数理解析研究所講究録
第2035巻 2017年 1-9

CERTAIN SEMINORM AND NORM CONDITIONS ON MAPS BETWEEN FUNCTION SPACES

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1. Introduction

For a locally compact Hausdorff space X, let $C_0(X)$ be the Banach space of continuous complex-valued functions on X vanishing at infinity endowed with the supremum norm $\|\cdot\|_X$. A *uniform algebra* (function algebra) on X is a closed subalgebra of $C_0(X)$ which separates strongly the points of X, that is for distinct points $x, y \in X$, there exists $f \in A$ with $|f(x)| \neq |f(y)|$. In compact case, $C(X)$ denotes the Banach space of continuous complex-valued functions on X and all uniform algebras on X are assumed to contain constants.

Preserving maps (linear or not) between various Banach algebras have been studied extensively. For algebras of functions, the preserving condition mostly deals with the norm, spectrum and some parts of the spectrum. In most cases, under certain natural conditions, such maps are forced to be linear, multiplicative or weighted composition operators.

There is a vast literature on describing isometries between spaces of functions as weighted composition operators. Classical results concerning linear surjective isometries are Banach-Stone theorem and de Leeuw-Wermer-Rudin theorem, see the books [6, 7] for more results. Here, we refer to some of these extensions which are closely related to our results.

Extensions of Banach-Stone theorem concerning isometries between subspaces of continuous functions proved in, for instance, [2, 22] A Banach-Stone type theorem for function A-modules of the form Af proved in [3] shows that for uniform algebras A and B on compact Hausdorff spaces X, Y, respectively and strictly positive functions $f_1 \in C(X)$ and $f_2 \in C(Y)$ any surjective linear isometry $T : Af_1 \longrightarrow Bf_2$ can be described as

$$T(ff_1)(y) = (f \circ \varphi)(y) h f_2(y) \quad (f \in A, y \in Y)$$

[1] This note has been written based on the author's joint works with R.S. Ghodrat, A. Golbaharan and A. Jamshidi

where h is an invertible element of B and φ is a homeomorphism from Y onto a subset of the maximal ideal space $M(A)$ of A. Moreover, if $A = B$, then T is an A-module isometry if and only if φ is the identity map.

Some recent generalizations of Banach-Stone theorem are devoted to the case of not necessarily linear isometries. Real-linear isometries and multiplicatively norm preserving maps (as well as multiplicatively range preserving maps) between algebras of continuous complex-valued functions are examples of such maps. The study of multiplicatively range preserving maps between $C(X)$-spaces was initiated by Molnár in [21] and have been studied in different settings such as uniform algebras and multiplicative subsets of functions rather than $C(X)$, see [11, 13, 20, 24, 25]. Real-linear isometries between uniform algebras have been characterized in [19]. In general, such an isometry has a description as a weighted composition operator on some points and a conjugate weighted composition operator on the other points of the Choquet boundary. This result has been generalized in [18] for isometries between certain complex subspaces of $C_0(X)$ and $C_0(Y)$, where X and Y are locally compact Hausdorff spaces.

The main subject of this manuscript, is to investigate the extensions of some recent results concerning preserving maps between uniform algebras to function spaces with application to function modules of the form Af, where A is a uniform algebra and f is a strictly positive continuous function. First we state some results on real-linear isometries, norm-multiplicative and norm-additive in modulus maps between certain subspaces of continuous functions. The results can be applied for function modules of the form Af, whenever A is a certain uniform algebra. We also give a description of a (not necessarily linear) diameter preserving map between dense subspaces of continuous functions. Finally, we give a characterization of a bijective Hölder seminorm preserving map, not assumed to be linear, between little Lipschitz functions.

2. Preliminaries

As we noted before, for a compact Hausdorff space X, $C(X)$ is the Banach algebra of continuous complex-valued functions on X. Given an arbitrary Banach algebra $\mathcal{A}$, a *function $\mathcal{A}$-module* is a closed subspace $\mathcal{M}$ of $C(X)$, for some compact Hausdorff space X, which is closed under the multiplication induced by a unital homomorphism $\pi : \mathcal{A} \longrightarrow C(X)$, i.e. $\pi(a)f \in \mathcal{M}$ for all $a \in \mathcal{A}$ and $f \in \mathcal{M}$.

Typical examples of function modules are subspaces of the form Af where A is a uniform algebra on a compact Hausdorff spaces X and $f \in C(X)$ is strictly positive. More generally, subspaces of the form $\overline{Af_1 + Af_2 + \cdots + Af_n}$, where A is a uniform algebra on X and each f_i is a strictly positive continuous function on X, are all function A-modules.

Let X be a locally compact Hausdorff space and A be a subspace of $C_0(X)$. For $f \in A$, we set $M(f) = \{x \in A : |f(x)| = \|f\|_X\}$. A *boundary* for A is a subset $E \subseteq X$ containing at least one point of each $M(f)$, $f \in A$. The *Choquet boundary* of A is defined as the set of all points $x \in X$ such that the evaluation functional δ_x at x is an extreme point of the unit ball of A^*. Here, A^* is the dual space of $(A, \|\cdot\|_X)$.

A *strong boundary point* of A is a point $x \in X$ such that for each neighborhood U of x and $0 < \epsilon \leq 1$ there exists $f \in A$ with $f(x) = 1 = \|f\|_X$ and $|f| < \epsilon$ on $X \backslash U$. A closed subset E of X is called a *peak set* for A if there exists $f \in A$ such that $f|_E = 1 = \|f\|_X$ and $|f| < 1$ on $X \backslash E$. For a subspace A of $C_0(X)$, we denote the set of strong boundary points of A by $\Theta(A)$ and its Choquet boundary by $\mathrm{ch}(A)$. Then $\mathrm{ch}(A)$ is a boundary for A. It is well-known that in uniform algebra case, $\Theta(A) = \mathrm{ch}(A)$, however, when A is a subspace of $C_0(X)$ for some locally compact Hausdorff space X, we have $\Theta(A) \subseteq \mathrm{ch}(A)$, see [4, Theorem 2.2.1] for compact case and [14] for locally compact case.

If A is a uniform algebra on a compact Hausdorff space X, then by [3], for strictly positive $f \in C(X)$,

$$\mathrm{ch}(A) \subseteq \Theta(Af) \subseteq \mathrm{ch}(Af).$$

In particular, in the case that $\mathrm{ch}(A) = X$, we have $\Theta(Af) = \mathrm{ch}(Af)$.

3. Norm-multiplicative and norm-additive in modulus maps

In this section, we first recall some of our recently published results on real-linear isometries between certain function spaces. Then we give some results concerning maps whose domains are subspaces (rather than uniform algebras) satisfying some norm conditions.

We should note that in the study of nonlinear isometries, the additive version of Bishop's lemma has an important role. By this lemma, for each element f in a uniform algebra A and $x_0 \in \mathrm{ch}(A)$ with $f(x_0) = 0$ there exists $u \in A$ with $u(x_0) = 1 = \|u\|_X$ and $f + 2u(x_0) = 2 = \|f + 2u\|_X$. However, if A is a closed subspace of $C_0(X)$, then the same result holds for all $f \in A$ and points $x_0 \in \Theta(A)$, see [14].

Next theorem gives a description of a nonlinear, not necessarily surjective, isometry between subspaces of continuous functions.

Theorem 3.1. [14] *For locally compact Hausdorff spaces X and Y, a complex subspace A of $C_0(X)$ with $\Theta(A) \neq \emptyset$ and a real-linear isometry $T : A \longrightarrow C_0(Y)$ there exists a subset Y_0 of Y and continuous functions $\Phi : Y_0 \longrightarrow \Theta(\overline{A})$, $\alpha : Y_0 \longrightarrow [-1, 1]$ and $w : Y_0 \longrightarrow \mathbb{T}$, where Φ is surjective, such that*

$$Tf(y) = w(y) \cdot \Big(\mathrm{Re}(f(\Phi(y))) + \alpha(y) i \,\mathrm{Im}(f(\Phi(y)))\Big) \quad (f \in A, y \in Y_0).$$

If, furthermore, $\Theta(A) = \mathrm{ch(A)}$ and $T(A)$ is a complex subspace of $C_0(Y)$, then $\alpha(y) \in \{1, -1\}$ and Y_0 is a boundary for $T(A)$.

An improvement of the above result for the case that $T(A)$ satisfies a separation property, called $\mathbb{T}$-separating, is given in the next theorem. By a $\mathbb{T}$-separating real subspace of $C_0(X)$ we mean a real subspace B of $C_0(Y)$ such that for distinct points $x, x' \in X$ and scalars $\lambda, \lambda' \in \mathbb{T}$, there exists a function $g \in B$ such that $\|g\|_X = 1$, $g(x) = \lambda$, $g(x') = \lambda'$. Meanwhile, $\tau(A)$ stands for the points $x \in X$ such that for each neighborhood U of x there exists a function $f \in A$ with $f(x) = 1 = \|f\|_X$ and $|f| < 1$ on $X \backslash U$.

Theorem 3.2. [14] *Let X, Y be locally compact Hausdorff spaces, A be a complex subspace of $C_0(X)$ with $\tau(A) \cap \mathrm{ch}(A) \neq \emptyset$, B be a $\mathbb{T}$-separating real subspace of $C_0(Y)$ and $T : A \longrightarrow B$ be a surjective real-linear isometry. Then there exist a subset Y_1 of $\tau(B)$, a clopen subset K of Y_1 and continuous functions $\Phi : Y_1 \longrightarrow \tau(A) \cap \mathrm{ch}(A)$ and $w : Y_1 :\longrightarrow \mathbb{T}$, where Φ is bijective, such that*

$$(Tf)(y) = w(y) \begin{cases} f(\Phi(y)) & y \in K, \\ \overline{f(\Phi(y))} & y \in Y_1 \backslash K, \end{cases}$$

for each $f \in A$ and $y \in Y_1$.

Motivated by the above results, one may ask about the description of some (not necessarily linear) preserving maps on subspaces rather than algebras and, in particular, on function modules.

In the sequel we investigate some preserving maps on function spaces. We begin with norm-multiplicative and norm-additive in modulus maps. For subspaces A and B of $C_0(X)$ and $C_0(Y)$, where X and Y are locally compact Hausdorff spaces, we say that a map $T : A \longrightarrow B$ is *norm-multiplicative* if $\|TfTg\|_Y = \|fg\|_X$, for all $f, g \in A$, and *norm-additive in modulus* if $\| \, |Tf| + |Tg| \, \|_Y = \| \, |f| + |g| \, \|_X$ for all $f, g \in A$.

Norm-multiplicative mappings between uniform algebras, Banach subalgebras of $C_0(X)$ and, in general, multiplicative subsets of $C_0(X)$ have been investigated, for instance in [12, 20, 26]. However, the known results in this context cannot be applied for function modules of the form Af.

We state our results concerning norm-multiplicative and norm-additive in modulus conditions, for maps between good subspaces defined below, with applications to certain function modules.

Definition 3.3. *Let X be a compact Hausdorff space. We call a subspace A of $C(X)$ a* good subspace *if one of the following conditions holds:*

(i) A contains constants, or

(ii) A is a subalgebra of $C(X)$, or

(iii) $\mathrm{ch}(A)$ *is a closed subset of X.*

For simplicity we set $\Phi_+(\lambda,\mu)=\lambda\mu$ and $\Phi_\times(\lambda,\mu)=|\lambda|+|\mu|$, for $\lambda,\mu\in\mathbb{C}$ and in the sequel we assume that either $\Phi=\Phi_+$ or $\Phi=\Phi_\times$.

Theorem 3.4. [9] *Let X,Y be compact Hausdorff spaces, A,B be good subspaces of $C(X)$ and $C(Y)$, not assumed to be closed, with $\mathrm{ch}(A)=\Theta(A)$ and $\mathrm{ch}(B)=\Theta(B)$. Then for every surjective map $T:A\longrightarrow B$ satisfying the norm condition*

$$\|\Phi((Tf)^s,(Tg)^t)\|_Y=\|\Phi(f^s,g^t)\|_X \quad (f,g\in A)$$

where, $s,t\in\mathbb{N}$, there exists a homeomorphism $\varphi:\mathrm{ch}(B)\longrightarrow\mathrm{ch}(A)$ such that $|Tf(y_0)|=|f(\varphi(y_0))|$ for all $f\in A$ and $y_0\in\mathrm{ch}(B)$.

We should note that similar results for locally compact case can be easily deduced from compact case, when A and B are either subalgebras of $C_0(X)$ and $C_0(Y)$ for some locally compact Hausdorff spaces X and Y or their Choquet boundaries are compact.

Remark. (i) In our results we do not assume that the subspaces A and B are closed in $C_0(X)$ and $C_0(Y)$, or complete under some norm.

(ii) The result for the case $\Phi=\Phi_+$ improves [26] (stated for $s=t=1$) for subspaces and without assuming that T is $\mathbb{R}^+$-homogeneous.

As the next corollary shows, the above results can be applied for certain function modules of the form Af.

Corollary 3.5. *Let $A_1,\ldots,A_n$ be uniform algebras on a compact Hausdorff space X with $\cup_{i=1}^n\mathrm{ch}(A_i)=X$. Let $s,t\in\mathbb{N}$, $f_1,\ldots,f_n$ be strictly positive functions in $C(X)$ and $B=A_1f_1+\cdots+A_nf_n$. Then for each surjective map $T:B\longrightarrow B$ satisfying $\|\Phi((Tf)^s,(Tg)^t)\|_X=\|\Phi(f^s,g^t)\|_X$ for all $f,g\in B$, where $\Phi=\Phi_+$ or $\Phi_\times$, there exists a homeomorphism $\varphi:X\longrightarrow X$ such that $|Tf(y)|=|f(\varphi(y))|$ for all $f\in B$ and $y\in X$.*

Now in the next theorem, as in uniform algebra case, we improve the above result under a certain peripheral range condition. We recall that for a compact Hausdorff space X and $f\in C(X)$, the peripheral range of f is defined as $R_\pi(f)=\{\lambda\in f(X):|\lambda|=\|f\|_X\}$.

Theorem 3.6. [9] *Under the assumptions of the above theorem, if A is closed and contains constants, then for any surjective map $T:A\longrightarrow B$ satisfying $R_\pi((Tf)^s(Tg)^t)\subseteq R_\pi(f^sg^t)$, $f,g\in A$, where $s,t\in\mathbb{N}$, there exist a homeomorphism $\varphi:\mathrm{ch}(B)\longrightarrow\mathrm{ch}(A)$ and a continuous function $\gamma:\mathrm{ch}(B)\longrightarrow\mathbb{T}$ such that*

$$Tf^d(y)=\gamma(y)f^d(\varphi(y)) \quad (f\in A, y\in\mathrm{ch}(B))$$

where d is the greatest common divisor of s and t.

Corollary 3.7. *Let X be a compact Hausdorff space and A be a closed subspace of $C(X)$ containing constants with $\Theta(A)=\mathrm{ch}(A)$. Then for any surjective map T on A satisfying*

$(fg)(X) = (TfTg)(X)$ for all $f, g \in A$ there exist a homeomorphism $\varphi : \mathrm{ch}(B) \longrightarrow \mathrm{ch}(A)$ and a continuous function $\gamma : \mathrm{ch}(B) \longrightarrow \mathbb{T}$ such that

$$Tf(y) = \gamma(y)f(\varphi(y)) \quad (f \in A, y \in \mathrm{ch}(B)).$$

4. Diameter seminorm and Hölder seminorm preserving maps

In this section we give a description of (not necessarily linear) diameter seminorm preserving maps between dense subspaces of continuous functions and Hölder seminorm preserving maps between little Lipschitz algebras.

For a compact Hausdorff spaces X and $f \in C(X)$ by $\mathrm{diam}(f)$ we mean the diameter of $f(X)$. Given compact Hausdorff spaces X and Y and subspaces A and B of $C(X)$ and $C(Y)$, respectively containing constants, let $T : A \longrightarrow B$ be a surjective (not assumed to be linear) diameter preserving map in the following sense

$$\mathrm{diam}(Tf - Tg) = \mathrm{diam}(f - g) \quad (f, g \in A).$$

For each $f \in C(X)$ let $\widetilde{f} \in C(X \times X)$ be defined by $\tilde{f}(x, y) = f(x) - f(y)$, $x, y \in X$. Then clearly T induces a surjective sup-norm isometry $\widetilde{T} : \widetilde{A} \longrightarrow \widetilde{B}$ between subspaces $\widetilde{A} = \{\widetilde{f} : f \in A\}$ and $\widetilde{B} = \{\widetilde{g} : g \in B\}$ of $C(X \times X)$ and $C(Y \times Y)$, respectively. Hence by the Mazur-Ulam theorem $\widetilde{T}$ is real-linear.

There are many interesting results concerning linear diameter preserving maps, see for instance [1, 5, 8, 10, 23]. However, motivated by the Mazur-Ulam theorem, here we consider nonlinear case. Although such a preserving map induces a real-linear isometries between subspaces of $C(X \times X)$ and $C(Y \times Y)$ but, because of some required separating properties, the previous known results cannot be applied for this case. The following diameter versions of additive Bishop's Lemma have important role in our proofs.

For a compact Hausdorff space X and distinct points $x, x' \in X$, we fix the following notation

$$V_{x,x'}(A) = \{u \in A : u(x) - u(x') = 1 = \mathrm{diam}(u)\}.$$

Similarly, for each $\epsilon > 0$ and distinct points $x, x' \in X$, we consider the following subset of A

$$V^{\epsilon}_{x,x'}(A) = \{u \in A : |u(x) - u(x') - 1| \leq \epsilon, |\mathrm{diam}(u) - 1| \leq \epsilon\}.$$

Clearly, if for distinct points $x, x' \in X$ there exists a function $f \in A$ such that $f(x) = 1$, $f(x') = 0$ and $0 \leq f \leq 1$, then $V_{x,x'} \neq \emptyset$.

Lemma 4.1. [15] *Let X be a compact Hausdorff space and let $x_1, x_2 \in X$ be distinct. If $f \in C(X)$ such that $0 \leq f \leq 1$ and $f(x_1) = f(x_2)$, then*

(i) there exists a function $g \in V_{x_2,x_1}(C(X))$ such that $f + 2g \in 2V_{x_2,x_1}(C(X))$,

(ii) for each $\epsilon > 0$ there exists a function $g \in V_{x_2,x_1}(C(X))$ such that $\mathrm{diam}(if + 2g) \leq 2 + \epsilon$.

Lemma 4.2. [15] *Let $x_1, x_2 \in X$ be distinct, and $f \in A$ such that $\|f\|_X = 1$ and $f(x_1) = f(x_2)$. Then for each $0 < \epsilon < 1$ there exist four elements $f'_1, ..., f'_4 \in A$ such that $f'_1, f'_2, if'_3, if'_4 \in 2(V^{\epsilon}_{x_1,x_2}(A) - V^{\epsilon}_{x_1,x_2}(A))$ and $\|f - (f'_1 - f'_2 + if'_3 - if'_4)\|_X \leq \epsilon$.*

Next theorem characterizes nonlinear diameter preserving maps.

Theorem 4.3. [15] *Let X, Y be compact Hausdorff spaces and A, B be complex subspaces of $C(X)$ and $C(Y)$, respectively containing the constants such that A is dense in $C(X)$, B is point separating and the evaluation functionals are linearly independent on B. Let $T : A \longrightarrow B$ be a surjective, not necessarily linear, diameter-preserving map and $T_1 = T - T0$. Then there exist a subset Y_0 of Y and a continuous bijective map $\Psi : Y_0 \longrightarrow X$ such that for any fixed point $y_1 \in Y_0$, either*

$$T_1 f(y) - T_1 f(y_1) = \beta\big(f(\Psi(y)) - f(\Psi(y_1))\big) \quad (f \in A, y \in Y_0 \backslash \{y_1\}),$$

or

$$T_1 f(y) - T_1 f(y_1) = \beta\big(\overline{f(\Psi(y))} - \overline{f(\Psi(y_1))}\big) \quad (f \in A, y \in Y_0 \backslash \{y_1\})$$

holds for some $\beta \in \mathbb{T}$.

For a compact metric space (X, d) and $\alpha \in (0,1)$, let $\mathrm{lip}^{\alpha}(X)$ be the space of little Lipschitz functions of order α on X and for each $f \in \mathrm{lip}^{\alpha}(X)$, $L_{\alpha}(f)$ be its Hölder seminorm defined by $L_{\alpha}(f) := \sup_{x \neq y} \frac{|f(x)-f(y)|}{d^{\alpha}(x,y)}$. For compact metric spaces X, Y and $0 < \alpha < 1$, we say that an arbitrary map $T : \mathrm{lip}^{\alpha}(X) \longrightarrow \mathrm{lip}^{\alpha}(Y)$ is Hölder seminorm preserving if $L_{\alpha}(Tf - Tg) = L_{\alpha}(f - g)$ for all $f, g \in \mathrm{lip}^{\alpha}(X)$.

Linear Hölder seminorm preserving maps between little Lipschitz spaces were characterized in [17] and it was shown that for compact metric spaces X and Y and $\alpha \in (0,1)$, any linear bijective Hölder seminorm preserving map $T : \mathrm{lip}^{\alpha}(X) \longrightarrow \mathrm{lip}^{\alpha}(Y)$ is of the form $Tf(y) = (\tau/k^{\alpha}) f \circ \phi + \mu(f)$, $f \in \mathrm{lip}^{\alpha}(X)$, where $\tau = e^{i\theta}$ for some $\theta \in [0, \pi)$, $\phi : Y \longrightarrow X$ is a bi-Lipschitz map such that $\frac{d(\phi(x),\phi(y))}{d(x,y)} = k$ for all $x, y \in Y$, and μ is a linear functional on $\mathrm{lip}^{\alpha}(X)$. We consider the arbitrary case that T is not assumed to be complex-linear. In the lack of linearity assumption, again we need to prove a Hölder seminorm version of Bishop's lemma (Theorem 4.4).

For distinct points $x, x' \in X$, and distinct points $y, y' \in Y$, we set

$$V_{x,x'} = \{f \in \mathrm{lip}^{\alpha}(X) : \frac{f(x) - f(x')}{d^{\alpha}(x, x')} = 1 = L_{\alpha}(f)\},$$

$$W_{y,y'} = \{g \in \mathrm{lip}^{\alpha}(Y) : \frac{g(y) - g(y')}{d^{\alpha}(y, y')} = 1 = L_{\alpha}(g)\}.$$

We should note that the above defined sets are non-empty. For instance, for each $\beta \in (\alpha, 1)$ and distinct points $x, x' \in X$ the function $g(z) = \frac{d^\beta(z,x) - d^\beta(z,x')}{2d^{\beta-\alpha}(x,x')}$, $z \in X$, is an element of $V_{x,x'}$ (see [17, Page 57]).

Theorem 4.4. [16] *Let $x_1, x_2 \in X$ be distinct and $f \in \mathrm{lip}^\beta(X)$ for some $\beta \in (\alpha, 1)$. If $f(x_1) = f(x_2)$, then there exists $g \in V_{x_1,x_2}$ such that $sf + g \in V_{x_1,x_2}$ for some $s > 0$.*

Theorem 4.5. [16] *Let X and Y be compact metric spaces, $\alpha \in (0,1)$ and $T : \mathrm{lip}^\alpha(X) \longrightarrow \mathrm{lip}^\alpha(Y)$ be a bijective, not assumed to be linear, Hölder seminorm preserving map. Then there exist a bijective Lipschitz map $\Psi : Y \longrightarrow X$, scalars $K_0 > 0$ and $\tau \in U^+$ and a function $\Lambda : \mathrm{lip}^\alpha(X) \longrightarrow \mathbb{C}$ (which is linear resp. real-linear if T is so) such that $\frac{d^\alpha(y,z)}{d^\alpha(\Psi(y),\Psi(z))} = K_0$, for all $y, z \in Y$, and either*

$$T_1 f(y) = \overline{\tau} K_0\, f(\Psi(y)) + \Lambda(f) \qquad (f \in \mathrm{lip}^\alpha(X), y \in Y)$$

or

$$T_1 f(y) = \overline{\tau} K_0\, \overline{f(\Psi(y))} + \Lambda(f) \qquad (f \in \mathrm{lip}^\alpha(X), y \in Y),$$

where $T_1 = T - T0$.

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