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Algebraic isomorphisms in the descriptions of generalized isometries on spaces of positive definite matrices

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1 Introduction

This article is of survey character. Our aim is to present certain algebraic approaches what we recently developed to describe maps that can be called generalized isometries on the set of all positive definite complex matrices. In this paper isometries (i.e., distance preserving maps) and generalized isometries are *assumed to be surjective.*

The study of symmetries of mathematical structures of any kinds is one of the most classical general problems in mathematics. In algebra a lot of work is done on the automorphism groups of algebraic structures while in geometry, if a given object is endowed with a sort of distance measure, one naturally considers transformations which leave that measure invariant.

Below we will see that sometimes those two types of transformations (i.e., automorphisms and isometries) are rather closely related. In fact, this phenomenon was not noticed only recently. We recall the following well-known results. Let X, Y be compact Hausdorff topological spaces and let $C(X), C(Y)$ denote the corresponding algebras of all continuous real valued functions on X and Y, respectively. The algebra isomorphisms of $C(X)$ onto $C(Y)$ are known to be exactly the maps of the form $f \mapsto f \circ \varphi$, where $\varphi : Y \to X$ is a homeomorphism, while, by the famous Banach-Stone theorem, the isometries of $C(X)$ onto $C(Y)$ (with respect to the distance coming from the supremum norm) which send 0 to 0 (a harmless assumption) are of the form $f \mapsto \tau \cdot f \circ \varphi$, where φ is as above and $\tau \in C(X)$ is some given function with values in $\{-1, 1\}$. Therefore, one can see that in the case of those function algebras any algebra isomorphism is an isometry and, conversely, any isometry is "almost" an algebra isomorphism.

We point out that nowadays Banach-Stone theorem is usually formulated for linear isometries. But in the case of normed real linear spaces concerning isometries which send 0 to 0 we get the linearity for free. This is the consequence of the famous Mazur-Ulam theorem [11] which will play an important role in what follows.

Theorem 1 (Mazur-Ulam (1932)) *Let X, Y be normed real linear spaces. Then every isometry $\phi : X \to Y$ is affine, i.e., respects the operation of convex combinations. Consequently, if ϕ is assumed to send 0 to 0, then it is real linear.*

This is a very nice result but if we would like to determine the precise structure of the isometries between given normed linear spaces, it does not help much. We mean that there are "too many" linear transformations on a real linear space, so it is nonsense to say that what we have to do in order to determine the isometries is that we simply select those bijective linear transformations which are isometries. Indeed, for example, in Banach-Stone theorem not the linearity plays the key role but something else which is more purely algebraic in nature: namely, the fact that the extreme points of the unit ball are exactly the functions whose square is identically 1 and that extremes points are mapped onto extreme point. Although Mazur-Ulam theorem does not do the main job in the descriptions of isometries in the setting of normed linear spaces, we will see below that that type of results can be of fundamental help in describing the isometries of certain noncommutative structures.

In what follows we are interested in the determination of the structure of maps on the set of all positive definite matrices which preserve certain so-called generalized distance measures. We also call these maps generalized isometries and again we emphasize that we always assume that they are *surjective transformations.*

Let us fix the notation. In what follows $\mathbb{M}_n$ denotes the space of all $n \times n$ complex matrices, $n \geq 2$, $\mathbb{H}_n$ stands for the space of all Hermitian elements of $\mathbb{M}_n$ and $\mathbb{P}_n$ denotes the cone of all positive definite $n \times n$ complex matrices.

Below we consider several norms on $\mathbb{M}_n$. In particular, $\|.\|$ denotes the spectral norm (in another words operator norm, i.e., the one which equals the largest singular value of a matrix), $\|.\|_2$ stands for the Frobenius norm (in another words Hilbert-Schmidt norm, i.e., the l_2-norm of the singular values) and $\|.\|_1$ denotes the trace norm (i.e., the l_1-norm of the singular values).

2 Generalized Mazur-Ulam theorems, divergence preservers I

In this part of the article we are mainly concerned with the description of the structure of isometries and generalized isometries corresponding to certain non-Euclidean geometries on the cone of positive definite matrices. In the solutions abstract Mazur-Ulam type results play a fundamental role. Below we explain how and why.

In the paper [14] we determined the structure of all Thompson isometries of the positive definite cone of the full operator algebra over a complex Hilbert space. The Thompson metric is very important for the role it plays in several areas e.g., in the theory of nonlinear matrix equations, optimization and control, etc.

Since in the present article would like to keep the presentation at an easily accessible level, hence we do not want to treat infinite dimensions, we restrict our considerations to the finite dimensional case of matrices. The Thompson metric (which is usually defined in a much more general setting, see e.g. [4]) on $\mathbb{P}_n$ is given by the formula

$$d_T(A,B) = \|\log A^{-1/2}BA^{-1/2}\|, \quad A,B \in \mathbb{P}_n. \tag{1}$$

(A few more comments regarding this metric will be given below.) Our aim was to describe the corresponding isometries of $\mathbb{P}_n$. The key step of the proof in [14] was the following interesting observation. Adapting the arguments in the very nice proof of Mazur-Ulam theorem due to Väisälä [23], we proved that every Thompson isometry preserves the geometric mean, i.e., it is an automorphism of $\mathbb{P}_n$ under the operation

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}, \quad A,B \in \mathbb{P}_n. \tag{2}$$

Recall that in the original version of Mazur-Ulam theorem, the point is that the isometries between normed real linear spaces preserve the arithmetic mean which operation is connected to the additive structure. We see in the formula (1) above that the Thompson metric on $\mathbb{P}_n$ is in close connection with multiplication, so one can indeed expect that the Thompson isometries may show some features of respecting the multiplicative structure. We mention that we had to be very careful in [14] when adapting the proof of Mazur-Ulam theorem: the additive structure of a normed real linear space is commutative while multiplication is highly noncommutative.

So it turned out that the Thompson isometries are geometric mean preservers. But the geometric mean is a rather complicated operation. How can we go further? One could try to determine that sort of preservers directly (we mention that this had previously been done on the positive semidefinite cone in [13]) but better to observe the following. By Anderson-Trapp theorem (for original source, see [2]), the geometric mean $A\#B$ is the unique solution X of the equation $XA^{-1}X = B$ in $\mathbb{P}_n$. It then easily follows that any map which respects the geometric mean necessarily respects the operation $AB^{-1}A$, too. We call this latter object the inverted Jordan triple product of A and B. Therefore, we had that any Thompson isometry on $\mathbb{P}_n$ is an automorphism under that product. In [14], applying former results we determined those automorphims and found that all of them were actually Thompson isometries. This gave the solution of our problem.

Since the inverted Jordan triple product can be defined in any group (this does not hold for the geometric mean, have a look at (2)), what we have discussed above gave us the idea to try to deduce general Mazur-Ulam theorems in general groups or even more general algebraic structures. In the joint work [7] with Hatori, Hirasawa, Miura we presented our first such result and some applications.

Later we got to an even higher level of generality and in what follows we exhibit the presently most general (to our knowledge) such result in the setting of so-called point-reflection geometries. This concept is due to Manara and Marchi [10] and its definition is as follows.

Definition 2 *Let X be a set equipped with a binary operation $\diamond$ which satisfies the following conditions:*

(a1) $a \diamond a = a$ holds for every $a \in X$;

(a2) $a \diamond (a \diamond b) = b$ holds for any $a, b \in X$;

(a3) the equation $x \diamond a = b$ has a unique solution $x \in X$ for any given $a, b \in X$.

In this case the pair $(X, \diamond)$ (or X itself) is called a point-reflection geometry.

The trivial example of a point-reflection geometry is, of course, the Euclidean plane with the operation $a \diamond b$ being the reflection of the point b with respect to the one a. Apparently, this can be extended to any linear space X by defining $a \diamond b = 2a - b$, $a, b \in X$.

Now, a highly nontrivial example is $\mathbb{P}_n$ equipped with the operation $A \diamond B = AB^{-1}A$ for any $A, B \in \mathbb{P}_n$. Indeed, the conditions (a1), (a2) above are trivial to check. As for (a3), it is just the consequence of the Anderson-Trapp theorem what we already mentioned.

We make one further step in generality, our Mazur-Ulam type result is formulated not only for metrics but for a much more general notion defined as follows.

Definition 3 *Given an arbitrary set X, the function $d : X \times X \to [0, \infty[$ is called a generalized distance measure if it has the property that for an arbitrary pair $x, y \in X$ of points we have $d(x, y) = 0$ if and only if $x = y$.*

Generalized distance measures are also called divergences. One may think that such a general notion, so far more general than that of a metric, is artificial and not useful. But this thought is completely false. We recall that, for example, in quantum information theory that kind of distinguishability measures between states (e.g., the many different notions of relative entropy) play a very important role.

And now, our general Mazur-Ulam theorem reads as follows [17].

Theorem 4 *Let X, Y be sets equipped with binary operations $\diamond, \star$, respectively, with which they form point-reflection geometries. Let $d : X \times X \to [0, \infty[$, $\rho : Y \times Y \to [0, \infty[$ be generalized distance measures. Pick $a, b \in X$, set*

$$L_{a,b} = \{x \in X \, : \, d(a, x) = d(x, b \diamond a) = d(a, b)\}$$

and assume the following:

(b1) $d(c \diamond x, c \diamond x') = d(x', x)$ *holds for all* $c, x, x' \in X$*;*

(b1') $\rho(d \star y, d \star y') = \rho(y', y)$ *holds for all* $d, y, y' \in Y$*;*

(b2) $\sup\{d(x, b) \, : \, x \in L_{a,b}\} < \infty$*;*

(b3) there exists a constant $K > 1$ *such that* $d(x, b \diamond x) \geq K d(x, b)$ *holds for every* $x \in L_{a,b}$*.*

Let $\phi : X \to Y$ be a surjective map such that

$$\rho(\phi(x), \phi(x')) = d(x, x'), \quad x, x' \in X.$$

Then we have

$$\phi(b \diamond a) = \phi(b) \star \phi(a).$$

The proof of this general result is based on extensions of the ideas of Väisälä [23] that appeared in his proof of the original Mazur-Ulam theorem.

Two remarks should certainly be made here. First, observe that the above result is "local" in the sense that it guarantees that ϕ respects the operations $\diamond, \star$ only for a given pair a, b of elements having particular properties. Fortunately, in practice we can show in many cases that either the related conditions are satisfied for all pairs of elements or from the local respectfulness we can somehow deduce that the maps under considerations (i.e., the generalized isometries in question) respect the operations globally. Therefore, in those cases we can show that the generalized isometries are necessarily isomorphisms with respect to the pair $\diamond, \star$ of operations.

The other simple but important remark is that the above result trivially includes the original Mazur-Ulam theorem. To see this, take normed real linear spaces X, Y and an isometry $\phi : X \to Y$. Define the operation $\diamond$ on X by

$$x \diamond x' = 2x - x', \qquad x, x' \in X$$

and the operation $\star$ on Y similarly. Let d, ρ be the metrics corresponding to the norms on X and Y. Selecting any pair a, b of points in X, it is apparent that all conditions in the theorem are fulfilled and hence we have

$$\phi(2b - a) = 2\phi(b) - \phi(a).$$

It easily implies that ϕ respects the operation of the arithmetic mean from which it follows that ϕ respects all dyadic convex combinations and finally, by the continuity of ϕ, we conclude that ϕ is affine.

After these let us now see how Mazur-Ulam theorems help in the practice to determine certain generalized isometries. The general scheme of our approach is the following.

Assuming the conditions in Theorem 4 are fulfilled, by our Mazur-Ulam type result we deduce that the generalized isometries, the structure of which we are interested in, are in fact automorphisms of certain algebraic structures. Next, we try to determine the structure of those automorphisms. Clearly, this may be a highly nontrivial task but there we face an algebraic problem. This means that we can "compute" which has many advantages over considering distances. If, in the fortunate case, the automorphisms in question are described, we finally try to select those ones which are true generalized isometries.

Again, we emphasize that this scheme is not functioning in all cases, as we have seen it is basically useless when normed spaces and their corresponding isometries are studied. However, we will see below that in certain cases (highly noncommutative ones) this approach do work really fine.

Our main objects are metrical structures on the set $\mathbb{P}_n$ of all positive definite $n \times n$ matrices. We introduce a large collection of generalized distance measures on $\mathbb{P}_n$ as follows. Apparently, for any norm N on $\mathbb{M}_n$ and continuous function $f\colon]0,\infty[\to \mathbb{R}$ with the property that $f(y) = 0$ holds if and only if $y = 1$, the formula

$$d_{N,f}(A,B) = N(f(A^{-1/2}BA^{-1/2})), \qquad A,B \in \mathbb{P}_n, \tag{3}$$

defines a generalized distance measure. Of course, in most cases $d_{N,f}$ is not a true metric.

The above collection of distance measures has elements which are connected to important Finsler-type differential geometrical structures on $\mathbb{P}_n$. The positive definite cone $\mathbb{P}_n$ is a differentiable manifold in $\mathbb{H}_n$, the tangent space at any point of $\mathbb{P}_n$ can be identified with $\mathbb{H}_n$. Let N be a unitarily invariant norm on $\mathbb{M}_n$. For each $A \in \mathbb{P}_n$ we define

$$N(X)_A = N(A^{-1/2}XA^{-1/2}), \qquad X \in \mathbb{H}_n$$

and obtain a Finsler-type structure on $\mathbb{P}_n$, see [5]. It turned out in [5] that the shortest path distance $d_N(A,B)$ between $A,B \in \mathbb{P}_n$ can be computed by the formula

$$d_{N,\log}(A,B) = N(\log A^{-1/2}BA^{-1/2}), \qquad A,B \in \mathbb{P}_n.$$

The probably most important case is the one where N equals the Frobenius norm $\|.\|_2$. Then we are given a Riemannian geometry on $\mathbb{P}_n$ which has a lot of applications, see, e.g., Chapter 6 in [3]. Observe that when N equals the spectral norm $\|.\|$ we in fact obtain the Thompson metric mentioned above. Clearly, these metrics belong to the class defined in (3). But there are additional important elements of that class. A new interesting metric was recently introduced by Sra [22]. In fact, his aim was to find a substitute for the above mentioned Riemannian shortest path distance which is similarly useful but much less computation demanding. In order to define Sra's metric we need the following other notions. For any pair $A,B \in \mathbb{P}_n$, the Stein's loss $l(A,B)$ is defined by

$$l(A,B) = \mathrm{Tr}(AB^{-1}) - \log\det(AB^{-1}) - n.$$

This plays a very important role e.g. in multivariate analysis. The Jensen-Shannon symmetrization of $l(A,B)$ is the quantity

$$S_{JS}(A,B) = \frac{1}{2}\left(l\left(A,\frac{A+B}{2}\right) + l\left(B,\frac{A+B}{2}\right)\right)$$

which is called symmetric Stein divergence. It is easy to see that

$$S_{JS}(A,B) = \log\det\left(\frac{A+B}{2}\right) - \frac{1}{2}\log\det(AB), \qquad A,B\in\mathbb{P}_n.$$

In [22] Sra proved the very interesting and nice fact that

$$\delta_S(A,B) = \sqrt{S_{JS}(A,B)}, \qquad A,B\in\mathbb{P}_n,$$

is a true metric on $\mathbb{P}_n$. Now, one can see that

$$\delta_S(A,B)^2 = S_{JS}(A,B) = \operatorname{Tr}\log(Y+I)(2Y^{1/2})^{-1} = \left\|\log(Y+I)(2Y^{1/2})^{-1}\right\|_1$$

holds with $Y = A^{-1/2}BA^{-1/2}$ where, as before, $\|.\|_1$ denotes the trace-norm on $\mathbb{M}_n$. Therefore, we have $\delta_S^2 = d_{N,f}$ with $N = \|.\|_1$ and $f(y) = \log((y+1)/(2\sqrt{y}))$, $y>0$ meaning that the square of Sra's metric also belongs to our class (3).

The above distance measures are true metrics. It is easy to verify that the next theorem can be applied for them and for many other generalized distance measures, too. In particular, we mention Stein's loss, Jeffrey's Kullback-Leibler divergence, log-determinant α-divergence, etc, see [19].

Theorem 5 *Let N be a unitarily invariant norm on $\mathbb{M}_n$. Assume that $f\colon]0,\infty[\to\mathbb{R}$ is a continuous function such that*

(c1) $f(y)=0$ holds if and only if $y=1$;

(c2) there exists a number $K>1$ such that

$$|f(y^2)| \geq K|f(y)|, \qquad y\in]0,\infty[.$$

If $\phi\colon \mathbb{P}_n\to\mathbb{P}_n$ is a surjective map which satisfies

$$d_{N,f}(\phi(A),\phi(B)) = d_{N,f}(A,B), \qquad A,B\in\mathbb{P}_n, \tag{4}$$

then there exist an invertible matrix $T\in\mathbb{M}_n$ and a real number $c\neq -1/n$ such that ϕ is of one of the following forms

(1) $\phi(A) = (\det A)^c TAT^*, \quad A\in\mathbb{P}_n$;

(2) $\phi(A) = (\det A)^c TA^{-1}T^*, \quad A\in\mathbb{P}_n$;

(3) $\phi(A) = (\det A)^c TA^tT^*, \quad A\in\mathbb{P}_n$;

(4) $\phi(A) = (\det A)^c T{A^t}^{-1}T^*, \quad A\in\mathbb{P}_n$.

The idea of the proof of this result is simple, we sketch it in what follows. For details see [19]. We equip $\mathbb{P}_n$ with the point-reflection geometrical structure

$$A \diamond B = AB^{-1}A, \quad A, B \in \mathbb{P}_n$$

and apply our general Mazur-Ulam result, Theorem 4. We obtain that the generalized isometries under considerations are inverted Jordan triple automorphisms of $\mathbb{P}_n$. With some efforts, using the properties of f and the fact that any two norms on $\mathbb{M}_n$ are equivalent, we can also show that the fact that ϕ preserves the distance measure $d_{N,f}$ implies that it is continuous with respect to the operator norm. We next see that $\psi : A \longmapsto \phi(I)^{-1/2}\phi(A)\phi(I)^{-1/2}$ is a continuous inverted Jordan triple automorphism of $\mathbb{P}_n$ which also satisfies $\psi(I) = I$. Easy computation gives us that ψ is a Jordan triple automorphism of $\mathbb{P}_n$, i.e., a bijective map which respects the product ABA.

The next step of the proof of Theorem 5 is to determine the structure of those automorphisms. This was done in [16], also see [21]. In fact, there we described not only the corresponding automorphisms but all continuous Jordan triple endomorphisms, too.

Theorem 6 *Assume $n \geq 2$. Let $\phi : \mathbb{P}_n \to \mathbb{P}_n$ be a continuous map which is a Jordan triple endomorphism, i.e., ϕ is a continuous map which satisfies*

$$\phi(ABA) = \phi(A)\phi(B)\phi(A), \quad A, B \in \mathbb{P}_n. \tag{5}$$

Then there exist a unitary matrix $U \in \mathbb{M}_n$, a real number c, a set $\{P_1, \dots, P_n\}$ of mutually orthogonal rank-one projections in $\mathbb{M}_n$, and a set $\{c_1, \dots, c_n\}$ of real numbers such that ϕ is of one of the following forms

(1) $\phi(A) = (\det A)^c UAU^*, \quad A \in \mathbb{P}_n$;

(2) $\phi(A) = (\det A)^c UA^{-1}U^*, \quad A \in \mathbb{P}_n$;

(3) $\phi(A) = (\det A)^c UA^tU^*, \quad A \in \mathbb{P}_n$;

(4) $\phi(A) = (\det A)^c U{A^t}^{-1}U^*, \quad A \in \mathbb{P}_n$;

(5) $\phi(A) = \sum_{j=1}^n (\det A)^{c_j} P_j, \quad A \in \mathbb{P}_n$.

The main steps of the proof of this result in the case where $n \geq 3$ were the following, for details see [16]. First we could show that ϕ is a Lipschitz map in a neighbourhood of I. Then, using this property and the multiplicativity of ϕ in the sense of (5) we proved that the map $F : \mathbb{H}_n \to \mathbb{H}_n$ defined by $F(A) = \log \phi(e^A)$, $A \in \mathbb{H}_n$ is linear and preserves the commutativity meaning that it maps commuting matrices to commuting ones. Next we applied an old result on the structure of those important linear preservers and concluded the proof.

Obviously, we immediately have the corollary that the continuous Jordan triple automorphisms of $\mathbb{P}_n$ are of one of the forms (1)-(4) right above. Putting all the information together one can easily finish the proof of Theorem 5 for the case $n \geq 3$.

In the argument above we have assumed $n \geq 3$. The reason for this is the use of a structural result on linear commutativity preservers on $\mathbb{H}_n$ which is available only under that condition. But what happens if $n = 2$? Using a rather different, totally 2-dimensional approach, we proved that the conclusions in Theorem 6 are valid also in that case. This was done in [21]. Therefore, we have that the conclusion in Theorem 5 holds for $n = 2$, too.

Let us next present a serious application of the just mentioned result in [21]. This is the description of the endomorphism semigroup of the Einstein velocity addition which is a fundamental operation in the special theory of relativity. That operation is defined in the following way. Let $\mathbb{B} = \{u \in \mathbb{R}^3 : \|u\| < 1\}$ be the open unit ball in the 3-dimensional Euclidean space (in physics they consider the radius equal to c, the speed of light, but in mathematics we can use that simple normalization). The velocity addition (or relativistic sum) on $\mathbb{B}$ is given by

$$\oplus : \mathbb{B} \times \mathbb{B} \to \mathbb{B};\ (u,v) \mapsto u \oplus v := \frac{1}{1+\langle u,v\rangle}\left(u + \frac{1}{\gamma_u}v + \frac{\gamma_u}{1+\gamma_u}\langle u,v\rangle u\right)$$

where $\gamma_u = (1 - \|u\|^2)^{-\frac{1}{2}}$ is the so-called Lorentz factor. Our result in [20] describing the structure of all continuous endomorphisms of this structure reads as follows.

Theorem 7 *Let $\beta : \mathbb{B} \to \mathbb{B}$ be a continuous endomorphism with respect to the operation $\oplus$. Then it is either identically zero or there is an orthogonal matrix $O \in \mathbb{M}_3(\mathbb{R})$ such that*

$$\beta(v) = Ov, \quad v \in \mathbb{B}.$$

Let us go back to our original problem on generalized isometries. We emphasize that Theorem 5 tells that every considered generalized isometry must be of one of some particular forms, but it is not a precise result. Indeed, it does not assert that the maps of those forms are all generalized isometries. Clearly, one cannot even expect such an "if and only if" statement due to the generality of the norm N and the function f. If the function f is the logarithmic function, we can have a slightly more precise result which is still not of "if and only if" type due to the generality of the norm N. It was published in [16] and reads as follows.

Theorem 8 *Suppose $n \geq 2$. Let N be a unitarily invariant norm on $\mathbb{M}_n$ and $\phi : \mathbb{P}_n \to \mathbb{P}_n$ a surjective isometry relative to the metric $d_{N,\log}$. Assume $n \geq 3$ and N is not a scalar multiple of the Frobenius norm. If $n \neq 4$, then there exists an invertible matrix $T \in \mathbb{M}_n$ such that ϕ is of one of the following forms*

(1) $\phi(A) = TAT^*, \quad A \in \mathbb{P}_n;$

(2) $\phi(A) = TA^{-1}T^*, \quad A \in \mathbb{P}_n;$

(3) $\phi(A) = TA^tT^*, \quad A \in \mathbb{P}_n$;

(4) $\phi(A) = TA^{t^{-1}}T^*, \quad A \in \mathbb{P}_n$.

If $n = 4$, then beside (1)-(4) the following additional possibilities can occur

(5) $\phi(A) = (\det A)^{-2/n}TAT^*, \quad A \in \mathbb{P}_n$;

(6) $\phi(A) = (\det A)^{2/n}TA^{-1}T^*, \quad A \in \mathbb{P}_n$;

(7) $\phi(A) = (\det A)^{-2/n}TA^tT^*, \quad A \in \mathbb{P}_n$;

(8) $\phi(A) = (\det A)^{2/n}TA^{t^{-1}}T^*, \quad A \in \mathbb{P}_n$.

In the case where $n \geq 3$ and N is a scalar multiple of the Frobenius norm, ϕ is of one of the forms (1)-(8). Finally, if $n = 2$, then ϕ can be written in one of the forms (1)-(4).

We close this section with a precise result concerning Sra's metric [16].

Theorem 9 *Assume $n \geq 2$. Let $\phi : \mathbb{P}_n \to \mathbb{P}_n$ be a surjective map. It is an isometry relative to the metric δ_S if and only if there is a nonsingular matrix $T \in \mathbb{M}_n$ such that ϕ is of one of the following forms*

(1) $\phi(A) = TAT^*, \quad A \in \mathbb{P}_n$;

(2) $\phi(A) = TA^{-1}T^*, \quad A \in \mathbb{P}_n$;

(3) $\phi(A) = TA^tT^*, \quad A \in \mathbb{P}_n$;

(4) $\phi(A) = TA^{t^{-1}}T^*, \quad A \in \mathbb{P}_n$.

Further specific precise results on the structures of particular generalized distance measures can be found in [19]. We remark that related infinite dimensional results (concerning operator algebras) were given in [8], [17], see also [9]. We also call the attention to another approach to general Mazur-Ulam theorems based on the concept of generalized gyrovector spaces that was developed in [1].

3 Order automorphisms, divergence preservers II

In the present and closing section we consider some other types of divergences and the corresponding generalized isometries. We first recall the definition of the so-called Bregman divergences.

Definition 10 *For a differentiable strictly convex function $f :]0, \infty[\to \mathbb{R}$, the Bregman f-divergence on $\mathbb{P}_n$ is defined by*

$$H_f(A, B) = \operatorname{Tr}\left(f(A) - f(B) - f'(B)(A - B)\right), \quad A, B \in \mathbb{P}_n.$$

The next concept we consider here is that of the Jensen divergences.

Definition 11 *For a strictly convex function $f:]0,\infty[\to \mathbb{R}$ and for a given number $\lambda \in]0,1[$, the Jensen $\lambda - f$-divergence on $\mathbb{P}_n$ is defined by*

$$J_{f,\lambda}(A,B) = \operatorname{Tr}\left(\lambda f(A) + (1-\lambda)f(B) - f\left(\lambda A + (1-\lambda)B\right)\right), \quad A,B \in \mathbb{P}_n.$$

The reason for assuming above that f is not only convex but strictly convex is that in that case H_f and $J_{f,\lambda}$ are generalized distance measures in the sense of Definition 3 meaning that they are nonnegative and take the value 0 only at equal variables.

The probably most important examples of the above divergences are the following.

- The Bregman divergences corresponding to the generating functions $x \mapsto x\log x$ and $x \mapsto -\log x$ are called Umegaki relative entropy and Stein's loss, respectively. This latter one is also an example of the divergences of the form (3).

- The Jensen divergence with parameter $\lambda = 1/2$ and generating function $x \mapsto -\log x$ is called the symmetric Stein divergence. In general, for any $-1 \leq \alpha \leq 1$, the Jensen divergence with parameter $\lambda = (1-\alpha)/2$ and the same generating function $x \mapsto -\log x$ is called log-determinant α-divergence which is also a divergence of the form (3), see [19].

Hence, as we can see, there are overlaps between these sorts of divergences and the class of generalized distance measures which appeared in the second section of the article. In fact, it was proved that the "intersection" of the collection of Bregman divergences (resp. Jensen divergences) and that of the ones of the form $d_{N,f}$ in (3) contains only functions of the form $x \mapsto a\log x + bx + c$, where a,b,c are reals, see [18].

Now, the natural question we are interested in is the following. What are the surjective maps of $\mathbb{P}_n$ which preserve the Bregman or Jensen divergences? It is easy to see that transformations of the forms $A \longmapsto UAU^*$ or $A \longmapsto UA^tU^*$ with unitary $U \in \mathbb{M}_n$ have that property. In [18] we proved that under certain conditions on the function f, the converse statements are also true. Namely, we obtained the following.

Theorem 12 *Let f be a differentiable convex function on $]0,\infty[$ with the following conditions: f' is bounded from below and unbounded from above. Let $\phi : \mathbb{P}_n \to \mathbb{P}_n$ be a bijective map which satisfies*

$$H_f\left(\phi(A),\phi(B)\right) = H_f\left(A,B\right), \quad A,B \in \mathbb{P}_n.$$

Then there exists a unitary matrix $U \in \mathbb{M}_n$ such that ϕ is of one of the following forms

(1) $\phi(A) = UAU^*, \quad A \in \mathbb{P}_n$;

(2) $\phi(A) = UA^tU^*, \quad A \in \mathbb{P}_n$.

The corresponding result concerning Jensen divergences reads as follows.

Theorem 13 *Let f be a strictly convex function on $]0,\infty[$ with the following conditions: $\lim_{x\to 0+} f(x)$ exists and finite, f is differentiable and f' is unbounded from above. Pick $\lambda \in]0,1[$. If $\phi : \mathbb{P}_n \to \mathbb{P}_n$ is a surjective map which satisfies*

$$J_{f,\lambda}(\phi(A), \phi(B)) = J_{f,\lambda}(A, B), \quad A, B \in \mathbb{P}_n,$$

then there exists a unitary matrix $U \in \mathbb{M}_n$ such that ϕ is of one of the following forms

(1) $\phi(A) = UAU^*, \quad A \in \mathbb{P}_n$;

(2) $\phi(A) = UA^tU^*, \quad A \in \mathbb{P}_n$.

As for the proofs of the above results we note the following. Similarly to the case of divergences of the form $d_{N,f}$ in (3) considered in the previous section of the paper, the proofs here also heavily rely on the structure of certain automorphisms of $\mathbb{P}_n$. Namely, this time those are the order automorphisms. By order we mean the usual partial order coming from positive semidefiniteness. I.e., for $A, B \in \mathbb{H}_n$ we write $A \leq B$ if and only if the matrix $B - A$ is positive semidefinite. The next result appeared in [15], cf. [12].

Theorem 14 *Let $\phi : \mathbb{P}_n \to \mathbb{P}_n$ be a bijective map which is an order automorphism, i.e., assume that for any $A, B \in \mathbb{P}_n$ we have $A \leq B$ if and only if $\phi(A) \leq \phi(B)$. Then there exists a nonsingular matrix $T \in \mathbb{M}_n$ such that ϕ is of one of the following forms*

(1) $\phi(A) = TAT^*, \quad A \in \mathbb{P}_n$;

(2) $\phi(A) = TA^tT^*, \quad A \in \mathbb{P}_n$.

Now the proof of Theorem 12 (and also that of Theorem 13) is essentially based on a characterization of the order $\leq$ on $\mathbb{P}_n$ expressed by the divergences in question. Namely, one can verify that for any given $B, C \in \mathbb{P}_n$, the set

$$\{H_f(B, A) - H_f(C, A) | A \in \mathbb{P}_n\}$$

of real numbers is bounded from below if and only if $B \leq C$. Having this characterization in mind, it is apparent that the transformation ϕ in Theorem 12 is an order automorphism of $\mathbb{P}_n$. One can apply Theorem 14 and then prove that T is necessarily unitary.

Unfortunately, because of its conditions, Theorem 12 does not cover the cases of the important convex functions $x \mapsto x \log x - x$ and $x \mapsto -\log x$. As we have learned, the former function gives rise to the Umegaki relative entropy, the latter one to the Stein's loss. But it was also noted above that Stein's loss is a member of the collection of generalized distance measures defined in (3), so the corresponding preservers are described due to the results given in the previous section. As for the Umegaki relative entropy, the corresponding generalized ismetries can still be described by using order automorphisms, see [18].

As for Theorem 13, it does not cover the case of the function $x \mapsto -\log x$, i.e., where the Jensen divergence is the symmetric Stein divergence, or, more generally a log-determinant α-divergence. Again, we can point out that those divergences belong to the class considered in the previous section of the paper and therefore their preservers can be determined using the results presented there.

We finish the paper mentioning that the structural result Theorem 14 on the order automorphisms of $\mathbb{P}_n$ was used in the descriptions of the preservers of further divergences, too. For example, in our very recent paper [6] in a similar manner we determined the bijective maps of $\mathbb{P}_n$ which preserve the quantum Rényi divergences.

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