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An infinite dimensional Birhkoff's Theorem, a majorization relation for two density matrices and LOCC-convertibility

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1 Introduction

In quantum mechanics, *entanglement* between bipartite system is known as quantum correlations which do not arise in classical systems. With entanglement, we can consider useful tasks that can never be accomplished by classical systems, such as quantum teleportation and quantum dense coding. For this reason, entanglement has been regarded as a resource in quantum information theory.

If a state $|\psi\rangle\langle\psi| \in \mathcal{S}(\mathcal{H}\otimes\mathcal{K})$ on a bipartite system $\mathcal{H}\otimes\mathcal{K}$ is incomplete as an entanglement resource, one may want to convert it into a more entangled form $|\phi\rangle\langle\phi| \in \mathcal{S}(\mathcal{H}\otimes\mathcal{K})$. However, if two particles are far apart from each other, it is difficult to apply full quantum operations that is allowed theoretically in the composite system. Instead, as a practical class of quantum operations, local operations and classical communications (LOCC) play an important role in this situation.

For the LOCC-convertibility states, Nielsen proved [11, 12] in 1999 that the following statements are equivalent (the Nielsen's theorem).

(i) The initial state $|\psi\rangle$ can be converted to the target state $|\phi\rangle$ by LOCC.

(ii) The Schmidt coefficients of the initial state $|\psi\rangle$ is majorized by those of the target state $|\phi\rangle$.

Mathematically, the Nielsen's theorem can be written as follows.

Theorem 1.1. *(Nielsen[11, 12]) Let $\mathcal{H}$ and $\mathcal{K}$ be finite dimensional Hilbert spaces, and let ψ, $\phi \in \mathcal{H}\otimes\mathcal{K}$ be unit vectors. Then, the following are equivalent.*

- *One can convert $|\psi\rangle$ to $|\phi\rangle$ by the following LOCC: there exist a POVM $\{M_i\}_i$ on $\mathcal{H}$ and a set of unitary operators $\{U_i\}_i$ on $\mathcal{K}$ such that*

$$|\phi\rangle\langle\phi| = \sum_i (M_i\otimes U_i)|\psi\rangle\langle\psi|(M_i^*\otimes U_i^*), \tag{1}$$

  where the sum is finite sum.

- *The following majorization relation holds:*

$$\mathrm{Tr}_{\mathcal{K}}\,|\psi\rangle\langle\psi| \prec \mathrm{Tr}_{\mathcal{K}}\,|\phi\rangle\langle\phi|.$$

Namely, the majorization condition (ii) fully characterizes the LOCC-convertibility of pure states in finite dimensional systems.

Subsequently, in 2006, Owari *et al.* [13] extended the Nielsen's theorem to infinite dimensional systems. They proved that the implication (i) $\Rightarrow$ (ii) (necessary condition for the LOCC-convertibility) holds in the same form as finite dimensional systems. Moreover, Owari *et al.* [13] introduced a notion of ϵ-convertibility by LOCC in infinite dimensional systems and proved that ϵ-convertibility for LOCC gives a characterization of the sufficient condition.

However, it has been open whether the implication (ii) $\Rightarrow$ (i) (the sufficient condition for LOCC-convertibility) holds or not in infinite dimensional systems.

To solve this problem, in [1], we develop an infinite dimensional analogue of Birkhoff's theorem [Theorem 2.2] and use this to prove the following theorem.

Theorem 1.2. *(Asakura [1]) Let $\mathcal{H}$ and $\mathcal{K}$ be infinite dimensional separable Hilbert spaces, and let ψ, $\phi \in \mathcal{H} \otimes \mathcal{K}$ be full rank unit vectors. Then, the following are equivalent.*

- *There exist a Borel set I of a certain of metric space, a probability mesure μ on I, a set of densely defined (not necessarily bounded) operators $\{M_i\}_{i\in I}$ on $\mathcal{H}$, a dense subspace $\mathcal{H}_0 \subset \mathcal{H}$, and a set of unitary operators $\{U_i\}_{i\in I}$ on $\mathcal{K}$ such that*

$$|\psi\rangle \in D(M_i \otimes U_i), \text{ for } i \in I, \tag{2}$$

$$(\mathrm{Tr}_{\mathcal{K}}|\psi\rangle\langle\psi|)\mathcal{H}_0 \subset \mathcal{H}_0, i.e., \{(\mathrm{Tr}_{\mathcal{K}}|\psi\rangle\langle\psi|)\,|\eta\rangle : \eta \in \mathcal{H}_0\} \subset \mathcal{H}_0, \tag{3}$$

$$D(M_i) \supset \mathcal{H}_0, \text{ for any } i \in I \tag{4}$$

$$\int_I \langle\eta|\, {M_i}^* M_i \,|\xi\rangle\, d\mu(i) = \langle\eta|\,\xi\rangle, \text{ for } \eta, \xi \in \mathcal{H}_0, \tag{5}$$

$$I \ni i \mapsto (M_i \otimes U_i)|\psi\rangle\langle\psi|(M_i^* \otimes U_i^*) \in \mathfrak{C}_1(\mathcal{H}) \text{ is integrable, and} \tag{6}$$

$$|\phi\rangle\langle\phi| = \int_I (M_i \otimes U_i)|\psi\rangle\langle\psi|(M_i^* \otimes U_i^*) d\mu(i), \text{ in } \mathfrak{C}_1(H). \tag{7}$$

- $\mathrm{Tr}_{\mathcal{K}}|\psi\rangle\langle\psi| \prec \mathrm{Tr}_{\mathcal{K}}|\phi\rangle\langle\phi|$ *holds.*

Remark 1.3. *Note that* (7) *becomes* (1) *and* (5) *becomes an equality for a POVM with finite cardinality in the case I is finite set and M_i are all bounded.*

In this paper, we introduce a new characterization for majorization relation between two density matrices, which is the characterization derived from our infinite dimensional analogue of Birkhoff's theorem [Theorem 2.2].

This paper is organized as follows. In Section 2, we introduce an infinite dimensional analogue of Birkhoff's theorem. In Section 3, we give a sketch of the proof of the sufficient condition of Theorem 1.1. In Section 4, we give a new characterization for majorization relation between two density matrices.

2 Infinite dimensional Birkhoff's theorem with WOT

Let $\mathcal{H}$ be a separable Hilbert space and $(|i\rangle)_{i=1}^{\infty}$ be a CONS in $\mathcal{H}$. We use the following notation.

$$\mathcal{P}(\mathcal{H}^{(|i\rangle)}) := \left\{ \sum_{i,j=1}^{\infty} a_{ij}|i\rangle\langle j| \in \mathfrak{B}(\mathcal{H}) \;\middle|\; a_{ij} = 0 \text{ or } 1, \sum_{j=1}^{\infty} a_{ij} = 1, \sum_{i=1}^{\infty} a_{ij} = 1 \text{ (for any } i,j) \right\},$$

$$\mathcal{D}(\mathcal{H}^{(|i\rangle)}) := \left\{ \sum_{i,j=1}^{\infty} a_{ij}|i\rangle\langle j| \in \mathfrak{B}(\mathcal{H}) \;\middle|\; a_{ij} \in [0,1], \sum_{j=1}^{\infty} a_{ij} = 1, \sum_{i=1}^{\infty} a_{ij} = 1 \text{ (for any } i,j) \right\}.$$

Remark 2.1. *When $\mathcal{H} = \mathbb{C}^n$ and $(|i\rangle)_i$ is a standard basis $(e_i)_i$ in $\mathbb{C}^n$, $\mathcal{P}(\mathcal{H}^{(|i\rangle)})$ is equal to the set of all $n \times n$ permutation matrices and $\mathcal{D}(\mathcal{H}^{(|i\rangle)})$ is equal to the set of all $n \times n$ doubly stochastic matrices. In the sequel, we abbreviate $\mathcal{D}({\mathbb{C}^n}^{(|e_i\rangle)})$ as $\mathcal{D}(\mathbb{C}^n)$.*

Using the notations in the previous section, Birkhoff's theorem [5] can be written as follows:

(1) $\mathrm{ex}\,\mathcal{D}(\mathbb{C}^n) = \mathcal{P}(\mathbb{C}^n)$,

(2) Let $\{P_i\}_{i=1}^{n!} := \mathcal{P}(\mathbb{C}^n)$. Then for any $D \in \mathcal{D}(\mathbb{C}^n)$, there exists a probability mass function $\{p_i\}_{i=1}^{n!}$ such that

$$D = \sum_{i=1}^{n!} p_i P_i,$$

(3) $\mathcal{D}(\mathbb{C}^n) = \mathrm{co}\,\mathcal{P}(\mathbb{C}^n)$.

Note that it is known that the three assertions are equivalent to each other, by Carathéodry theorem; see [4, Section II.2].

For the property (2), we proved the following theorem.

Theorem 2.2. *(Asakura [1]) For any $D \in \mathcal{D}(\mathcal{H}^{(|i\rangle)})$, there exists a probability measure μ_D on $\mathcal{P}(\mathcal{H}^{(|i\rangle)})$ such that*

$$D = w\text{-}\int_{\mathcal{P}(\mathcal{H}^{(|i\rangle)})} X d\mu_D(X). \tag{8}$$

where w- means the convergence of the weak operator topology (WOT) and $\mathcal{P}(\mathcal{H}^{(|i\rangle)})$ is a Borel set of a metric space $(\mathfrak{B}(\mathcal{H})_1, WOT)$.

Remark 2.3. *An infinite dimensional analogue of Birkhoff's theorem is known as* Birkhoff's Problem111. *For Birkhoff's Problem111, see [8, Section20] and [9, Section14.8].*

We remark that no one treated in any study (ii) in infinite dimensional space with operator topologies.

This theorem immediately implies the following theorem, which is the key tool to prove the sufficient condition of Theorem 1.2.

Theorem 2.4. *(Asakura [1]) Let ρ and σ be density matrices on $\mathcal{H}$ having same eigenbasis $(|i\rangle)_{i=1}^{\infty}$. If $\rho \prec \sigma$, there exist a $D \in \mathcal{D}(\mathcal{H}^{(|i\rangle)})$ and a probability measure μ_D on $\mathcal{P}(\mathcal{H}^{(|i\rangle)})$ corresponding to D such that*

$$\rho = \int_{\mathcal{P}(\mathcal{H}^{(|i\rangle)})} X\sigma X^* d\mu_D(X), \text{ in } \mathfrak{C}_1(H), \tag{9}$$

where in $\mathfrak{C}_1(H)$ means the convergence of the trace norm $\|\cdot\|_1$.

Proof. From [16, Theorem 3], we only have to show that the integral in (9) converges to ρ in WOT. By assumption, there exist $a = (a_n)_{n=1}^{\infty}$, $b = (b_n)_{n=1}^{\infty} \in \{(a_i)_{i=1}^{\infty} \in \ell^1 | a_i \geq 0,\ \sum_{i=1}^{\infty} a_i = 1,\ a_i \geq a_{i+1}\ (i \in \mathbb{N})\}$ such that $a \prec b$ and

$$\rho := \sum_{n=1}^{\infty} a_n |i_n\rangle\langle i_n|,\ \sigma := \sum_{n=1}^{\infty} b_n |i_n\rangle\langle i_n|,$$

where the infinite sums converge in the trace norm.

By [7, Theorem 4.7, Corolloary 6.1], there exists an infinite matrix $\tilde{D} = (d_{ij}) \in \mathcal{D}(\ell^2)$ such that $|a\rangle = \tilde{D}\,|b\rangle$ in ℓ^2. From Theorem 2.4, there exists a probability measure $\mu_{\tilde{D}}$ on $\mathcal{P}(l^2)$ such that

$$a_n = \int_{\mathcal{P}(l^2)} \langle e_n|\, X\, |b\rangle\, d\mu_{\tilde{D}}(X), \quad \text{for any } n \in \mathbb{N}, \tag{10}$$

where $(|e_n\rangle)_{n=1}^{\infty}$ is a standard basis in l^2. Let $D \in \mathcal{D}(\mathcal{H}^{(|i\rangle)})$ be $D = \sum_{i,j} d_{ij} |i\rangle\langle j|$. Then, from Theorem 2.4, there exists a probability measure μ_D on $\mathcal{P}(\mathcal{H}^{(|i\rangle)})$ such that (8) holds. Thus, we have $\langle n|\,\rho\,|n\rangle = a_n = \int_{\mathcal{P}(\mathcal{H}^{(|i\rangle)})} \langle n|\, X\sigma X^*\, |n\rangle\, d\mu_D(X)$. This implies that the integral in (9) converges to ρ in WOT. □

3 Sketch of the proof of the sufficient condition of Theorem 1.2.

We assume that $\mathrm{Tr}_{\mathcal{K}}|\psi\rangle\langle\psi| \prec \mathrm{Tr}_{\mathcal{K}}|\phi\rangle\langle\phi|$. Then, there exist unitaries $U_{\mathcal{H}}$ and $U_{\mathcal{K}}$ such that ψ and $\tilde{\phi} := (U_{\mathcal{H}} \otimes U_{\mathcal{K}})\phi$ have a same Schmidt basis, i.e.,

$$\psi = \sum_{i=1}^{\infty} \sqrt{a_i}\, |i\rangle_{\mathcal{H}}\, |i\rangle_{\mathcal{K}},\ \tilde{\phi} = \sum_{i=1}^{\infty} \sqrt{b_i}\, |i\rangle_{\mathcal{H}}\, |i\rangle_{\mathcal{K}}$$

with some $a = (a_i) \prec b = (b_i) \in \ell_1^{\downarrow}$ and some CONSs $(|i\rangle_{\mathcal{H}})_{i=1}^{\infty}$ and $(|i\rangle_{\mathcal{K}})_{i=1}^{\infty}$. Moreover, from Theorem 2.4, there exist a $D \in \mathcal{D}(\mathcal{H}^{(|i\rangle)})$ and a probability measure μ_D on $\mathcal{P}(\mathcal{H}^{(|i\rangle)})$ corresponding to D such that

$$\rho_{\psi} = \int_{\mathcal{P}(\mathcal{H}^{(|i\rangle)})} X\rho_{\tilde{\phi}}X^* d\mu_D(X), \text{ in } \mathfrak{C}_1(H), \tag{11}$$

For any $X \in \mathcal{P}(\mathcal{H}^{(|i\rangle)})$, let define a densely defined operator M_X on $\mathcal{H}$ by

$$\begin{cases} M_X := U_{\mathcal{H}}{}^* \sqrt{\rho_{\tilde{\phi}}} X^* (\rho_{\psi}{}^{-\frac{1}{2}}), \\ D(M_X) := D(\rho_{\psi}{}^{-\frac{1}{2}}), \end{cases}$$

and let $U_X := U_{\mathcal{K}}{}^* X^*$. Then, we have $(M_X \otimes U_X)\,|\psi\rangle = |\phi\rangle$ for any X, and then the function $\mathcal{P}(\mathcal{H}) \ni X \mapsto (M_X \otimes U_X)|\psi\rangle\langle\psi|(M_X{}^* \otimes U_X{}^*)$ is constant function. In particular, $\int_{\mathcal{P}(\mathcal{H}^{(|i\rangle)})}(M_X \otimes U_X)|\psi\rangle\langle\psi|(M_X{}^* \otimes U_X{}^*) = |\phi\rangle\langle\phi|$ holds.

Let $\mathcal{H}_0 := \mathrm{lin}\{|i\rangle\}_{i=1}^{\infty}$, then the dense subspace $\mathcal{H}_0 \subset$ satisfies the conditions (3) and (4). Moreover, for any X, we have

$$M_X{}^* M_X = \rho_\psi{}^{-\frac{1}{2}}(X\rho_\phi X^*)\rho_\psi{}^{-\frac{1}{2}} \text{ on } D(M_X) \supset \mathcal{H}_0.$$

Thus, $\{M_X\}_X$ satisfies the condition (5).

Putting it all together, the conditions (2), (3), (4), (5), (6) and (7) for a Borel set $I = \mathcal{P}(\mathcal{H})$ of a metric space $(\mathfrak{B}(\mathcal{H})_1, WOT)$ are satisfied. □

Remark 3.1. *The densely defined operator M_X in the above proof is equal to $U_{\mathcal{H}}\Delta_{\rho_\phi,\rho_\psi}(X^*)$, where $\Delta_{\rho_\phi,\rho_\psi}$ is a kind of relative modular operator [2, 3].*

4 Majorization relation between two density matrices

For majorization relation between two density matrices, it has been known that the following theorems hold.

Theorem 4.1. *([14, Section 4.3]) Let $\mathcal{H}$ be an finite dimensional Hilbert space. For ρ, $\sigma \in \mathcal{S}(\mathcal{H})$, the following are equivalent.*

(a) $\rho \prec \sigma$.

(B) *There exists a mixed unitary map Φ such that $\Phi(\sigma) = \rho$, i.e., $\rho \in \mathrm{co}U(\sigma)$. Here, $U(\sigma) := \{U\sigma U^* | U \text{ is unitary}\}$ is the unitary orbit of σ.*

(C) *There exists a unital and completely positive-trace preserving (CP-TP) map Φ such that $\Phi(\sigma) = \rho$.*

Remark 4.2. *A linear map Φ on $\mathfrak{C}_1(\mathcal{H})$ is called mixed unitary channel, if $\Phi(X) = \sum_{i=1}^{n} p_i U_i{}^* X U_i$, where $n < \infty$, the U_i are all unitary operators and $p_i > 0$, $\sum_{i=1}^{n} p_i = 1$.*

By definition, any mixed unitary channel is unital and CP-TP.

Theorem 4.3. *([10, Theorem 3.3], [6, Theorem 2.5(1)]) Let $\mathcal{H}$ be an infinite dimensional Hilbert space. For ρ, $\sigma \in \mathcal{S}(\mathcal{H})$, the following are equivalent.*

(a) $\rho \prec \sigma$.

(b) $\rho \in \overline{\mathrm{co}}U(\sigma)$.

(c) *There exists an unital CP-TP map Φ such that $\Phi(\sigma) = \rho$.*

(d) *There exist a sequence of mixed unitary channels $\{\Phi_n\}_{n=1}^{\infty}$ and an unital CP-TP map Φ such that*

$$\|\Phi_n(X) - \Phi(X)\|_1 \to 0 \text{ for all } X \in \mathfrak{C}_1(\mathcal{H}), \quad \Phi(\sigma) = \rho.$$

Using Theorem 2.4, we add a new characterization to Theorem 4.3 as follows.

Theorem 4.4. *(Asakura) For density matrices ρ, $\sigma \in \mathcal{S}(\mathcal{H})$, the conditions (a)~(e) are equivalent.*

(e) *There exist a Borel set I of a certain of metric space, a probability measure μ on I and a set of partial isometry operators $\{V_X\}_{X\in I}$ such that*

$$\rho = \int_I V_X \sigma V_X{}^* d\mu(X), \text{ in } \mathfrak{C}_1(H). \tag{12}$$

Proof. (a) $\Rightarrow$ (e). There exist two partial isometry V, W such that $V^*\rho V$ and $W\sigma W^*$ have same eigenbasis $(|i\rangle)_{i=1}^{\infty}$. Thus, by Theorem 2.4, letting $I := \mathcal{P}(\mathcal{H}^{(|i\rangle)})$ and $V_X := VXW$, the equality (12) holds.

(e) $\Rightarrow$ (a). From Weyl's eigenvalue theorem [15], we only have to show that $\mathrm{Tr}\,\rho P \geq \mathrm{Tr}\,\sigma P$ for any nonnegative operator P; see also [10, page 8]. Since $\mathrm{Tr}\,\rho P\sigma \geq \mathrm{Tr}\,\rho P V_X \sigma V_X{}^*$ for any $X \in I$, we have

$$\mathrm{Tr}\,\rho P \geq \int_I \mathrm{Tr}[V_X\sigma V_X{}^* P] d\mu(X) = \mathrm{Tr}[(\int_I V_X \sigma V_X{}^* d\mu(X))P] = \mathrm{Tr}\,\sigma P,$$

where several "interchanges" in the equalities are all legitimate from [17, V.5.]. □

Remark 4.5. *From this proof, for full-rank density matrices ρ, $\sigma \in \mathcal{S}(\mathcal{H})$, $\rho \prec \sigma$ if and only if there exist (I, μ) as above and a set of unitary operators $\{U_X\}_{X \in I}$ such that*

$$\rho = \int_I U_X \sigma U_X{}^* d\mu(X), \quad in\, \mathfrak{C}_1(H).$$

Note that this characterization is a natural generalization of $(a) \iff (C)$ in Theorem 4.1.

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