric properties for positive invertible operators. In this invertible case, note that (KE) is equivalent to a simple equation by the relative operator entoropy

$$(**) \qquad 0 = \sum_{j=1}^{n} \omega_j \mathsf{S}\,(X|A_j) = \mathsf{A}(\omega_j; \mathsf{S}(X|A_j)),$$

which also makes sense for non-invertible A_j. But this equation always has a trivial solution $X = 0$ since $\mathsf{S}(0|A_j) = 0$. For the case of Corollary 3.8, the entropy is $\mathsf{S}(P|P_j) = P \log P_j - P \log P = 0$, and hence P is indeed a solution of the equation (**). But this consideration shows that each projection Q with $0 \le Q \le P$ is a solution of (**). Thereby, a reasonable extension of (KE) is the following *EKE(Extended Karcher equation) with the kernel condition* under the existence of each $\mathsf{S}(X|A_j)$:

$$\textbf{(EKE)} \qquad 0 = \sum_{j=1}^{n} \omega_j \mathsf{S}\,(X|A_j) = \mathsf{A}(\omega_j; \mathsf{S}(X|A_j)) \quad \text{with} \quad \ker X = \bigvee_{\omega_j > 0} \ker A_j.$$

Remark 3.2. If operators A_j are commuting for all $j = 1, 2, \ldots, n$, then $X_0 = \mathsf{G_K}(\omega_j; A_j) = \prod_j A_j^{\omega_j}$ and X_0 is a solution of (EKE). But, if the kernel condition is removed, the following example gives another solution X even if X commutes with all A_j.

Example 1. For diagonal matrices $A = \operatorname{diag}(a, b, c, 0)$ and $B = \operatorname{diag}(\frac{1}{a}, b, 0, d)$, take $X_1 = \operatorname{diag}(0, b, 0, 0)$. Then $\ker X_1 \neq \ker A \vee \ker B$ and all matrices are commuting and

$$\begin{aligned}
&\mathsf{S}(X_1|A) + \mathsf{S}(X_1|B) \\
&= -2X_1 \log X_1 + X_1 \log A + X_1 \log B \\
&= \operatorname{diag}(0, -2b \log b, 0, 0) + \operatorname{diag}(0, b \log b, 0, 0) + \operatorname{diag}(0, b \log b, 0, 0) = 0.
\end{aligned}$$

So X_1 is a solution while $X_0 = \operatorname{diag}(1, b, 0, 0)$ is a solution similarly and $X_1 \neq X_0$.

Remark 3.3. For the case of projections $A_j = P_j$ for $j = 1, 2, \ldots, n$, the above P in Corollary 3.8 is a unique solution of (EKE). In fact, suppose Y is another solution. Then the kernel condition $\ker Y = \bigvee \ker P_j \supset \bigwedge \ker P_j = \ker P$ shows $PYP = Y$ and hence we have $YP_j = P_jY = Y$ and $Y \log P_j = YP_j \log P_j = 0$. Therefore

$$0 = \sum_j \omega_j \mathsf{S}(Y|P_j) = \sum_j \omega_j \left(Y \log P_j - Y \log Y\right) = \sum_j \omega_j(-Y \log Y) = -Y \log Y,$$

which implies that Y must be a projection and consequently $Y = P$ by the kernel condition.

Moreover note that $\mathsf{S}(A|B)$ does not always exist as a bounded self-adjoint operator as in the preceding section. But $\mathsf{S}(X_0|A_j)$ indeed exists for the Karcher mean $X_0 = \mathsf{G}_\mathsf{K}(\omega_j; A_j)$:

Lemma 3.9. *Let A_j be positive operators for $j = 1, 2, \ldots, n$ and $\{\omega_j\}$ a weight. For the Karcher mean $X_0 = \mathsf{G}_\mathsf{K}(\omega_j; A_j)$, each entropy $\mathsf{S}(X_0|A_j)$ exists for $\omega_j > 0$. For $\omega_k > 0$, bounds are expressed by*

$$-\frac{M_k}{\omega_k} \leq \mathsf{S}(X_0|A_k) \leq M_k$$

for $M_k = \max_{j \neq k} \|A_j\| + 1$.

So far, we have not showed that our Karcher mean $X_0 = \text{s-lim}_{\varepsilon \to 0} X_\varepsilon$ satisfies (EKE) for general positive operators. Here we can obtain only the inequality:

Lemma 3.10. *Let A_j be positive operators for $j = 1, 2, \ldots, n$ and $\{\omega_j\}$ a weight. Then*

$$0 \leq \sum_{j=1}^{n} \omega_j \mathsf{S}\left(X_0|A_j\right) \qquad \text{with} \qquad \ker X_0 = \bigvee_{\omega_j > 0} \ker A_j.$$

Theorem 3.11. *For positive operators A and B and $t \in (0, 1)$, the original geometric mean $A \#_t B$ satisfies (EKE).*

Recall that a non-invertible positive operator A has the closed range if and only if 0 is an isolated point in $\sigma(A)$. Any positive semi-definite matrix has the closed range. Finally in this section, we show that the Karcher mean for such operators is a unique solution of (EKE). To see this, we verify the following fact:

Lemma 3.12. *If A_j $(j = 1, 2, \ldots, n)$ are positive opereators whose ranges are closed, then so is $X_0 = \mathsf{G}_\mathsf{K}(\omega_j; A_j)$.*

So we have a unique solution of (EKE):

Theorem 3.13. *If A_j $(j = 1, 2, \ldots, n)$ are positive opereators whose ranges are closed, the Karcher mean $X_0 = \mathsf{G_K}(\omega_j; A_j)$ is a unique solution of* (EKE).

Corollary 3.14. *For positive semi-definite matrices A_j for $j = 1, 2, \ldots, n$ and a weight $\{\omega_j\}$, the Karcher mean $X = \mathsf{G_K}(\omega_j; A_j)$ is the unique solution of* (EKE)*:*

$$0 = \sum_{j=1}^{n} \omega_j \mathsf{S}\,(X|A_j) \quad \textit{with} \quad \ker X = \bigvee_{\omega_j > 0} \ker A_j.$$

4 Power means for non-invertible operators

In [20], Lawson and Lim established that the Karcher mean of positive invertible operators on a Hilbert space is the strong-operator limit of power means of positive invertible operators as $t \downarrow 0$. In this section, we show that the Karcher mean of positive operators is the strong-operator limit of power means of positive operators as $t \downarrow 0$.

Let A_j be positive operators for $j = 1, 2, \ldots, n$ and $\{\omega_j\}$ a weight. For each $\varepsilon > 0$, similarly to the Karcher mean $X_\varepsilon = \mathsf{G_K}(\omega_j; A_j + \varepsilon)$, the power means $\mathsf{P}_t(\omega_i; A + \varepsilon)$ for $t \in (0, 1]$ is the unique positive invertible solution of the power mean equation

$$X = \sum_{j=1}^{n} \omega_j (X \#_t (A_j + \varepsilon)).$$

For the negative case, the power means $\mathsf{P}_{-t}(\omega_j; A_j + \varepsilon)$ for $t \in (0, 1]$ are defined by $\mathsf{P}_{-t}(\omega_j; A_j + \varepsilon) = \mathsf{P}_t(\omega_j; (A_j + \varepsilon)^{-1})^{-1}$. In addition, we extended the range of the definition of the power means to the open interval $(-2, 2)$ in [25].

Then the Karcher mean for invertible case is the strong-operator limit of the power means:

$$\operatorname*{s-lim}_{t \to 0} \mathsf{P}_t(\omega_j; A_j + \varepsilon) = X_\varepsilon.$$

For $t \in (0, 1]$, the power means $\mathsf{P}_t(\omega_j; A_j + \varepsilon)$ are monotone decreasing for $\varepsilon \downarrow 0$ by [20, Proposition 3.6 (4)] and lower bounded by the zero operator. Hence $\mathsf{P}_t(\omega_j; A_j) = \inf_{\varepsilon > 0} \mathsf{P}_t(\omega_j; A_j + \varepsilon)$ exists and

$$\mathsf{P}_t(\omega_j; A_j) = \operatorname*{s-lim}_{\varepsilon \downarrow 0} \mathsf{P}_t(\omega_j; A_j + \varepsilon)$$

in the strong operator topology and so $\mathsf{P}_t(\omega_j; A_j)$ is a solution of the power mean equation

$$X = \sum_{j} \omega_j (X \#_t A_j) \tag{6}$$

for $t \in (0, 1]$ by the upper semi-continuity of $\#_t$.

Then we immediately obtain the similar properties: For $\mathbb{A} = (A_1, \dots, A_n)$, put the k-copy $\mathbb{A}^{(k)} = (\mathbb{A}, \dots, \mathbb{A})$ and the corresponding weight $\omega^{(k)} = \frac{1}{k}(\omega, \dots, \omega)$. Then, Theorem 4.3 guarantees that power means preserve the following properties for non-invertible case. In particular, a proof of (5') is given by a similar way as in ones of Theorem 3.1. The other proofs follows from the definition of $\mathsf{P}_t(\omega_j; A_j)$ and [20, Proposition 3.6]:

Lemma 4.1. *Let A_j be positive operators for $j = 1, 2, \dots, n$ and $\{\omega_j\}$ a weight. If $a_j \in (0, \infty)^n$ and $s, t \in (0, 1]$, then*

(1) $\mathsf{P}_t(\omega_j; A_j) = \left(\sum_{j=1}^n \omega_j A_j^t\right)^{1/t}$ *if A_j's commute.*

(2) $\mathsf{P}_t(\omega_j; a_j A_j) = (\sum_{j=1}^n a_j^t)^{1/t} \mathsf{P}_t\left(\frac{\omega_j a_j^t}{\sum_i \omega_i a_i^t}; A_j\right)$.

(3) $\mathsf{P}_t(\omega_{\sigma(j)}; A_{\sigma(j)}) = \mathsf{P}_t(\omega_j; A_j)$ *for any permutation σ.*

(4) $\mathsf{P}_t(\omega_j; A_j) \le \mathsf{P}_t(\omega_j; B_j)$ *if $A_j \le B_j$ for all $j = 1, 2, \dots, n$.*

(5) $T^* \mathsf{P}_t(\omega_j; A_j) T \le \mathsf{P}_t(\omega_j; T^* A_j T)$ *for all T (the equality holds for invertible T).*

(5') $\Phi(\mathsf{P}_t(\omega_j; A_j)) \le \mathsf{P}_t(\omega_j; \Phi(A_j))$ *for all normal positive linear maps Φ.*

(6) $\mathsf{P}_t(\omega_j; A_j) + \mathsf{P}_t(\omega_j; B_j) \le \mathsf{P}_t(\omega_j; A_j + B_j)$.

(6') $(1-u)\mathsf{P}_t(\omega_j; A_j) + u\mathsf{P}_t(\omega_j; B_j) \le \mathsf{P}_t(\omega_j; (1-u)A_j + uB_j)$ *for any $u \in [0, 1]$.*

(7) $\mathsf{H}(\omega_j; A_j) \le \mathsf{G_K}(\omega_j; A_j) \le \mathsf{P}_t(\omega_j; A_j) \le \mathsf{A}(\omega_j; A_j)$ *for $t \in (0, 1]$.*

(8) $\mathsf{P}_t(\omega^{(k)}; \mathbb{A}^{(k)}) = \mathsf{P}_t(\omega_j; A_j)$ *for any $k \in \mathbb{N}$.*

(9) $\mathsf{P}_t\left(\omega_j; \bigoplus_m A_{j,m}\right) = \bigoplus_m \mathsf{P}_t(\omega_j; A_{j,m})$.

(10) $\mathsf{P}_t(\omega_j; A_j) \le \mathsf{P}_s(\omega_j; A_j)$ *for $0 < t < s < 1$.*

Moreover, the power means $\mathsf{P}_t(\omega_j; A_j)$ for $t \in (0, 1]$ satisfy the following kernel condition:

Theorem 4.2. *Let A_j be positive operators for $j = 1, 2, \dots, n$. If a weight $\{\omega_j\}$ is proper, then*

$$\ker \mathsf{P}_t(\omega; A_j) = \bigcap_j \ker A_j \qquad \text{for } t \in (0, 1].$$

Remark 4.1. For non-proper case, it is easy to see $\ker \mathsf{P}_t(\omega_j; A_j) = \cap_{w_j > 0} \ker A_j$.

Similarly to the Karcher equation for positive operators, the power mean equation (6) always has a trivial solution $X = 0$. Thereby, we consider the following EPE (Extended Power mean equation) with the kernel condition:

$$\textbf{(EPE)} \qquad X = \sum_j \omega_j X \#_t A_j \qquad \text{with} \quad \ker X = \bigcap_{w_j > 0} \ker A_j.$$

Theorem 4.3. *Let A_j be positive operators for $j = 1, 2, \ldots, n$ and $\{\omega_j\}$ a weight. Then the power means $\mathsf{P}_t(\omega_j; A_j)$ for $t \in (0, 1]$ satisfy* (EPE) *and*

$$\mathsf{P}_t(\omega_j; A_j) \searrow \mathsf{G}_{\mathsf{K}}(\omega_j; A_j) \qquad \text{as } t \downarrow 0.$$

To observe the relations between $\mathsf{A}(\omega_j; \mathsf{S}(X|A_j))$ and the solution of (EKE), we reformulate Yamazaki's inequality [28, Theorem 1] in our situation:

Theorem Y (Yamazaki). *For positive operators A_j ($j = 1, 2, \ldots, n$) and X, and $\{\omega_j\}$ a weight, the inequality $\mathsf{A}(\omega_j; \mathsf{S}(X|A_j)) \geq 0$ implies $\mathsf{G}_{\mathsf{K}}(\omega_j; A_j) \geq X$. Moreover, for positive invertible operators A_j and X, the inequality $\mathsf{A}(\omega_j; \mathsf{S}(X|A_j)) \leq 0$ implies $\mathsf{G}_{\mathsf{K}}(\omega_j; A_j) \leq X$.*

This theorem formally shows the uniqueness of the Karcher solution for invertible case. But unfortunately, Yamazaki's proof depends on this uniqueness itself.

To show this uniqueness, Lawson-Lim [20] used the implicit function theorem of Banach spaces, but it was a little complicated. On the other hand, the uniqueness of the power mean depends on the Banach fixed point theorem, which is simple and natural. The following result follows from the uniqueness of the power mean and is an extension of Theorem Y. Moreover it will be shown in the next section that the Karcher solution for the invertible case is unique.

Theorem 4.4. *For positive operators A_j ($j = 1, 2, \ldots, n$) and X and a weight $\{\omega_j\}$, the inequality $X \leq \sum_j \omega_j X \#_t A_j = \mathsf{A}(\omega_j; X \#_t A_j)$ implies $X \leq \mathsf{P}_t(\omega_j; A_j)$. Moreover, if A_j and X are invertible, then the inequality $X \geq \sum_j \omega_j X \#_t A_j = \mathsf{A}(\omega_j; X \#_t A_j)$ implies $X \geq \mathsf{P}_t(\omega_j; A_j)$.*

Remark 4.2. This is an extension of Theorem Y. Indeed, suppose $\mathsf{A}(\omega_j; \mathsf{S}(X|A_j)) \geq 0$. Then

$$0 \leq \mathsf{A}(\omega_j; \mathsf{S}(X|A_j)) \leq \frac{\sum_j \omega_j X \#_t A_j - X}{t}.$$

Then, by the above theorem, $X \leq \mathsf{P}_t(\omega_j; A_j)$ for all $0 < t \leqq 1$. Taking limit as $t \downarrow 0$, we have $X \leq \mathsf{G}_{\mathsf{K}}(\omega_j; A_j)$. Another part is obtained by this result.

Remark 4.3. The invertibility in Theorem 4.4 cannot be removed. In fact, for a nontrivial projection P, let $X = A_1 = P$, $A_2 = P^\perp$ and $t = \omega_1 = \omega_2 = \frac{1}{2}$. Then we have

$$\frac{X\#A_1 + X\#A_2}{2} = \frac{P+O}{2} = \frac{1}{2}P \le P = X,$$

while

$$\mathsf{P}_{\frac{1}{2}}\left(\frac{1}{2},\frac{1}{2};P,P^\perp\right) = \left(\frac{P^{\frac{1}{2}} + (P^\perp)^{\frac{1}{2}}}{2}\right)^2 = \left(\frac{1}{2}I\right)^2 = \frac{1}{4}I \not\le P = X.$$

In the Kubo-Ando mean, the *adjoint sub-additivity*

$$(A:C)\,\mathrm{m}\,(B:D) \le (A\,\mathrm{m}\,B):(C\,\mathrm{m}\,D)$$

holds for the parallel addition ':' defined by (†), which is nothing but the sub-additivity of the adjoint mean m^*. Since G_K is selfadjoint, the Karcher mean satisfies the adjoint sub-additivity:

$$\mathsf{G}_\mathsf{K}(\omega_j; A_j : B_j) \le \mathsf{G}_\mathsf{K}(\omega_j; A_j) : \mathsf{G}_\mathsf{K}(\omega_j; B_j).$$

To observe the adjoint of power mean in the next section, we confirm this property for P_t:

Theorem 4.5. *The power mean satisfies the adjoint sub-additivity:*

$$\mathsf{P}_t(\omega_j; A_j : B_j) \le \mathsf{P}_t(\omega_j; A_j) : \mathsf{P}_t(\omega_j; B_j)$$

for $t \in (0,1]$, *where* : *is the parallel addition defined by* (†).

5 General operator mean and its adjoint

Since it is somewhat hard to handle the negative power means P_{-t} for $t \in (0,1]$, we also use Lawson-Lim's negative mean (say, P_t^* later). In this section, we want to show that it is a legal operator mean. For this purpose, we generalize the Kubo-Ando mean and its adjoint. Here for positive operators A_j $(j = 1,2,\ldots,n)$ and a weight $\{\omega_j\}$, we define an (n-variable) *general operator mean* $\mathsf{M}(\omega_j; A_j)$ as an n-ary operation on positive invertible operators on a Hilbert space $\mathcal{H}$ satisfying the following properties:

(M1) $T^*\mathsf{M}(\omega_j; A_j)T = \mathsf{M}(\omega_j; T^*A_jT)$ for all invertible T.

(M1') $\mathsf{M}(\omega_j; tA_j) = t\mathsf{M}(\omega_j; A_j)$ for $t > 0$.

(M2) $\mathsf{M}(\omega_j; A,\ldots,A) = A$.

(M3) $A_j \le B_j$ for all $j = 1,\ldots,n$ implies $\mathsf{M}(\omega_j; A_j) \le \mathsf{M}(\omega_j; B_j)$.

(M4) $\mathsf{M}(\omega_j; A_j + B_j) \geq \mathsf{M}(\omega_j; A_j) + \mathsf{M}(\omega_j; B_j)$.

(M5) $\mathsf{M}(\omega_j; A_j : B_j) \leq \mathsf{M}(\omega_j; A_j) : \mathsf{M}(\omega_j; B_j)$.

(M6) $\mathsf{M}(\omega_j; \bigoplus_m A_j^{(m)}) = \bigoplus_m \mathsf{M}(\omega_j; A_j^{(m)})$.

In addition, we define

$$\mathsf{M}(\omega_j; A_j) = \operatorname*{s-lim}_{\varepsilon \to 0} \mathsf{M}(\omega_j; (A_j + \varepsilon))$$

for (non-invertible) positive operators A_j and hence the above properties are preserved. For $t \in [0, 1]$, note that

(M7) **joint concavity:** $\mathsf{M}(\omega_j; (1-t)A_j + tB_j) \leq (1-t)\mathsf{M}(\omega_j; A_j) + t\mathsf{M}(\omega_j; B_j)$

follows from the sub-additivity (M4) and homogeneity (M1').

Similarly to the proof for upper semi-continuity in Theorem 3.2 based on Lemma 2.7, the sub-additivity and monotonicity imply the following:

Theorem 5.1. *A general operator mean* M *is upper semi-continuous:*
(M8) **upper semi-continuity:**

$$A_j^{(\delta)} \searrow A_j \text{ implies } \mathsf{M}(\omega_j; A_j^{(\delta)}) \searrow \mathsf{M}(\omega_j; A_j) \quad \text{as } \delta \downarrow 0.$$

Moreover in general, the transformer inequality holds. To show this, we see the case of projections:

Lemma 5.2. *Let* A_j *be positive operators for* $j = 1, 2, \ldots, n$ *and* $\{\omega_j\}$ *a weight. Then*

$$P\mathsf{M}(\omega_j; A_j)P \leq \mathsf{M}(\omega_j; PA_jP) \qquad \text{for all projections } P.$$

Theorem 5.3. *A general operator mean* M *satisfies*
(M9) **transformer inequality:**

$$T^*\mathsf{M}(\omega_j; A_j)T \leq \mathsf{M}(\omega_j; T^*A_jT) \quad \text{for all operators } T.$$

The transformer inequality also implies the joint concavity. Moreover its operator version like the Kubo-Ando means is obtained:

Corollary 5.4. *Let* $A_{j,m}$ *be positive operators for* $j = 1, \ldots, n$ *and* $m = 1, \ldots, k$, *and* $\{\omega_j\}$ *a weight. If* $\sum_{m=1}^k C_m^* C_m = I$, *then*

$$\sum_{m=1}^k C_m^* \mathsf{M}(\omega_j; A_{j,m}) C_m \leq \mathsf{M}(\omega_j; \sum_{m=1}^k C_m^* A_{j,m} C_m).$$

This inequality also implies the sub-additivity.

Now we study the adjoint of general operator means:

Lemma 5.5. *For a general operator mean* M*, the relation* $\mathsf{M}^*(\omega_j; A_j) = \mathsf{M}(\omega_j; A_j^{-1})^{-1}$ *for invertible* A_j *induces also a general operator mean for all positive operators* A_j*.*

The operator mean introduced above

$$\mathsf{M}^*(\omega_j; A_j) = \operatorname*{s-lim}_{\varepsilon\to 0} \mathsf{M}^*(\omega_j; A_j + \varepsilon) = \operatorname*{s-lim}_{\varepsilon\to 0} \mathsf{M}(\omega_j; (A_j + \varepsilon)^{-1})^{-1}$$

is called the *adjoint* M^* named after Kubo-Ando [19].

Now we observe the general operator mean $\mathsf{P}_t^*(\omega_j; A_j)$ for $t \in (0, 1]$. In the invertible case, it coincides with the power mean $\mathsf{P}_{-t}(\omega_j; A_j)$ with negative parameter in [20]. Since $\mathsf{P}_t = \mathsf{P}_t(\omega_j; A_j)$ is a general operator mean, we have:

Theorem 5.6. *For each* $t \in (0, 1]$*, the adjoint power mean* P_t^* *is a general operator mean.*

Remark 5.1. The joint concavity for P_t^* also holds though it is not shown in [20]. Like P_t, all the properties in Lemma 4.1 except (5') are hold for P_{-t}.

We recall $\mathsf{G_K}^* = \mathsf{G_K}$ in Theorem 3.1 (5), $\mathsf{A}^* = \mathsf{P}_1^* = \mathsf{H}$ and $\mathsf{H}^* = \mathsf{A}$. The following properties are clear since $\mathsf{M}^{**} = \mathsf{M}$:

Lemma 5.7. *Let* M*,* M' *and* M_n *be general operator means. Then* $\mathsf{M} \le \mathsf{M}'$ *if and only if* $\mathsf{M}^* \ge (\mathsf{M}')^*$*.* $\mathsf{M}_n \searrow \mathsf{M}$ *if and only if* $\mathsf{M}_n^* \nearrow \mathsf{M}^*$ *as* $n \to \infty$*.*

We have already shown that $\mathsf{P}_t \searrow \mathsf{G_K}$ as $t \searrow 0$, so that $\mathsf{P}_t^* \nearrow \mathsf{G_K}^* = \mathsf{G_K}$. Thus we have $\text{s-lim}_{t\to 0} \mathsf{P}_t = \text{s-lim}_{t\to 0} \mathsf{P}_t^* = \mathsf{G_K}$:

Theorem 5.8. *For each* $t \in (0, 1]$*, the adjoint power mean* P_t^* *converges increasingly to* $\mathsf{G_K}$ *as* $t \downarrow 0$*.*

Taking adjoint, we have the counter part of Theorem 4.4:

Theorem 5.9. *For positive operators* A_j $(j = 1, 2, \ldots, n)$ *and* X*, the inequality* $X \le \mathsf{H}(\omega_j; X\#_t A_j)$ *implies* $X \le \mathsf{P}_t^*(\omega_j; A_j)$*. Moreover, if* X *and* A_j *are invertible and* $X \ge \mathsf{H}(\omega_j; X\#_t A_j)$*, then* $X \ge \mathsf{P}_t^*(\omega_j; A_j)$*.*

For a solution X of the Karcher equation and $t \in (0, 1]$, we have

$$0 = \sum_j \omega_j \mathsf{S}(X|A_j) \le \sum_j \omega_j \frac{X\#_t A_j - X}{t} = \frac{\mathsf{A}(\omega_j; X\#_t A_j) - X}{t},$$

so that $X \le \mathsf{A}(\omega_j; X\#_t A_j)$. Also, for invertible case,

$$0 = -\sum_j \omega_j \log\left(X^{\frac{1}{2}} A_j^{-1} X^{\frac{1}{2}}\right) \ge X^{\frac{1}{2}} \left(\sum_j \omega_j \frac{X^{-1}\#_t A_j^{-1} - X^{-1}}{-t}\right) X^{\frac{1}{2}}$$
$$= \frac{X^{\frac{1}{2}} \mathsf{H}(\omega_j; X\#_t A_j)^{-1} X^{\frac{1}{2}} - I}{-t}.$$

Thus

$$I \le X^{\frac{1}{2}} \mathsf{H}(\omega_j; X\#_t A_j)^{-1}) X^{\frac{1}{2}} \quad \text{that is,} \quad X \ge \mathsf{H}(\omega_j; X\#_t A_j).$$

Therefore Theorems 4.4 and 5.9 say $\mathsf{P}_t^*(\omega_j; A_j) \le X \le \mathsf{P}_t(\omega_j; A_j)$ for all $t \in (0,1]$ for invertible A_j and X. By taking $t \downarrow 0$, we have

Corollary 5.10. *In positive invertible operators, the Karcher equation has a unique solution* $\mathsf{G}_{\mathsf{K}}(\omega_j; A_j)$.

As a final remark, we observe the corresponding equation for the power mean $\mathsf{P}_{-t} = \mathsf{P}_t^*(\omega_j; A_j)$ for $t \in (0,1]$. For the Lawson-Lim equation $X^{-1} = \sum_j \omega_j (X\#_t A_j)^{-1}$, it should be reformed into $X = \mathsf{H}(\omega_j; X\#_t A_j)$ to avoid invertibility of operators. Then we have

Lemma 5.11. $\ker \mathsf{P}_t^*(\omega_j; A_j) = \bigvee_{\omega_j > 0} \ker A_j.$

Now the required equation for the adjoint power mean $\mathsf{P}_t^*(\omega_j; A_j)$ for $t \in (0,1]$ is

(EPE*) $$X = \mathsf{H}(\omega_j; X\#_t A_j) \quad \text{with} \quad \ker X = \bigvee_{\omega_j > 0} \ker A_j.$$

Then from the upper semicontinuity for H and $\#_t$, we have

Theorem 5.12. *The adjoint power mean* $\mathsf{P}_t^*(\omega_j; A_j)$ *is the solution of* (EPE*) *for* $t \in (0,1]$.

Though the properties for power means are given, the following general problems are still not answered:

Conjecture. *For non-invertible positive operators on a Hilbert space, the Karcher mean satisfies* (EKE) *and it is a unique solution of* (EKE).

Conjecture 2. *For non-invertible positive operators on a Hilbert space, each power mean* $\mathsf{P}_t(\omega_j; A_j)$ *(resp.* $\mathsf{P}_t^*(\omega_j; A_j)$ *) for* $t \in (0,1]$ *is a unique solution of* (EPE) *(resp.* (EPE*)).

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