数理解析研究所講究録
第2035巻 2017年 53-59

Isometries of Grassmann spaces

Peter Šemrl
Faculty of Mathematics and Physics
University of Ljubljana

1 Introduction

In this short survey we will deal with isometries of Grassmannians on Hilbert spaces. We will first introduce the notation and then explain the problem and the motivation. In the second section we will formulate known results and briefly explain the main ideas of the proofs. The last section will be devoted to open problems.

Let H be a real or complex Hilbert space. Recall that a bounded linear operator $P: H \to H$ is called a projection if $P^2 = P$ and $P^* = P$. Such a projection is uniquely determined by its image $\operatorname{Im} P$. Indeed, if P is a projection then the kernel of P is the orthogonal complement of the image of P. Therefore H is the orthogonal direct sum of the image of P and the kernel of P and if x is any vector in H, then there are unique vectors $x_1 \in \operatorname{Im} P$ and $x_2 \in \operatorname{Ker} P$ such that $x = x_1 + x_2$. We have $Px = x_1$.

In other words, each closed subspace of H can be identified with the orthogonal projection onto this subspace. Let n be a positive integer. Then the Grassmann space of all n-dimensional subspaces of H can be identified with $P_n(H)$, the set of all projections of rank n. Clearly, $P_n(H) \subset B(H)$, where $B(H)$ denotes the Banach algebra of all bounded linear operators on H equipped with the usual operator norm. Hence, the Grassmann space $P_n(H)$ of all n-dimensional subspaces of H is a metric space with the distance function defined by $d(P,Q) = \|P - Q\|$, $P, Q \in P_n(H)$. This distance is called the gap metric.

Problem 1.1 *Describe the general form of isometries of the Grass-*

mann space of all n-dimensional subspaces of H with respect to the gap metric.

In the case when $n = 1$ this problem has been solved long ago and is actually one of the basic results in the mathematical foundations of quantum mechanics. Indeed, in quantum physics the Grassmann space $P_1(H)$ of all rank one projections is used to represent the set of pure states of the quantum system, and the quantity $\operatorname{tr}(PQ)$, the trace of the product PQ, is the so-called transition probability between two pure states. The classical Wigner's theorem describes those transformations of $P_1(H)$ which preserve the transition probability.

One can easily obtain the following equation:

$$\|P - Q\| = \sqrt{1 - \operatorname{tr} PQ}, \quad P, Q \in P_1(H).$$

Therefore, Wigner's theorem characterizes isometries of $P_1(H)$ with respect to the gap metric, and in fact it states that these maps are induced by linear or conjugate-linear isometries of the underlying space H. Let us note that in its original version, Wigner's theorem describes surjective mappings of this kind, but as was shown later in several papers, the above conclusion holds for non-surjective transformations as well.

2 Known results

In this section we will present four structural results for isometries of Grassmann spaces that were proved in [5]. Somewhat weaker versions of two of them had been previously obtained in [3].

Theorem 2.1 *Let H be an infinite-dimensional complex Hilbert space and n a positive integer. Assume that a surjective map $\phi : P_n(H) \to P_n(H)$ is an isometry with respect to the gap metric. Then there exists either a unitary or an antiunitary operator U on H such that*

$$\phi(P) = UPU^*$$

for every $P \in P_n(H)$.

Theorem 2.2 *Let H be an infinite-dimensional real Hilbert space and n a positive integer. Assume that a surjective map $\phi : P_n(H) \to P_n(H)$ is an isometry with respect to the gap metric. Then there exists an orthogonal operator U on H such that*

$$\phi(P) = UPU^*$$

for every $P \in P_n(H)$.

Let us briefly explain the main ideas of the proof of the above two theorems. For any two projections $P, Q \in P_n(H)$ we define the following set:

$$M(P,Q) = \left\{ R \in P_n(H) \ : \ \|R - P\| \leq \frac{1}{\sqrt{2}} \text{ and } \|R - Q\| \leq \frac{1}{\sqrt{2}} \right\}.$$

Since $\phi : P_n(H) \to P_n(H)$ is an isometry we have

$$\phi(M(P,Q)) = M(\phi(P), \phi(Q)).$$

Consider the case when $P, Q \in P_n(H)$ are orthogonal, that is, $PQ = 0$ (and then, clearly, $QP = 0$). With respect to the orthogonal direct sum decomposition $H = \operatorname{Im} P \oplus \operatorname{Im} Q \oplus H_0$ the projections P, Q have the following matrix representations:

$$P = \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By I_n we have denoted the $n \times n$ identity matrix. It is then not too difficult to check that

$$M(P,Q) = \left\{ \begin{bmatrix} \frac{1}{2} I_n & \frac{1}{2} U & 0 \\ \frac{1}{2} U^* & \frac{1}{2} I_n & 0 \\ 0 & 0 & 0 \end{bmatrix} \ : \ U \in \mathcal{U}_n \right\}.$$

Here, $\mathcal{U}_n$ denotes the group of all $n \times n$ unitary matrices in the complex case and the group of all $n \times n$ orthogonal matrices in the real case. In particular, $M(P,Q)$ is a compact manifold.

If on the other hand, P and Q are not orthogonal, then one can use Halmos' two projections theorem to show that $M(P,Q)$ is not compact. Since ϕ is a bijective isometry, it maps compact sets to compact sets. It follows that for every pair $P, Q \in P_n(H)$ we have

$$P \perp Q \iff \phi(P) \perp \phi(Q).$$

The desired conclusion is now a straightforward consequence of the Györy - Šemrl's description of the general form of orthogonality preserving transformations on the set of n-dimensional subspaces of a Hilbert space [6, 7].

Theorem 2.3 *Let H be a finite-dimensional complex Hilbert space and n a positive integer, $n < \dim H$. Assume that a map $\phi : P_n(H) \to P_n(H)$ is an isometry with respect to the gap metric. If $\dim H \neq 2n$, then there exists either a unitary or an antiunitary operator U on H such that ϕ is of the following form:*

$$\phi(P) = UPU^*, \qquad P \in P_n(H). \tag{1}$$

In the case when $\dim H = 2n$, we have either (1), or the following additional possibility occurs:

$$\phi(P) = U(I - P)U^*, \qquad P \in P_n(H).$$

Theorem 2.4 *Let H be a finite-dimensional real Hilbert space and n a positive integer, $n < \dim H$. Assume that a map $\phi : P_n(H) \to P_n(H)$ is an isometry with respect to the gap metric. If $\dim H \neq 2n$, then there exists an orthogonal operator U on H such that ϕ is of the following form:*

$$\phi(P) = UPU^*, \qquad P \in P_n(H). \tag{2}$$

In the case when $\dim H = 2n$, we have either (2), or the following additional possibility occurs:

$$\phi(P) = U(I - P)U^*, \qquad P \in P_n(H).$$

There is an essential difference between the first two theorems and the last two: in the finite-dimensional case we get the description of isometries of Grassmann spaces without assuming surjectivity.

Let us briefly explain the main ideas of the proof of Theorems 2.3 and 2.4. First we need to see that ϕ is surjective. This is easy.

Indeed, $P_n(H)$ is a compact manifold, and therefore $\phi(P_n(H))$ is also compact. On the other hand, the domain invariance theorem ensures that the range of ϕ is open as well. Since $P_n(H)$ is connected, we conclude that ϕ is surjective.

Now, we have the same assumptions as in the infinite-dimensional case. It is somewhat surprising that the finite-dimensional case is much more difficult. We have to distinguish several cases.

The first one is that $2n < \dim H < \infty$. We first verify that for any pair $P, Q \in P_n(H)$ with $\|P - Q\| = 1$ the following are equivalent:

- P and Q are orthogonal.
- $M(P, Q)$ is a compact manifold.

We already know that if P and Q are orthogonal then $M(P, Q)$ is homeomorphic to either the unitary group, or the orthogonal group, and consequently, it is a compact manifold. To prove the converse one needs to show that if P and Q are not orthogonal, then $M(P, Q)$ is not a compact manifold. Of course, $M(P, Q)$ is closed, and hence compact. Thus, one needs to verify that it is not a manifold and this makes this part of the proof much more involved than in the infinite-dimensional case. Once this is done, we know that ϕ preserves orthogonality. Unfortunately, Győry - Šemrl's result [6, 7] describes the general form of orthogonality preserving transformations on the set of n-dimensional subspaces only on infinite-dimensional Hilbert spaces. But it is not too difficult to extend it to the case when $\dim H > 2n$. And then the proof in our first case is done.

As orthogonality preserving maps on $P_n(H)$ may behave badly when $\dim H = 2n$, this special case needs to be treated separately. The main idea is to apply a theorem of Blunck and Havlicek [2] on complementarity preservers. Note that $P, Q \in P_n(H)$ are complementary if H is a direct sum of the images of P and Q. This is easily seen to be equivalent to the condition

$$\|(I - P) - Q\| < 1.$$

Hence, isometries preserve complementarity and are of a nice form by Blunck-Havlicek theorem.

In the remaining case when $n < d := \dim H < 2n$ we first observe that $\|P - Q\| = \|(I - P) - (I - Q)\|$, $P, Q \in P_n(H)$. Thus the map $\tilde{\phi} : P_{d-n}(H) \to P_{d-n}(H)$ defined by $\tilde{\phi}(I - P) = I - \phi(P)$ is also an isometry, but on the Grassmann space $P_{d-n}(H)$. Therefore we can apply our result in the first case to complete the proof.

3 Open problems

1. So far the isometries of Grassmann spaces with respect to the gap metric induced by the usual operator norm have been studied. What happens if we replace the operator norm with other unitarily invariant norms? Note that we are dealing with the distance between finite rank projections and thus we are interested in matrix unitarily invariant norms. An interested reader can find the complete description of unitarily invariant norms of matrices in [1].

2. Do we need the surjectivity assumption in Theorem 2.1? The description of not necessarrily surjective isometries of Grassmann spaces with respect to the gap metric in the special case when $n = 1$ is known. Of course, we need to replace unitary or antiunitary operators in the conclusion of the statement with linear or conjugate-linear (not necessarily surjective) isometries of the underlying Hilbert space H. For a short proof we refer to [4]. Of course, the same question can be asked for Theorem 2.2.

3. It would be nice to have an analogue of Theorems 2.1 and 2.2 with $P_\infty(H)$ instead of $P_n(H)$. Here, $P_\infty(H)$ denotes the set of all projections P on H such that both the image of P and the kernel of P are infinite-dimensional.

References

[1] R. Bhatia, *Matrix analysis*, Graduate Texts in Mathematics **169**, Springer-Verlag, New York, 1997. xii+347 pp.

[2] A. Blunck and H. Havlicek, On bijections that preserve complementarity of subspaces, *Discrete Math.* **301** (2005), 46-56.

[3] F. Botelho, J. Jamison, and L. Molnár, Surjective isometries on Grassmann spaces, *J. Funct. Anal.* **265** (2013), 2226-2238.

[4] G.P. Gehér, An elementary proof for the non-bijective version of Wigner's theorem, *Phys. Lett. A* **378** (2014), 2054 - 2057.

[5] G.P. Gehér and P. Šemrl, Isometries of Grassmann spaces, *J. Funct. Anal.* **270** (2016), 1585-1601.

[6] M. Győry, Transformations on the set of all n-dimensional subspaces of a Hilbert space preserving orthogonality, *Publ. Math. Debrecen* **65** (2004), 233-242.

[7] P. Šemrl, Orthogonality preserving transformations on the set of n-dimensional subspaces of a Hilbert space, *Illinois J. Math.* **48** (2004), 567–573.

Faculty of Mathematics and Physics
University of Ljubljana
Jadranska 19
SI-1000 Ljubljana
Slovenia
peter.semrl@fmf.uni-lj.si