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Rationality problem for fields of invariants: some examples

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In this article, we give a short survey of some known results and our recent results in [HKK13], [CHHK15], [Hos16], [HKY16] (with Huah Chu, Shou-Jen Hu, Ming-chang Kang, Boris E. Kunyavskii and Aiichi Yamasaki) about rationality problem for fields of invariants and unramified Brauer (cohomology) groups.

1 Introduction

Let k be a field and G be a finite group acting on the rational function field $k(x_g : g \in G)$ by k-automorphisms $h(x_g) = x_{hg}$ for any $g, h \in G$. We denote the fixed field $k(x_g : g \in G)^G$ by $k(G)$. Emmy Noether [Noe13, Noe17] asked whether $k(G)$ is rational (= purely transcendental) over k. This is called Noether's problem for G over k, and is related to the inverse Galois problem, to the existence of generic G-Galois extensions over k, and to the existence of versal G-torsors over k-rational field extensions (see Saltman [Sal82a], Swan [Swa83], Manin and Tsfasman [MT86], Garibaldi, Merkurjev and Serre [GMS03, Section 33.1, page 86]), Colliot-Thélène and Sansuc [CTS07]).

Theorem 1.1 (Fischer [Fis15], see also Swan [Swa83, Theorem 6.1]). *Let G be a finite abelian group with exponent e. Assume that* (i) *either* char $k = 0$ *or* char $k = p \nmid e$, *and* (ii) *k contains a primitive e-th root of unity. Then $k(G)$ is k-rational. In particular, $\mathbb{C}(G)$ is $\mathbb{C}$-rational.*

Theorem 1.2 (Kuniyoshi [Kun54], [Kun55], [Kun56], see also Gaschütz [Gas59]). *Let k be a field with* char $k = p > 0$ *and G be a finite p-group. Then $k(G)$ is k-rational.*

We now recall some relevant definitions of k-rationality of fields.

Definition 1.3. Let K/k and L/k be finitely generated extensions of fields.
(1) K is said to be *rational* over k (for short, *k-rational*) if K is purely transcendental over k, i.e. $K \simeq k(x_1, \ldots, x_n)$ for some algebraically independent elements $x_1, \ldots, x_n$ over k;
(2) K is said to be *stably k-rational* if $K(y_1, \ldots, y_m)$ is k-rational for some algebraically independent elements $y_1, \ldots, y_m$ over K;
(3) K and L are said to be *stably k-isomorphic* if $K(y_1, \ldots, y_m) \simeq L(z_1, \ldots, z_n)$ for some algebraically independent elements $y_1, \ldots, y_m$ over K and $z_1, \ldots, z_n$ over L;
(4) (Saltman, [Sal84b, Definition 3.1]) K is said to be *retract k-rational* if there exists a k-algebra A contained in K such that (i) K is the quotient field of A, (ii) there exist a non-zero polynomial $f \in k[x_1, \ldots, x_n]$ and k-algebra homomorphisms $\varphi\colon A \to k[x_1, \ldots, x_n][1/f]$ and $\psi\colon k[x_1, \ldots, x_n][1/f] \to A$ satisfying $\psi \circ \varphi = 1_A$;
(5) K is said to be *k-unirational* if $k \subset K \subset k(x_1, \ldots, x_n)$ for some integer n.

In Saltman's original definition of retract k-rationality ([Sal82b, page 130], [Sal84b, Definition 3.1]), a base field k is required to be infinite in order to guarantee the existence of sufficiently many k-specializations. We now assume that k is an infinite field. Then if K and L are stably k-isomorphic and K is retract k-rational, then L is also retract k-rational (see [Sal84b, Proposition 3.6]), and it is not difficult to verify the following implications:

$$k\text{-rational} \Rightarrow \text{stably } k\text{-rational} \Rightarrow \text{retract } k\text{-rational} \Rightarrow k\text{-unirational}.$$

Note that $k(G)$ is retract k-rational if and only if there exists a generic G-Galois extension over k (see [Sal82a, Theorem 5.3], [Sal84b, Theorem 3.12]). In particular, if k is a Hilbertian field, e.g. number field, and $k(G)$ is retract k-rational, then inverse Galois problem for G over k has a positive answer, i.e. there exists a Galois extension K/k with $\mathrm{Gal}(K/k) \simeq G$.

Swan [Swa69] gave the first negative solution to Noether's problem. He proved that if $p = 47$, 113 or 233, then $\mathbb{Q}(C_p)$ is not $\mathbb{Q}$-rational, where C_p is the cyclic group of order prime p, by using Masuda's idea of Galois descent [Mas55, Mas68].

Noether's problem for abelian groups was studied extensively by Masuda, Kuniyoshi, Swan, Voskresenskii, Endo and Miyata, etc. Eventually, Lenstra [Len74] gave a necessary and sufficient condition to Noether's problem for finite abelian groups. For details, see Swan's survey paper [Swa83], Voskresenskii's book [Vos98, Section 7] or [Hos15]. On the other hand, just a handful of results about Noether's problem are obtained when the groups are non-abelian.

Theorem 1.4 (Maeda [Mae89, Theorem, page 418]). *Let k be a field and A_5 be the alternating group of degree 5. Then $k(A_5)$ is k-rational.*

Theorem 1.5 (Serre [GMS03, Chapter IX], see also Kang [Kan05]). *Let G be a finite group with a 2-Sylow subgroup which is cyclic of order ≥ 8 or the generalized quaternion Q_{16} of order 16. Then $\mathbb{Q}(G)$ is not stably $\mathbb{Q}$-rational.*

Theorem 1.6 (Plans [Pla09, Theorem 2]). *Let A_n be the alternating group of degree n. If $n \geq 3$ is odd integer, then $\mathbb{Q}(A_n)$ is rational over $\mathbb{Q}(A_{n-1})$. In particular, if $\mathbb{Q}(A_{n-1})$ is $\mathbb{Q}$-rational, then so is $\mathbb{Q}(A_n)$.*

However, it is an open problem whether $k(A_n)$ is k-rational for $n \geq 6$.

From now on, we restrict ourselves to the case where G is a p-group. By Theorem 1.1 and Theorem 1.2, we may focus on the case where G is a non-abelian p-group and k is a field with char $k \neq p$. For p-groups of small order, the following results are known.

Theorem 1.7 (Chu and Kang [CK01]). *Let p be any prime and G be a p-group of order $\leq p^4$ and of exponent e. If k is a field containing a primitive e-th root of unity, then $k(G)$ is k-rational.*

Theorem 1.8 (Chu, Hu, Kang and Prokhorov [CHKP08]). *Let G be a group of order 32 and of exponent e. If k is a field containing a primitive e-th root of unity, then $k(G)$ is k-rational.*

For more recent results, see e.g. [HK10], [Kan11], [KMZ12].

Saltman introduced a notion of retract k-rationality (see Definition 1.3) and the unramified Brauer group. Recall that the implications for an infinite field k: k-rational $\Rightarrow$ stably k-rational $\Rightarrow$ retract k-rational. Hence if $k(G)$ is not retract k-rational, then it is not k-rational.

Definition 1.9 (Saltman [Sal84a, Definition 3.1], [Sal85, page 56]). Let K/k be an extension of fields. The *unramified Brauer group* $\mathrm{Br}_{\mathrm{nr}}(K/k)$ of K over k is defined to be

$$\mathrm{Br}_{\mathrm{nr}}(K/k) = \bigcap_R \mathrm{Image}\{\mathrm{Br}(R) \to \mathrm{Br}(K)\}$$

where $\mathrm{Br}(R) \to \mathrm{Br}(K)$ is the natural map of Brauer groups and R runs over all the discrete valuation rings R such that $k \subset R \subset K$ and K is the quotient field of R. We omit k from the notation and write just $\mathrm{Br}_{\mathrm{nr}}(K)$ when the base field k is clear from the context.

Proposition 1.10 (Saltman [Sal84a], [Sal85, Proposition 1.8], [Sal87]). *If K is retract k-rational, then $\mathrm{Br}(k) \xrightarrow{\sim} \mathrm{Br}_{\mathrm{nr}}(K)$. In particular, if k is an algebraically closed field and K is retract k-rational, then $\mathrm{Br}_{\mathrm{nr}}(K) = 0$.*

Theorem 1.11 (Bogomolov [Bog88, Theorem 3.1], Saltman [Sal90, Theorem 12]). *Let G be a finite group and k be an algebraically closed field with $\mathrm{char}\, k = 0$ or $\mathrm{char}\, k = p \nmid |G|$. Then $\mathrm{Br}_{\mathrm{nr}}(k(G)/k)$ is isomorphic to the group $B_0(G)$ defined by*

$$B_0(G) = \bigcap_A \mathrm{Ker}\{\mathrm{res} : H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(A, \mathbb{Q}/\mathbb{Z})\}$$

where A runs over all the bicyclic subgroups of G (a group A is called bicyclic if A is either a cyclic group or a direct product of two cyclic groups).

Remark 1.12. For a smooth projective variety X over $\mathbb{C}$ with function field K, $\mathrm{Br}_{\mathrm{nr}}(K/\mathbb{C})$ is isomorphic to the birational invariant $H^3(X, \mathbb{Z})_{\mathrm{tors}}$ which was used by Artin and Mumford [AM72] to provide some elementary examples of k-unirational varieties which are not k-rational (see also [Bog88, Theorem 1.1 and Corollary]).

Following Kunyavskii [Kun10], we call $B_0(G)$ the Bogomolov multiplier of G. Note that $B_0(G)$ is a subgroup of $H^2(G, \mathbb{Q}/\mathbb{Z})$ which is isomorphic to the Schur multiplier $H_2(G, \mathbb{Z})$ of G (see Karpilovsky [Kar87]). Because of Theorem 1.11, we will not distinguish $B_0(G)$ and $\mathrm{Br}_{\mathrm{nr}}(k(G)/k)$ when k is an algebraically closed field, and $\mathrm{char}\, k = 0$ or $\mathrm{char}\, k = p \nmid |G|$.

Using the Bogomolov multiplier $B_0(G)$, Saltman and Bogomolov gave counter-examples to Noether's problem for non-abelian p-groups over algebraically closed field.

Theorem 1.13 (Saltman [Sal84a], Bogomolov [Bog88]). *Let p be any prime and k be any algebraically closed field with $\mathrm{char}\, k \neq p$.*
(1) (Saltman [Sal84a, Theorem 3.6]) *There exists a meta-abelian group G of order p^9 such that $B_0(G) \neq 0$. In particular, $k(G)$ is not (retract, stably) k-rational;*
(2) (Bogomolov [Bog88, Lemma 5.6]) *There exists a group G of order p^6 such that $B_0(G) \neq 0$. In particular, $k(G)$ is not (retract, stably) k-rational.*

Colliot-Thélène and Ojanguren [CTO89] generalized the notion of the unramified Brauer group $\mathrm{Br}_{\mathrm{nr}}(K/k)$ to the unramified cohomology $H^i_{\mathrm{nr}}(K/k, \mu_n^{\otimes j})$ of degree $i \geq 1$, that is $F_n^{i,j}(K/k)$ in [CTO89, Definition 1.1].

Definition 1.14 (Colliot-Thélène and Ojanguren [CTO89], see also [CT95, Sections 2–4]). Let n be a positive integer and k be an algebraically closed field with $\mathrm{char}\, k = 0$ or $\mathrm{char}\, k = p$

$\not| n$. Let K/k be a function field, that is finitely generated as a field over k. The *unramified cohomology group* $H^i_{\mathrm{nr}}(K/k, \mu_n^{\otimes j})$ of K over k of degree $i \geq 1$ is defined to be

$$H^i_{\mathrm{nr}}(K/k, \mu_n^{\otimes j}) = \bigcap_R \mathrm{Image}\{H^i_{\text{ét}}(R, \mu_n^{\otimes j}) \to H^i_{\text{ét}}(K, \mu_n^{\otimes j})\}$$

where R runs over all the discrete valuation rings R of rank one such that $k \subset R \subset K$ and K is the quotient field of R. We write just $H^i_{\mathrm{nr}}(K, \mu_n^{\otimes j})$ when the base field k is clear.

Note that the unramified cohomology groups of degree two are isomorphic to the n-torsion part of the unramified Brauer group: ${}_n\mathrm{Br}_{\mathrm{nr}}(K/k) \simeq H^2_{\mathrm{nr}}(K/k, \mu_n)$.

Proposition 1.15. *Let k be an algebraically closed field with* char $k = 0$ *or* char $k = p \not| n$.
(1) (Colliot-Thélène and Ojanguren [CTO89, Proposition 1.2]) *If K and L are stably k-isomorphic, then $H^i_{\mathrm{nr}}(K/k, \mu_n^{\otimes j}) \xrightarrow{\sim} H^i_{\mathrm{nr}}(L/k, \mu_n^{\otimes j})$. In particular, K is stably k-rational, then $H^i_{\mathrm{nr}}(K/k, \mu_n^{\otimes j}) = 0$;*
(2) ([Mer08, Proposition 2.15], see also [CTO89, Remarque 1.2.2], [CT95, Sections 2–4], [GS10, Example 5.9]) *If K is retract k-rational, then $H^i_{\mathrm{nr}}(K/k, \mu_n^{\otimes j}) = 0$.*

Colliot-Thélène and Ojanguren [CTO89, Section 3] produced the first example of not stably $\mathbb{C}$-rational but $\mathbb{C}$-unirational field K with $H^3_{\mathrm{nr}}(K, \mu_2^{\otimes 3}) \neq 0$, where K is the function field of a quadric of the type $\langle\langle f_1, f_2\rangle\rangle = \langle g_1 g_2\rangle$ over the rational function field $\mathbb{C}(x, y, z)$ with three variables x, y, z for a 2-fold Pfister form $\langle\langle f_1, f_2\rangle\rangle$, as a generalization of Artin and Mumford [AM72]. Peyre [Pey93, Corollary 3] gave a sufficient condition for $H^i_{\mathrm{nr}}(K/k, \mu_p^{\otimes i}) \neq 0$ and produced an example of the function field K with $H^3_{\mathrm{nr}}(K/k, \mu_p^{\otimes 3}) \neq 0$ and $\mathrm{Br}_{\mathrm{nr}}(K/k) = 0$ using a result of Suslin [Sus91] where K is the function field of a product of some norm varieties associated to cyclic central simple algebras of degree p (see [Pey93, Proposition 7]). Using a result of Jacob and Rost [JR89], Peyre [Pey93, Proposition 9] also gave an example of $H^4_{\mathrm{nr}}(K/k, \mu_2^{\otimes 4}) \neq 0$ and $\mathrm{Br}_{\mathrm{nr}}(K/k) = 0$ where K is the function field of a product of quadrics associated to a 4-fold Pfister form $\langle\langle a_1, a_2, a_3, a_4\rangle\rangle$ (see also [CT95, Section 4.2]).

Take the direct limit with respect to n:

$$H^i(K/k, \mathbb{Q}/\mathbb{Z}(j)) = \varinjlim_n H^i(K/k, \mu_n^{\otimes j})$$

and we also define the unramified cohomology group

$$H^i_{\mathrm{nr}}(K/k, \mathbb{Q}/\mathbb{Z}(j)) = \bigcap_R \mathrm{Image}\{H^i_{\text{ét}}(R, \mathbb{Q}/\mathbb{Z}(j)) \to H^i_{\text{ét}}(K, \mathbb{Q}/\mathbb{Z}(j))\}.$$

Then we have $\mathrm{Br}_{\mathrm{nr}}(K/k) \simeq H^2_{\mathrm{nr}}(K/k, \mathbb{Q}/\mathbb{Z}(1))$.

Peyre [Pey08] was able to construct an example of a field K, as $K = \mathbb{C}(G)$, whose unramified Brauer group vanishes, but unramified cohomology of degree three does not vanish:

Theorem 1.16 (Peyre [Pey08, Theorem 3]). *Let p be any odd prime. Then there exists a p-group G of order p^{12} such that $B_0(G) = 0$ and $H^3_{\mathrm{nr}}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0$. In particular, $\mathbb{C}(G)$ is not (retract, stably) $\mathbb{C}$-rational.*

The idea of Peyre's proof is to find a subgroup K^3_{max}/K^3 of $H^3_{\mathrm{nr}}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$ and to show that $K^3_{\mathrm{max}}/K^3 \neq 0$ (see [Pey08, page 210]).

Asok [Aso13] generalized Peyre's argument [Pey93] and established the following theorem for a smooth proper model X (resp. a smooth projective model Y) of the function field of a product of quadrics of the type $\langle\langle s_1, \ldots, s_{n-1}\rangle\rangle = \langle s_n\rangle$ (resp. Rost varieties) over some rational function field over $\mathbb{C}$ with many variables.

Theorem 1.17 (Asok [Aso13], see also [AM11, Theorem 3] for retract $\mathbb{C}$-rationality).
(1) ([Aso13, Theorem 1]) *For any $n > 0$, there exists a smooth projective complex variety X that is $\mathbb{C}$-unirational, for which $H^i_{\mathrm{nr}}(\mathbb{C}(X), \mu_2^{\otimes i}) = 0$ for each $i < n$, yet $H^n_{\mathrm{nr}}(\mathbb{C}(X), \mu_2^{\otimes n}) \neq 0$, and so X is not $\mathbb{A}^1$-connected, nor (retract, stably) $\mathbb{C}$-rational;*
(2) ([Aso13, Theorem 3]) *For any prime l and any $n \geq 2$, there exists a smooth projective rationally connected complex variety Y such that $H^n_{\mathrm{nr}}(\mathbb{C}(Y), \mu_l^{\otimes n}) \neq 0$. In particular, Y is not $\mathbb{A}^1$-connected, nor (retract, stably) $\mathbb{C}$-rational.*

Namely, the triviality of the unramified Brauer group or the unramified cohomology of higher degree is just a necessary condition of $\mathbb{C}$-rationality of fields. It is unknown whether the vanishing of all the unramified cohomologies is a sufficient condition for $\mathbb{C}$-rationality. It is interesting to consider an analog of Theorem 1.17 for quotient varieties V/G, e.g. $\mathbb{C}(V_{\mathrm{reg}}/G) = \mathbb{C}(G)$.

Colliot-Thélène and Voisin [CTV12] established:

Theorem 1.18 (Colliot-Thélène and Voisin [CTV12], see also [Voi14, Theorem 6.18]). *For any smooth projective complex variety X, there is an exact sequence*

$$0 \to H^3_{\mathrm{nr}}(X, \mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} \to H^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Tors}(Z^4(X)) \to 0$$

where

$$Z^4(X) = \mathrm{Hdg}^4(X, \mathbb{Z})/\mathrm{Hdg}^4(X, \mathbb{Z})_{\mathrm{alg}}$$

and the lower index "alg" means that we consider the group of integral Hodge classes which are algebraic. In particular, if X is rationally connected, then we have

$$H^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}) \simeq Z^4(X).$$

Using Peyre's method [Pey08], we obtain the following theorem which is an improvement of Theorem 1.16 and gives an explicit counter-example to integral Hodge conjecture with the aid of Theorem 1.18.

Theorem 1.19 (Hoshi, Kang and Yamasaki [HKY16, Theorem 1.4]). *Let p be any odd prime. Then there exists a p-group G of order p^9 such that $B_0(G) = 0$ and $H^3_{\mathrm{nr}}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0$. In particular, $\mathbb{C}(G)$ is not (retract, stably) $\mathbb{C}$-rational.*

1.1 The case where G is a group of order p^5 ($p \geq 3$)

From Theorem 1.13 (2), Bogomolov [Bog88, Remark 1] raised a question to classify the groups of order p^6 with $B_0(G) \neq 0$. He also claimed that if G is a p-group of order $\leq p^5$, then $B_0(G) = 0$ ([Bog88, Lemma 5.6]). However, this claim was disproved by Moravec:

Theorem 1.20 (Moravec [Mor12, Section 8]). *Let G be a group of order 243. Then $B_0(G) \neq 0$ if and only if $G = G(3^5, i)$ with $28 \leq i \leq 30$, where $G(3^5, i)$ is the i-th group of order 243 in the GAP database* [GAP]. *Moreover, if $B_0(G) \neq 0$, then $B_0(G) \simeq C_3$.*

Moravec [Mor12] gave a formula for $B_0(G)$ by using a nonabelian exterior square $G \wedge G$ of G and an implemented algorithm **b0g.g** in computer algebra system GAP [GAP], which is available from his website `www.fmf.uni-lj.si/~moravec/b0g.g`. The number of all solvable groups G of order ≤ 729 apart from the orders 512, 576 and 640 with $B_0(G) \neq 0$ was given as in [Mor12, Table 1].

Hoshi, Kang and Kunyavskii [HKK13] determined p-groups G of order p^5 with $B_0(G) \neq 0$ for any $p \geq 3$. It turns out that they belong to the same isoclinism family.

Definition 1.21 (Hall [Hal40, page 133]). Let G be a finite group. Let $Z(G)$ be the center of G and $[G, G]$ be the commutator subgroup of G. Two p-groups G_1 and G_2 are called *isoclinic* if there exist group isomorphisms $\theta\colon G_1/Z(G_1) \to G_2/Z(G_2)$ and $\phi\colon [G_1, G_1] \to [G_2, G_2]$ such that $\phi([g, h]) = [g', h']$ for any $g, h \in G_1$ with $g' \in \theta(gZ(G_1))$, $h' \in \theta(hZ(G_1))$:

$$\begin{tikzcd} G_1/Z_1 \times G_1/Z_1 \arrow[r, "{(\theta,\theta)}"] \arrow[d, "{[\cdot,\cdot]}"'] \arrow[dr, phantom, "\circlearrowleft"] & G_2/Z_2 \times G_2/Z_2 \arrow[d, "{[\cdot,\cdot]}"'] \\ {[G_1, G_1]} \arrow[r, "\phi"] & {[G_2, G_2]}. \end{tikzcd}$$

For a prime p and an integer n, we denote by $G_n(p)$ the set of all non-isomorphic groups of order p^n. In $G_n(p)$, consider an equivalence relation: two groups G_1 and G_2 are equivalent if and only if they are isoclinic. Each equivalence class of $G_n(p)$ is called an *isoclinism family*, and the j-th isoclinism family is denoted by Φ_j.

For $p \geq 5$ (resp. $p = 3$), there exist $2p + 61 + \gcd\{4, p-1\} + 2\gcd\{3, p-1\}$ (resp. 67) groups G of order p^5 which are classified into ten isoclinism families $\Phi_1, \ldots, \Phi_{10}$ (see [Jam80, Section 4]). The main theorem of [HKK13] can be stated as follows:

Theorem 1.22 (Hoshi, Kang and Kunyavskii [HKK13, Theorem 1.12], [Kan14, page 424]). *Let p be any odd prime and G be a group of order p^5. Then $B_0(G) \neq 0$ if and only if G belongs to the isoclinism family Φ_{10}. Moreover, if $B_0(G) \neq 0$, then $B_0(G) \simeq C_p$.*

For the last statement, see [Kan14, Remark, page 424]. The proof of Theorem 1.22 was given by purely algebraic way. There exist exactly 3 groups which belong to Φ_{10} if $p = 3$, i.e. $G = G(243, i)$ with $28 \leq i \leq 30$. This agrees with Moravec's computational result (Theorem 1.20). For $p \geq 5$, the exist exactly $1 + \gcd\{4, p-1\} + \gcd\{3, p-1\}$ groups which belong to Φ_{10} ([Jam80, page 621]).

The following result for the k-rationality of $k(G)$ supplements Theorem 1.20 although it is unknown whether $k(G)$ is k-rational for groups G which belong to Φ_7:

Theorem 1.23 (Chu, Hoshi, Hu and Kang [CHHK15, Theorem 1.13]). *Let G be a group of order 243 with exponent e. If $B_0(G) = 0$ and k be a field containing a primitive e-th root of unity, then $k(G)$ is k-rational except possibly for the five groups G which belong to Φ_7, i.e. $G = G(243, i)$ with $56 \leq i \leq 60$.*

In [HKK13] and [CHHK15], not only the evaluation of the Bogomolov multiplier $B_0(G)$ and the k-rationality of $k(G)$ but also the k-isomorphisms between $k(G_1)$ and $k(G_2)$ for some groups G_1 and G_2 belonging to the same isoclinism family were given.

Bogomolov and Böhning [BB13] gave an answer to the question raised as [HKK13, Question 1.11] in the affirmative as follows.

Theorem 1.24 (Bogomolov and Böhning [BB13, Theorem 6]). *If G_1 and G_2 are isoclinic, then $\mathbb{C}(G_1)$ and $\mathbb{C}(G_2)$ are stably $\mathbb{C}$-isomorphic. In particular, $H^i_{\mathrm{nr}}(\mathbb{C}(G_1), \mu_n^{\otimes j}) \xrightarrow{\sim} H^i_{\mathrm{nr}}(\mathbb{C}(G_2), \mu_n^{\otimes j})$.*

A partial result of Theorem 1.24 was already given by Moravec. Indeed, Moravec [Mor14, Theorem 1.2] proved that if G_1 and G_2 are isoclinic, then $B_0(G_1) \simeq B_0(G_2)$.

1.2 The case where G is a group of order 64

The classification of the groups G of order p^6 with $B_0(G) \neq 0$ for $p = 2$ was obtained by Chu, Hu, Kang and Kunyavskii [CHKK10]. Moreover, they investigated Noether's problem for groups G with $B_0(G) = 0$. There exist 267 groups G of order 64 which are classified into 27 isoclinism families $\Phi_1, \ldots, \Phi_{27}$ by Hall and Senior [HS64] (see also [JNO90, Table I]). The main result of [CHKK10] can be stated in terms of the isoclinism families as follows.

Theorem 1.25 (Chu, Hu, Kang and Kunyavskii [CHKK10]). *Let $G = G(2^6, i)$, $1 \leq i \leq 267$, be the i-th group of order 64 in the GAP database* [GAP].
(1) ([CHKK10, Theorem 1.8]) *$B_0(G) \neq 0$ if and only if G belongs to the isoclinism family Φ_{16}, i.e. $G = G(2^6, i)$ with $149 \leq i \leq 151$, $170 \leq i \leq 172$, $177 \leq i \leq 178$ or $i = 182$. Moreover, if $B_0(G) \neq 0$, then $B_0(G) \simeq C_2$ (see* [Kan14, Remark, page 424] *for this statement);*
(2) ([CHKK10, Theorem 1.10]) *If $B_0(G) = 0$ and k is an quadratically closed field, then $k(G)$ is k-rational except possibly for five groups which belong to Φ_{13}, i.e. $G = G(2^6, i)$ with $241 \leq i \leq 245$.*

For groups G which belong to Φ_{13}, k-rationality of $k(G)$ is unknown. The following two propositions supplement the cases Φ_{13} and Φ_{16} of Theorem 1.25. For the proof, the case of $G = G(2^6, 149)$ is given in [HKK14, Proof of Theorem 6.3], see also [CHKK10, Example 5.11, page 2355] and the proof for other cases can be obtained by the similar manner.

Definition 1.26. Let k be a field with char $k \neq 2$ and $k(X_1, X_2, X_3, X_4, X_5, X_6)$ be the rational function field over k with variables $X_1, X_2, X_3, X_4, X_5, X_6$.
(i) *The field $L_k^{(0)}$* is defined to be $k(X_1, X_2, X_3, X_4, X_5, X_6)^H$ where $H = \langle \sigma_1, \sigma_2 \rangle \simeq C_2 \times C_2$ act on $k(X_1, X_2, X_3, X_4, X_5, X_6)$ by k-automorphisms

$$\sigma_1 : X_1 \mapsto X_3,\ X_2 \mapsto \frac{1}{X_1X_2X_3},\ X_3 \mapsto X_1,\ X_4 \mapsto X_6,\ X_5 \mapsto \frac{1}{X_4X_5X_6},\ X_6 \mapsto X_4,$$
$$\sigma_2 : X_1 \mapsto X_2,\ X_2 \mapsto X_1,\ X_3 \mapsto \frac{1}{X_1X_2X_3},\ X_4 \mapsto X_5,\ X_5 \mapsto X_4,\ X_6 \mapsto \frac{1}{X_4X_5X_6}.$$

(ii) *The field $L_k^{(1)}$* is defined to be $k(X_1, X_2, X_3, X_4)^{\langle \tau \rangle}$ where $\langle \tau \rangle \simeq C_2$ acts on $k(X_1, X_2, X_3, X_4)$ by k-automorphisms

$$\tau : X_1 \mapsto -X_1,\ X_2 \mapsto \frac{X_4}{X_2},\ X_3 \mapsto \frac{(X_4 - 1)(X_4 - X_1^2)}{X_3},\ X_4 \mapsto X_4.$$

Proposition 1.27 ([CHKK10, Proposition 6.3], see also [HY, Proposition 12.5]). *Let G be a group of order 64 which belongs to Φ_{13}, i.e. $G = G(2^6, i)$ with $241 \leq i \leq 245$. There exists a $\mathbb{C}$-injective homomorphism $\varphi : L_{\mathbb{C}}^{(0)} \to \mathbb{C}(G)$ such that $\mathbb{C}(G)$ is rational over $\rho(L_{\mathbb{C}}^{(0)})$. In particular, $\mathbb{C}(G)$ and $L_{\mathbb{C}}^{(0)}$ are stably $\mathbb{C}$-isomorphic and $B_0(G) \simeq \mathrm{Br}_{\mathrm{nr}}(L_{\mathbb{C}}^{(0)}) = 0$.*

Proposition 1.28 ([CHKK10, Example 5.11, page 2355], [HKK14, Proof of Theorem 6.3]). *Let G be a group of order 64 which belongs to Φ_{16}, i.e. $G = G(2^6, i)$ with $149 \leq i \leq 151$, $170 \leq i \leq 172$, $177 \leq i \leq 178$ or $i = 182$. There exists a $\mathbb{C}$-injective homomorphism $\varphi : L_{\mathbb{C}}^{(1)} \to \mathbb{C}(G)$ such that $\mathbb{C}(G)$ is rational over $\varphi(L_{\mathbb{C}}^{(1)})$. In particular, $\mathbb{C}(G)$ and $L_{\mathbb{C}}^{(1)}$ are stably $\mathbb{C}$-isomorphic, $B_0(G) \simeq \mathrm{Br}_{\mathrm{nr}}(L_{\mathbb{C}}^{(1)}) \simeq C_2$ and hence $\mathbb{C}(G)$ and $L_{\mathbb{C}}^{(1)}$ are not (retract, stably) $\mathbb{C}$-rational.*

Question 1.29 ([CHKK10, Section 6], see also [HY, Section 12]). *Is $L_k^{(0)}$ k-rational?*

1.3 The case where G is a group of order 128

There exist 2328 groups of order 128 which are classified into 115 isoclinism families $\Phi_1, \ldots, \Phi_{115}$ ([JNO90, Tables I, II, III]). Let $G(2^7, i)$ be the i-th group of order $2^7 = 128$ in the GAP database [GAP]. By using Moravec's algorithm **b0g.g** in [Mor12] of GAP, e.g.

"`for i in [1..2328] do Print([i,BOG(SmallGroup(128,i))],"\n");od;`",

we obtain the following theorem.

Theorem 1.30 (Moravec [Mor12, Section 8, Table 1]). *Let G be a group of order 128. Then $B_0(G) \neq 0$ if and only if G is one of the following 220 groups:*
(1) $G(2^7, i)$ *with* $i =$ 227,228,229,301,324,325,326,541,543,568,570,579,581,626,627,629,667,668, 670,675,676,678,691,692,693,695,703,704,705,707,724,725,727,1783,1784,1785,1786,1864,1865, 1866,1867,1880,1881,1882,1893,1894,1903,1904;
(2) $G(2^7, i)$ *with* $1345 \leq i \leq 1399$;
(3) $G(2^7, i)$ *with* $242 \leq i \leq 247$, $265 \leq i \leq 269$, $287 \leq i \leq 293$;
(4) $G(2^7, i)$ *with* $36 \leq i \leq 41$;
(5) $G(2^7, i)$ *with* $1924 \leq i \leq 1929$, $1945 \leq i \leq 1951$, $1966 \leq i \leq 1972$, $1983 \leq i \leq 1988$;
(6) $G(2^7, i)$ *with* $417 \leq i \leq 436$;
(7) $G(2^7, i)$ *with* $446 \leq i \leq 455$;
(8) $G(2^7, i)$ *with* $i =$ 950, 951, 952, 975, 976, 977, 982, 983, 987;
(9) $G(2^7, i)$ *with* $i =$ 144, 145;
(10) $G(2^7, i)$ *with* $i =$ 138, 139;
(11) $G(2^7, i)$ *with* $1544 \leq i \leq 1577$.
Moreover, if G is a group in (1)–(10) (resp. (11)), then $B_0(G) \simeq C_2$ (resp. $C_2 \times C_2$).

By [JNO90, Tables I, II, III], we can get the classification of 115 isoclinism families for groups G of order 128 in terms of the GAP database [GAP], see [Hos16, Table 2]. Using this, we see that the groups as in (1)–(11) of Theorem 1.30 correspond to the isoclinism families Φ_{16}, Φ_{31}, Φ_{37}, Φ_{39}, Φ_{43}, Φ_{58}, Φ_{60}, Φ_{80}, Φ_{106}, Φ_{114}, Φ_{30} respectively:

Corollary 1.31 (Moravec [Mor12, Section 8, Table 1]). *Let G be a group of order 128. Then $B_0(G) \neq 0$ if and only if G belongs to the isoclinism family Φ_{16}, Φ_{30}, Φ_{31}, Φ_{37}, Φ_{39}, Φ_{43}, Φ_{58}, Φ_{60}, Φ_{80}, Φ_{106} or Φ_{114}. Moreover, if $B_0(G) \neq 0$, then*

$$B_0(G) \simeq \begin{cases} C_2 & \text{if } G \text{ belongs to } \Phi_{16}, \Phi_{31}, \Phi_{37}, \Phi_{39}, \Phi_{43}, \Phi_{58}, \Phi_{60}, \Phi_{80}, \Phi_{106} \text{ or } \Phi_{114}, \\ C_2 \times C_2 & \text{if } G \text{ belongs to } \Phi_{30}. \end{cases}$$

In particular, $\mathbb{C}(G)$ is not (retract, stably) $\mathbb{C}$-rational.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	Total
Family	Φ_{16}	Φ_{31}	Φ_{37}	Φ_{39}	Φ_{43}	Φ_{58}	Φ_{60}	Φ_{80}	Φ_{106}	Φ_{114}	Φ_{30}	
$\exp(G)$	8	4	8	4 or 8	8	8	8	16	8	8	4	
$B_0(G)$					C_2						$C_2 \times C_2$	
$\#$ G's	48	55	18	6	26	20	10	9	2	2	34	220

Table 1: Isoclinism families Φ_j for groups G of order 128 with $B_0(G) \neq 0$

It is natural to ask the (stably) birational classification of $\mathbb{C}(G)$ for groups G of order 128. In particular, what happens to $\mathbb{C}(G)$ with $B_0(G) \neq 0$? The following theorem (Theorem 1.33) gives a partial answer to this question.

Definition 1.32. Let k be a field with char $k \neq 2$ and $k(X_1, X_2, X_3, X_4, X_5, X_6, X_7)$ be the rational function field over k with variables $X_1, X_2, X_3, X_4, X_5, X_6, X_7$.
(i) *The field* $L_k^{(2)}$ is defined to be $k(X_1, X_2, X_3, X_4, X_5, X_6)^{\langle\rho\rangle}$ where $\langle\rho\rangle \simeq C_4$ acts on $k(X_1, X_2, X_3, X_4, X_5, X_6)$ by k-automorphisms

$$\begin{aligned} \rho : {} & X_1 \mapsto X_2, X_2 \mapsto -X_1, X_3 \mapsto X_4, X_4 \mapsto X_3, \\ & X_5 \mapsto X_6, X_6 \mapsto \frac{(X_1^2X_2^2 - 1)(X_1^2X_3^2 + X_2^2 - X_3^2 - 1)}{X_5}. \end{aligned}$$

(ii) *The field* $L_k^{(3)}$ is defined to be $k(X_1, X_2, X_3, X_4, X_5, X_6, X_7)^{\langle\lambda_1,\lambda_2\rangle}$ where $\langle\lambda_1, \lambda_2\rangle \simeq C_2 \times C_2$ acts on $k(X_1, X_2, X_3, X_4, X_5, X_6, X_7)$ by k-automorphisms

$$\begin{aligned} \lambda_1 : {} & X_1 \mapsto X_1, X_2 \mapsto \frac{X_1}{X_2}, X_3 \mapsto \frac{1}{X_1X_3}, X_4 \mapsto \frac{X_2X_4}{X_1X_3}, \\ & X_5 \mapsto -\frac{X_1X_6^2 - 1}{X_5}, X_6 \mapsto -X_6, X_7 \mapsto X_7, \\ \lambda_2 : {} & X_1 \mapsto \frac{1}{X_1}, X_2 \mapsto X_3, X_3 \mapsto X_2, X_4 \mapsto \frac{(X_1X_6^2 - 1)(X_1X_7^2 - 1)}{X_4}, \\ & X_5 \mapsto -X_5, X_6 \mapsto -X_1X_6, X_7 \mapsto -X_1X_7. \end{aligned}$$

Theorem 1.33 (Hoshi [Hos16, Theorem 1.31]). *Let G be a group of order 128. Assume that $B_0(G) \neq 0$. Then $\mathbb{C}(G)$ and $L_{\mathbb{C}}^{(m)}$ are stably $\mathbb{C}$-isomorphic where*

$$m = \begin{cases} 1 & \text{if } G \text{ belongs to } \Phi_{16}, \Phi_{31}, \Phi_{37}, \Phi_{39}, \Phi_{43}, \Phi_{58}, \Phi_{60} \text{ or } \Phi_{80}, \\ 2 & \text{if } G \text{ belongs to } \Phi_{106} \text{ or } \Phi_{114}, \\ 3 & \text{if } G \text{ belongs to } \Phi_{30}. \end{cases}$$

In particular, $\mathrm{Br}_{\mathrm{nr}}(L_{\mathbb{C}}^{(1)}) \simeq \mathrm{Br}_{\mathrm{nr}}(L_{\mathbb{C}}^{(2)}) \simeq C_2$ *and* $\mathrm{Br}_{\mathrm{nr}}(L_{\mathbb{C}}^{(3)}) \simeq C_2 \times C_2$ *and hence the fields* $L_{\mathbb{C}}^{(1)}$, $L_{\mathbb{C}}^{(2)}$ *and* $L_{\mathbb{C}}^{(3)}$ *are not (retract, stably)* $\mathbb{C}$*-rational.*

For $m = 1, 2$, the fields $L_{\mathbb{C}}^{(m)}$ and $L_{\mathbb{C}}^{(3)}$ are not stably $\mathbb{C}$-isomorphic because their unramified Brauer groups are not isomorphic. However, we do not know whether the fields $L_{\mathbb{C}}^{(1)}$ and $L_{\mathbb{C}}^{(2)}$ are stably $\mathbb{C}$-isomorphic. If not, it is interesting to evaluate the higher unramified cohomologies. Unfortunately, a useful formula like Bogomolov's formula (Theorem 1.11) or Moravec's formula [Mor12, Section 3] for $B_0(G)$ is unknown for higher unramified cohomologies.

Theorem 1.33 gives another proof of $B_0(G) \simeq C_2$ to Theorem 1.30 when G belongs to Φ_{16}, Φ_{31}, Φ_{37}, Φ_{39}, Φ_{43}, Φ_{58}, Φ_{60} or Φ_{80}. Especially, this proof is based on the result of order 64 for Φ_{16} (Theorem 1.25) and it does not depend on the computer calculations of GAP.

Although Theorem 1.33 gives only the first step, the author hopes that it will stimulate further work towards a more complete understanding of the (stably) birational classification of $\mathbb{C}(G)$ for non-abelian groups G.

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