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A CHERNOFF PRODUCT FORMULA FOR GRADIENT FLOWS IN METRIC SPACES

NAOKI TANAKA

1. Introduction

Let (X,d) be a complete metric space and φ a proper lower semicontinuous functional from X into $(-\infty,\infty]$. Then the *evolution variational inequality problem* is stated as follows: Given $x\in X$ and $\tau>0$, find $u\in C([0,\tau];X)$ such that $u(0)=x$, $u(t)\in D(\varphi)$ for almost all $t\in(0,\tau)$, $\varphi\circ u\in L^1_{\mathrm{loc}}(0,\tau;\mathbb{R})$ and

$$\text{(1.1)}\qquad \frac{1}{2}\left(d(u(t),z)^2-d(u(s),z)^2\right)+\int_s^t\varphi(u(r))\,dr\le(t-s)\varphi(z)$$

for $0<s<t<\tau$ and $z\in D(\varphi)$. Such a function u is called an *integral solution to* (EVI;x) *on* $[0,\tau]$. A function $u\in C([0,\tau);X)$ where $\tau\in(0,\infty]$ is called an *integral solution to* (EVI;x) *on* $[0,\tau)$ if for any $0<b<\tau$ the restriction u to the interval $[0,b]$ is an integral solution to (EVI;x) on $[0,b]$.

We establish a Chernoff product formula for gradient flows and apply it to study the well-posedness of the evolution variational inequality (1.1) whose integral solutions u satisfy the growth condition

$$\text{(1.2)}\qquad \psi(u(t))\le m(t;\psi(x))\qquad\text{for }0\le t<\tau(\psi(x)),$$

where $\psi=(\psi_j)_{j=1}^N$ is an N-tuple of functionals satisfying the following conditions:

(ψ1) For $1\le j\le N$, the functional ψ_j maps X into $[0,\infty]$.
(ψ2) The set $D(\psi):=\{x\in X;\ \psi_j(x)<\infty\text{ for }1\le j\le N\}$ coincides with the effective domain $D(\varphi)$ of φ.
(ψ3) For $r\in\mathbb{R}^N_+$, the set $D_r(\psi):=\{x\in X;\psi(x)\le r\}$ is closed in X.
(ψ4) For each $r\in\mathbb{R}^N_+$, there exists $M\ge0$ such that $\psi(x)\le r$ implies $\varphi(x)\le M$.

Here and subsequently, the symbol $\mathbb{R}_+$ stands for the interval $[0,\infty)$, and the symbol $r\le\hat{r}$ in $\mathbb{R}^N_+$ means that $r_j\le\hat{r}_j$ for $1\le j\le N$, where $r=(r_j)_{j=1}^N$ and $\hat{r}=(\hat{r}_j)_{j=1}^N$. For $r\in\mathbb{R}^N_+$, the symbol $\tau(r)$ stands for the maximal existence time of the noncontinuable maximal solution $m(t;r)$ of the problem

$$p'(t)=g(p(t))\quad\text{for }t\ge0,\quad\text{and}\quad p(0)=r,$$

where $g\in C(\mathbb{R}^N_+;\mathbb{R}^N)$ satisfies the following conditions:

(g1) For $1\le j\le N$, $g_j(0)\ge0$.
(g2) For $1\le j\le N$, $g_j(r)$ is nondecreasing in r_k with $k\ne j$.

Such a function g is called a *comparison function.*

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2. A Chernoff product formula for gradient flows

The main theorem is given by

Theorem 2.1. *Let $\{C_h; h \in (0, h_0]\}$ be a family of operators from $D(\varphi)$ into itself. Assume that for any $\epsilon > 0$ and $r \in \mathbb{R}^N_+$ there exists $\delta_0 \in (0, h_0]$ such that for $h \in (0, \delta_0]$ and $v \in D(\varphi)$ with $\psi(v) \leq r$,*

$$\frac{1}{2h}\left(d(C_h v, z)^2 - d(v, z)^2\right) + \varphi(C_h v) \leq \varphi(z) + \epsilon \quad \text{for } z \in D(\varphi), \tag{2.1}$$

$$\psi(C_h v) \leq m^\epsilon(h; \psi(v)), \tag{2.2}$$

where for each $\epsilon \in (0, \epsilon_0]$ and $r \in \mathbb{R}^N_+$, the symbol $m^\epsilon(t; r)$ stands for the noncontinuable maximal solution of the problem

$$p'(t) = g^\epsilon(p(t)) \text{ for } t \geq 0, \text{ and } p(0) = r,$$

and $g^\epsilon \in C(\mathbb{R}^N_+; \mathbb{R}^N)$ is defined by

$$g^\epsilon_j(r) = g_j(r) + \epsilon \text{ for } 1 \leq j \leq N \text{ and } r \in \mathbb{R}^N_+.$$

Then for any $x \in D(\varphi)$ there exists a unique integral solution u to (EVI;x) *on $[0, \tau(\psi(x)))$ satisfying the growth condition* (1.2) *such that*

$$\lim_{h \downarrow 0} d(C_h^{[t/h]} x, u(t)) = 0 \tag{2.3}$$

for $t \in [0, \tau(\psi(x)))$, where the convergence is uniform on any compact subinterval of $[0, \tau(\psi(x)))$.

Remark 2.2. (i) Clément and Maas [2] recently pointed out that the results in [1] cannot be directly applied to Fokker-Plack equations and porous medium equations with a potential discussed in [6, 10] and proved a Trotter product formula for gradient flows in order to establish the convergence of the splitting method for such perturbed equations. The main theorem generalizes their result on Trotter product formula. (ii) In [1] the existence of a unique solution u with regularizing effect such that $\varphi(u(t))$ is nonincreasing in t is investigated. This is a special case where $g = 0$ and $\psi = \varphi^+$, where φ^+ denotes the positive part of φ. Other examples will be given in Corollary 2.3.

Proof. By (2.1) there exist $u_0 \in D(\varphi)$, $v_0 \in D(\varphi)$, $\eta_0 > 0$ and $\xi_0 > 0$ such that

$$\frac{1}{2\eta_0}\left(d(v_0, z)^2 - d(u_0, z)^2\right) + \varphi(v_0) \leq \varphi(z) + \xi_0 \tag{2.4}$$

for any $z \in D(\varphi)$. For $z \in D(\varphi)$ we set

$$M(z) = d(u_0, z)(d(u_0, v_0)/\eta_0) + (d(u_0, v_0)/\eta_0)^2/2 + (\varphi(z) - \varphi(v_0))^+ + \xi_0,$$

where $a^+ = \max\{a, 0\}$ for $a \in \mathbb{R}$.

We prove that for any $x \in D(\varphi)$, the limit $\lim_{h \downarrow 0} C_h^{[t/h]} x$ exists uniformly for t in any compact subinterval of $[0, \tau(\psi(x)))$. To do this, let $x \in D(\varphi)$ and set $\tau = \tau(\psi(x))$. Take $0 < T < \tau$ arbitrarily. Then there exist $r \in \mathbb{R}^N_+$ and $\epsilon_0 \in (0, 1/2]$ such that $\tau^\epsilon(\psi(x)) > T$ and $m^\epsilon(t; \psi(x)) \leq r$ for $t \in [0, T]$ and $\epsilon \in (0, \epsilon_0]$, where for each $\epsilon \in (0, \epsilon_0]$ and $r \in \mathbb{R}^N_+$, the symbol $\tau^\epsilon(r)$ stands for the maximal existence time of the maximal solution $m^\epsilon(t; r)$.

Let $\epsilon \in (0, \epsilon_0]$ and take $\delta_0 \in (0, h_0]$ so that conditions (2.1) and (2.2) hold for $h \in (0, \delta_0]$ and $v \in D(\varphi)$ with $\psi(v) \leq r$. Let $\eta_0 = \min\{\epsilon, \delta_0\}$ and set $K^h = [T/h]$

for $h \in (0, \eta_0]$. Then it can be inductively proved that $\psi(C_h^i x) \leq m^\epsilon(ih; \psi(x))$ for $h \in (0, \eta_0]$ and $0 \leq i \leq K^h$, and

$$\text{(2.5)} \qquad \frac{1}{2h}\left(d(C_h^i x, z)^2 - d(C_h^{i-1}x, z)^2\right) + \varphi(C_h^i x) \leq \varphi(z) + \epsilon$$

for $h \in (0, \eta_0]$, $z \in D(\varphi)$ and $1 \leq i \leq K^h$.

Let $\lambda, \mu \in (0, \eta_0]$ satisfy $2\lambda \leq 1$ and $2\mu \leq 1$. We prove by double induction that

$$\text{(2.6)} \qquad d(C_\lambda^i x, C_\mu^j x)^2 \leq 2\exp(2(i\lambda + j\mu))\left(M(x)D_{i,j}^{\lambda,\mu} + (i\lambda + j\mu)\epsilon\right)$$

for $0 \leq i \leq K^\lambda$ and $0 \leq j \leq K^\mu$, where the symbol $D_{i,j}^{\lambda,\mu}$ is defined by

$$D_{i,j}^{\lambda,\mu} = \{(i\lambda - j\mu)^2 + i\lambda^2 + j\mu^2\}^{1/2}$$

for $0 \leq i \leq K^\lambda$ and $0 \leq j \leq K^\mu$. In order to verify that the inequality (2.6) holds for $i = 0$, it suffices to show that

$$\text{(2.7)} \qquad d(C_\mu^j x, x)^2 \leq \exp(2j\mu)\,(2M(x)j\mu + 2j\mu\epsilon)$$

for $0 \leq j \leq K^\mu$. Clearly, the inequality (2.7) holds for $j = 0$. Now, let $1 \leq l \leq K^\mu$ and assume that the inequality (2.7) holds for $j = l - 1$. Combining the inequality (2.5) with $z = x$, $h = \mu$ and $i = l$ and the inequality (2.4) with $z = C_\mu^l x$, we have

$$\begin{aligned}\frac{1}{2\mu}\left(d(C_\mu^l x, x)^2 - d(C_\mu^{l-1}x, x)^2\right) &\leq \frac{1}{2\eta_0}\left(d(u_0, C_\mu^l x)^2 - d(v_0, C_\mu^l x)^2\right) \\ &\quad + (\varphi(x) - \varphi(v_0))^+ + \xi_0 + \epsilon.\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{2\eta_0}\left(d(u_0, C_\mu^l x)^2 - d(v_0, C_\mu^l x)^2\right) &\leq d(u_0, C_\mu^l x)\left(d(u_0, C_\mu^l x) - d(v_0, C_\mu^l x)\right)/\eta_0 \\ &\leq \left(d(u_0, x) + d(x, C_\mu^l x)\right) d(u_0, v_0)/\eta_0,\end{aligned}$$

we find that $\left(d(C_\mu^l x, x)^2 - d(C_\mu^{l-1}x, x)^2\right)/\mu \leq 2M(x) + d(C_\mu^l x, x)^2 + 2\epsilon$; hence

$$d(C_\mu^l x, x)^2 \leq \exp(2\mu)(d(C_\mu^{l-1}x, x)^2 + 2M(x)\mu + 2\epsilon\mu),$$

where we have used the fact that $(1-t)^{-1} \leq \exp(2t)$ for $t \in [0, 1/2]$. Substituting the inequality (2.7) with $j = l - 1$ into this inequality, we observe that the inequality (2.7) holds for $j = l$. This proves the inequality (2.6) holds for $i = 0$. Similarly, the inequality (2.6) is proved to be true for $j = 0$.

Now, let $1 \leq k \leq K^\lambda$ and $1 \leq l \leq K^\mu$, and assume that the inequality (2.6) hold for $(i, j) = (k - 1, l)$ and $(i, j) = (k, l - 1)$. Combining the two inequalities (2.5) with (h, i, z) replaced by $(\lambda, k, C_\mu^l x)$ and $(\mu, l, C_\lambda^k x)$, we find that

$$d(C_\lambda^k x, C_\mu^l x)^2 \leq \frac{\mu}{\lambda + \mu} d(C_\lambda^{k-1}x, C_\mu^l x)^2 + \frac{\lambda}{\lambda + \mu} d(C_\lambda^k x, C_\mu^{l-1}x)^2 + 4\frac{\lambda\mu}{\lambda + \mu}\epsilon.$$

We substitute the induction hypotheses into the first and second terms on the right-hand side of the above inequality and use the inequality

$$\frac{\mu}{\lambda + \mu} D_{k-1,l}^{\lambda,\mu} + \frac{\lambda}{\lambda + \mu} D_{k,l-1}^{\lambda,\mu} \leq D_{k,l}^{\lambda,\mu},$$

which follows from the Cauchy-Schwarz inequality (see also Kobayashi's argument used in proving [7, the inequality (2.10)]). This proves (2.6) with $(i, j) = (k, l)$. We conclude that the inequality (2.6) holds for any $0 \leq i \leq K^\lambda$ and $0 \leq j \leq K^\mu$.

By (2.6) we have

$$d(C_\lambda^{[s/\lambda]}x, C_\mu^{[t/\mu]}x)^2 \le 2\exp(4T)\left(M(x)\{(|t-s|+\lambda+\mu)^2+(\lambda+\mu)T\}^{1/2}+2T\epsilon\right)$$

for $\lambda,\mu \in (0,\eta_0]$ and $s,t \in [0,T]$. This implies that the family $\{C_h^{[t/h]}x\}$ converges to an X-valued measurable function u on $[0,T]$ in X uniformly for $t \in [0,T]$ as $h \downarrow 0$ and that $d(u(s),u(t))^2 \le 2\exp(4T)M(x)|t-s|$ for $t,s \in [0,T]$.

Since $\psi(C_h^{[t/h]}x) \le m^\epsilon([t/h]h;\psi(x))$ for $t \in [0,T]$ and $h \in (0,\eta_0]$, it follows from condition (ψ3) that $u(t) \in D(\psi) = D(\varphi)$ and $\psi(u(t)) \le m(t;\psi(x))$ for $t \in [0,T]$. Moreover, we have $\psi(C_h^{[t/h]}x) \le r$ for $t \in [0,T]$ and $h \in (0,\eta_0]$. Condition (ψ4) ensures the existence of $M_0 > 0$ such that $\varphi(C_h^{[t/h]}x) \le M_0$ for $t \in [0,T]$ and $h \in (0,\eta_0]$. Setting $z = C_h^{[t/h]}x$ in (2.4) and noting (2.3), we find a real number m_0 such that $\varphi(C_h^{t/h]}x) \ge m_0$ for $t \in [0,T]$ and $h \in (0,\eta_0]$. We use (2.5) to find that

$$\frac{1}{2}\left(d(C_h^l x,z)^2 - d(C_h^k x,z)^2\right) + \int_{(k+1)h}^{(l+1)h} \varphi(C_h^{[t/h]}x)\,dt \le (l-k)h(\varphi(z)+\epsilon)$$

for $z \in D(\varphi)$ and $0 \le k \le l \le K^h$. The lower semicontinuity of φ shows that $u(t) \in D(\varphi)$ and $\varphi(u(t)) \le M_0$ for $t \in [0,T]$ and that u satisfies the integral inequality (1.1). Since $\varphi \circ u$ is lower semicontinuous on $[0,T]$, it is bounded on $[0,T]$ from below. It follows that $\varphi \circ u \in L^\infty(0,T;X)$. Since $T \in (0,\tau(\psi(x)))$ is arbitrary, we conclude that the (EVI;x) has an integral solution u on $[0,\tau(\psi(x)))$ satisfying the growth condition (1.2). □

Theorem 2.1 generalizes some results in [2].

Corollary 2.3. ([2, Theorem 1.1 and Proposition 1.7]) *For $i = 1,2$, let φ^i be a lower semicontinuous functional from X into $(-\infty,\infty]$ satisfying $D(\varphi^1) \cap D(\varphi^2) \ne \emptyset$. Assume that the following conditions* (A1) *and* (A2) *hold:*

(A1) *For $i = 1,2$, the following variational inequality has a solution for any $h > 0$ and any $x \in \overline{D(\varphi^i)}$:*
Find $y \in D(\varphi^i)$ satisfying

$$\frac{1}{2h}\left(d(y,z)^2 - d(x,z)^2\right) + \frac{1}{2h}d(y,x)^2 + \varphi^i(y) \le \varphi^i(z)$$

for any $z \in D(\varphi^i)$.

(A2) *For any $h > 0$, $J_h^1\left(\overline{D(\varphi^1)} \cap D(\varphi^2)\right) \subset \overline{D(\varphi^2)}$ and $J_h^2\left(D(\varphi^1) \cap \overline{D(\varphi^2)}\right) \subset D(\varphi^1)$, where J_h^i is the resolvent of φ^i for $i = 1,2$.*

Suppose that φ^1 and φ^2 satisfy at least one of the following conditions:

(1) *There exists $c \ge 0$ such that $\varphi^1(J_h^2 x) \le \varphi^1(x) + ch$ for any $h > 0$ and $x \in D(\varphi^1) \cap D(\varphi^2)$.*
(2) *The functional φ^1 maps X to $[0,\infty]$ and there exists $\alpha \ge 0$ such that $\varphi^1(J_h^2 x) \le e^{\alpha h}\varphi^1(x)$ for any $h > 0$ and $x \in D(\varphi^1) \cap D(\varphi^2)$.*
(3) *The functional φ^2 maps X to $[0,\infty]$ and there exist $\alpha \ge 0$ and $c \ge 0$ such that $\varphi^1(J_h^2 x) \le \varphi^1(x) + ch\varphi^2(J_h^2 x)$ and $\varphi^2(J_h^1 x) \le e^{\alpha h}\varphi^2(x)$ for any $h > 0$ and $x \in D(\varphi^1) \cap D(\varphi^2)$.*

Then for any $x \in D(\varphi^1) \cap D(\varphi^2)$ there exists a unique integral solution u to (EVI;x) *on $[0,\infty)$ such that*

$$\lim_{h\downarrow 0}(J_h^2 J_h^1)^{[t/h]}x = u(t)$$

for $t \in [0,\infty)$, where the convergence is uniform on any compact subinterval of $[0,\infty)$.

Proof. Consider the functional φ defined by $\varphi(x) = \varphi^1(x) + \varphi^2(x)$ for $x \in D(\varphi) := D(\varphi^1) \cap D(\varphi^2)$ and the family $\{C_h; h > 0\}$ of operators from $D(\varphi)$ into itself defined by $C_h x = J_h^2 J_h^1 x$ for $x \in D(\varphi)$ and $h > 0$. Then the assumptions in Theorem 2.1 are satisfied with

(i) $\psi = \varphi^+$ and $g(r) = c$ for $r \in \mathbb{R}_+$ in case (1),
(ii) $\psi = (\varphi^+, \varphi^1)$ and $g(r) = (\alpha r_2, \alpha r_2)$ for $r = (r_1, r_2) \in \mathbb{R}_+^2$ in case (2),
(iii) $\psi = (\varphi^+, \varphi^2)$ and $g(r) = (\alpha r_1 + c r_2, \alpha r_2)$ for $r = (r_1, r_2) \in \mathbb{R}_+^2$ in case (3).

The conclusion follows from Theorem 2.1 (see [13] in detail). □

3. Concluding remark

In [13] the following characterization is established for the unique existence of integral solutions satisfying (1.2) and is used to prove the Chernoff product formula (Theorem 2.1).

Theorem 3.1. *For any $x \in D(\varphi)$ there exists a unique integral solution u to (EVI;x) on $[0, \tau(\psi(x)))$ satisfying the growth condition (1.2) if and only if the following condition is satisfied:*

(H) *For any $\epsilon > 0$ and $x \in D(\varphi)$ there exist $\delta \in (0,\epsilon]$ and $x_\delta \in D(\varphi)$ such that*
(i) $\frac{1}{2\delta}(d(x_\delta, z)^2 - d(x,z)^2) + \varphi(x_\delta) \le \varphi(z) + \epsilon$ *for* $z \in D(\varphi)$,
(ii) $\psi(x_\delta) \le m^\epsilon(\delta; \psi(x))$.

To prove the theorem we need to construct a family of approximate solutions described by countable ordinals (compare with [4, 3, 7, 8, 9]) and the proof is based on a transfinite induction argument similar to that used in [5, 11, 12].

Lemma 3.2. *Let $x_0 \in D(\varphi)$ and $\tau_0 = \tau(\psi(x_0))$. Assume that $\epsilon \in (0, 1/2]$, $\tau \in (0, \tau_0)$ and $r_0 \in \mathbb{R}_+^N$ satisfy $\tau^\epsilon(\psi(x_0)) > \tau$ and $m^\epsilon(t; \psi(x_0)) \le r_0$ for $t \in [0,\tau]$. Then there exist a countable ordinal κ, a set $\{t_\beta; 0 \le \beta \le \kappa\}$ in $[0,\tau]$ and a set $\{x_\beta; 1 \le \beta \le \kappa\}$ in $D(\varphi)$ satisfying the following conditions:*

(i) $0 = t_0 < t_\beta < t_\gamma < t_\kappa = \tau$ *for* $0 < \beta < \gamma < \kappa$.
(ii) *If β is a successor ordinal, then*
(ii-1) $h_\beta := t_\beta - t_{\beta-1} \le \epsilon$,
(ii-2) $\frac{1}{2h_\beta}\left(d(x_\beta, z)^2 - d(x_{\beta-1}, z)^2\right) + \varphi(x_\beta) \le \varphi(z) + \epsilon$ *for* $z \in D(\varphi)$.
(iii) *If β is a limit ordinal, then*

$$x_\beta = \lim_{n\to\infty} x_{\beta_n} \qquad \text{and} \qquad t_\beta = \lim_{n\to\infty} t_{\beta_n}$$

for any sequence $\{\beta_n\}$ of countable ordinals with $\beta = \lim_{n\to\infty}\beta_n$.

Moreover, the following inequalities hold:

(a) $\psi(x_\beta) \le m^\epsilon(t_\beta; \psi(x_0))$ *for* $0 \le \beta \le \kappa$.
(b) $d(x_\beta, x_0)^2 \le \exp(2t_\beta) N_0 t_\beta$ *for* $0 \le \beta \le \kappa$, *where*

$$N_0 = (d(x_1, x_0)/h_1)^2 + 2(\varphi(x_0) - \varphi(x_1))^+ + 2.$$

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Department of Mathematics, Faculty of Science, Shizuoka University, Shizuoka 422-8529, Japan

E-mail address: tanaka.naoki@shizuoka.ac.jp