数理解析研究所講究録
第2035巻 2017年 120-121

METRIZABILITY OF SPACES OF LIPSCHITZ FUNCTIONS

A. JIMÉNEZ-VARGAS

1. INTRODUCTION

Let $\mathrm{Lip}_0(E)$ be the linear space of all scalar-valued Lipschitz functions vanishing at 0 on a normed space E. Let τ be a locally convex topology on $\mathrm{Lip}_0(E)$ such that $\tau_0 \leq \tau \leq \tau_\delta$, where τ_0 and τ_δ denote the compact-open topology and the Nachbin–Couré topology on $\mathrm{Lip}_0(E)$.

We prove in this note that $(\mathrm{Lip}_0(E), \tau_0)$ is a metrizable space if and only if E has finite dimension. Motivated by a positive answer in the setting of holomorphic mappings, the following question is raised: Is it true that $(\mathrm{Lip}_0(E), \tau)$ is metrizable only if E is finite-dimensional ?

2. PRELIMINARIES

Let E be a normed space and let $\mathrm{Lip}_0(E)$ denote the linear space of all Lipschitz mappings f from E into $\mathbb{K}$ for which $f(0) = 0$. We refer the reader to Weaver's book [6] for the basic theory of $\mathrm{Lip}_0(E)$.

Let X be a topological space and let $C(X)$ be the linear space of all continuous mappings from X into $\mathbb{K}$. We recall the following topologies on $C(X)$.

The compact-open topology on $C(X)$ is the locally convex topology generated by the seminorms

$$|f|_K = \sup_{x\in K} |f(x)|, \qquad f \in C(X),$$

where K varies over the family of all compact subsets of X.

A seminorm p on $C(X)$ is ported by the compact subset K of X if for every open neighborhood V of K in X, there is a constant $c_V > 0$ such that $p(f) \leq c_V \sup_{x\in V} |f(x)|$ for all $f \in C(X)$. The Nachbin topology on $C(X)$ is the locally convex topology generated by the seminorms on $C(X)$ which are ported by the compact subsets of X.

The Nachbin–Couré topology on $C(X)$ is the locally convex topology generated by the seminorms p on $C(X)$ which satisfy the following property: for each increasing countable open cover $\{V_n\}_{n\in\mathbb{N}}$ of X, there are $m \in \mathbb{N}$ and $c_m > 0$ such that $p(f) \leq c_m \sup_{x\in V_m} |f(x)|$ for all $f \in C(X)$.

We will denote by τ_0, τ_γ and τ_δ the compact-open topology, the Nachbin-ported topology and the Nachbin–Couré topology on $C(X)$, or on any linear subspace of $C(X)$.

Now we prove the following result.

Theorem 2.1. *If E is a Banach space, then $(\mathrm{Lip}_0(E), \tau_0)$ is metrizable if and only if E has finite dimension.*

Proof. Suppose that $(\mathrm{Lip}_0(E), \tau_0)$ is metrizable. Then there exists a sequence $\{K_n\}_{n\in\mathbb{N}}$ of compact subsets of E, containing the origin, such that the sequence of seminorms $|\cdot|_{K_n}$ defines the topology τ_0 on $\mathrm{Lip}_0(E)$. We claim that there exists a constant $c > 0$ such that E is included in $\cup_{n\in\mathbb{N}} c\overline{\Gamma}(K_n)$, where $\overline{\Gamma}(K_n)$ denotes the closed, convex, balanced hull of K_n in E. Indeed, given $x \in E$, it is clear that $|\cdot|_{\{x\}}$ defined on $\mathrm{Lip}_0(E)$ is a continuous seminorm on $(\mathrm{Lip}_0(E), \tau_0)$, so there are $m \in \mathbb{N}$ and $c > 0$ such that $|f|_{\{x\}} \leq c|f|_{K_m}$ for all $f \in \mathrm{Lip}_0(E)$. It follows that $|f(x)| \leq c|f|_{\overline{\Gamma}(K_m)}$ for all $f \in \mathrm{Lip}_0(E)$. Notice that each $\overline{\Gamma}(K_n)$ is compact by the Mazur theorem. Since the dual space E' is a subset of $\mathrm{Lip}_0(E)$, we have $|f(x)| \leq c|f|_{\overline{\Gamma}(K_m)}$ for all $f \in E'$. By the Hahn–Banach separation theorem, we infer that x is in $c\overline{\Gamma}(K_m)$ as we wanted. Since E is a Baire space, our claim implies that there exists $p \in \mathbb{N}$ such that $\overline{\Gamma}(K_p)$ has no empty interior in E. Hence there is a compact neighborhood of 0 in E and therefore E has finite dimension by the Riesz theorem.

Conversely, if E is finite dimensional, then $(C(E), \tau_0)$ is metrizable (see the proof of [3, Theorem 16.9]) and therefore so is $(\mathrm{Lip}_0(E), \tau_0)$. ☐

The results on the metrizability of spaces of holomorphic functions have an interesting history. In 1968, Alexander [1] proved the following theorem for Banach spaces with Schauder basis, which was generalized by Chae (see [3, Theorem 16.10]): If U is an open subset of an infinite dimensional Banach space E and

τ is a topology on the space $H(U)$ of all holomorphic functions on U finer than the topology of pointwise convergence, then $(H(U), \tau)$ is not metrizable.

In 2007, this theorem probably motivated Ansemil and Ponte, whose paper [2] contains that if U is an open subset of an infinite-dimensional complex metrizable locally convex space E, then $(H(U), \tau_\gamma)$ is not metrizable. This answered a question stated by Mujica in [5, Problem 11.9] thirty years ago. It is known that $\tau_0 \leq \tau_\gamma \leq \tau_\delta$ on $H(U)$.

In 2009, López-Salazar [4] improved this result showing that if U is an open subset of a complex metrizable locally convex space E and τ is a locally convex topology on $H(U)$ such that $\tau_0 \leq \tau \leq \tau_\delta$, then $(H(U), \tau)$ is a metrizable space if and only if E has finite dimension.

Theorem 2.1 suggests to tackle the problem on the metrizability of $\mathrm{Lip}_0(E)$ equipped with other topologies, with an approach similar to that described above for spaces of holomorphic functions.

References

[1] H. Alexander, Analytic functions on Banach spaces, Thesis, University of California, Berkeley, 1968.
[2] José M. Ansemil and Socorro Ponte, Metrizability of spaces of holomorphic functions with the Nachbin topology, J. Math. Anal. Appl. **334** (2007), no. 2, 1146–1151.
[3] S. B. Chae, Holomorphy and Calculus in Normed Spaces, Marcel Dekker, 1985.
[4] Jerónimo López-Salazar, Metrizability of spaces of holomorphic functions, J. Math. Anal. Appl. **355** (2009), no. 1, 434–438.
[5] J. Mujica, Gérmenes holomorfos y funciones holomorfas en espacios de Fréchet, Publicaciones del Departamento de Teoría de Funciones, Universidad de Santiago, Spain, 1978.
[6] N. Weaver, Lipschitz Algebras, World Scientific Publishing Co., Singapore, 1999.

Departamento de Matemáticas, Universidad de Almería, 04120 Almería, Spain
E-mail address: ajimenez@ual.es