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# TOWARDS THE BILINEAR ISOMETRIES ON SPACES OF LIPSCHITZ FUNCTIONS

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## 1. Some results about linear isometries on spaces of continuous functions

The classical Banach–Stone theorem [4, 30] states that if $X$ and $Y$ are compact Hausdorff topological spaces and $T$ is a linear isometry from $C(X)$ onto $C(Y)$ (endowed with the supremum norm), then there exists a homeomorphism from $Y$ onto $X$ and a continuous function $\tau$ from $Y$ into $S_{\mathbb{K}}$ such that

$$Tf = \tau \cdot (f \circ \varphi) \quad (f \in C(X)).$$

An important generalization of the Banach-Stone theorem was given by Holsztyński in [14, 1966] by considering into isometries. His theorem asserts that if $T$ is a linear isometry from $C(X)$ into $C(Y)$, then there exists a closed subset $Y_0$ of $Y$, a continuous surjective map $\varphi$ from $Y_0$ onto $X$, and a norm-one element $\tau \in C(Y)$ with $|\tau(y)| = 1$ for all $y \in Y_0$ such that

$$Tf(y) = \tau(y)f(\varphi(y)) \quad (f \in C(X),\ y \in Y_0).$$

These results have been generalized in many ways. We can cite the works by Jeang and Wong [16, 1996] on spaces of continuous scalar-valued functions vanishing at infinity, by Araujo and Font [3, 1997] on certain subspaces of scalar-valued continuous functions, by Hatori and Miura [13, 2013] on unifomly closed function algebras, by Koshimizu, Miura, Takagi and Takahasi [23, 2014], etc.

In the vector-valued case, on the one hand, Jerison [17, 1950] extended the Banach–Stone theorem and, on the other hand, Cambern [8, 1978] improved the Holsztyński theorem by characterizing into linear isometries between spaces of vector-valued continuous functions. Subsequently, many other studies have been published on this subject (see the monograph [9]). To mention a recent one, we cite Kawamura's work [22, 2016] concerning surjective linear isometries between certain subspaces of vector-valued continuous functions.

This type of results can be very useful. For example, Botelho and Jamison [6, 2008] investigated the algebraic and topological reflexivity of $C(X)$ and $C(X,E)$ by using the representations of the into isometries given by Holsztyński and Cambern (extending, in this way, a theorem of Molnár and Zalar [25]).

## 2. Linear isometries on spaces of Lipschitz functions

Let us recall that a map $f\colon X \to Y$ between metric spaces is said to be *Lipschitz* if

$$L(f) = \sup\left\{\frac{d_Y(f(x),f(y))}{d_X(x,y)} : x,y \in X,\ x \neq y\right\} < \infty.$$

In such case, $L(f)$ is called the *Lipschitz constant* of $f$.

Given a metric space $X$ and a normed space $E$, we denote by $\mathrm{Lip}(X, E)$ the vector space of all bounded Lipschitz functions $f\colon X \to E$. If $E$ is the field of real or complex numbers, we shall write simply $\mathrm{Lip}(X)$.

On $\mathrm{Lip}(X, E)$, it is usually considered the norm $\|f\| = \max\{\|f\|_\infty, L(f)\}$, where $\|f\|_\infty$ is the supremum norm of $f$. If $E$ is a Banach space, then $(\mathrm{Lip}(X, E), \|\cdot\|)$ is a Banach space too.

The spaces of Lipschitz functions appear in the works of many authors. See for example [29, 20, 32, 21, 12, 1, 7]. In particular, the study of the surjective linear isometries between Lipschitz-spaces started with Roy [28, 1968], Vasavada [31, 1969] and Novinger [27, 1975]. Later Weaver [32, 1999] improved these results by taking complete and 1-connected metric spaces (a metric space is $r$-connected if it cannot be decomposed into two nonempty disjoint sets whose distance is greater than or equal to $r$). On the other hand, Mayer-Wolf [24, 1981] provided a description of the surjective linear isometries on spaces of Hölder functions different from a weighted composition operator.

For our part, we stated a Lipschitz version of the Holsztyński theorem for into linear isometries (not necessarily surjective) on Lipschitz spaces [18], only under the assumption that the linear isometry takes the constant function 1 into a contraction. Moreover we extended our result to the vector-valued case [19], obtaining in this way a Lipschitz version of Cambern's theorem.

More recently, Botelho, Fleming and Jamison [5, 2011] gave a description of the linear surjective isometries on $\mathrm{Lip}(X, E)$ under weaker conditions by using extreme points of the ball of the dual $\mathrm{Lip}(X, E)^*$.

Finally Araujo and Dubarbie [2, 2011] gave a complete description of surjective linear isometries in a very general setting (only strict convexity on the normed spaces $E$ and $F$ is assumed). They considered *standard isometries* and *purely nonstandard isometries*. A map $T\colon \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ is a *standard isometry* if it has the form

$$T(f)(y) = Jy(f(\varphi(y))) \qquad (f \in \mathrm{Lip}(X, E), y \in Y),$$

where $Jy\colon E \to F$ is a surjective linear isometry for each $y \in Y$, the map $J$ is constant on each 2-component of $Y$, and $\varphi\colon Y \to X$ satisfies that both $\varphi$ and $\varphi^{-1}$ preserve distances less than 2. The *purely nonstandard isometries*, however, are not weighted composition operators on a part of the metric space $Y$. Concretely, $S_\psi\colon \mathrm{Lip}(Y, F) \to \mathrm{Lip}(Y, F)$ is a *purely nonstandard isometry* if it can be described by

$$S_\psi(f)(y) = \begin{cases} f(y) & \text{if } y \in \mathcal{B}, \\ f(\psi(y)) - f(y) & \text{if } y \in \mathcal{U}. \end{cases}$$

where $\{\mathcal{B}, \mathcal{U}\}$ is certain partition of $Y$, and $\psi\colon \mathcal{U} \to \mathcal{B}$ is a map with certain metric properties. Araujo and Dubarbie proved that every nonstandard surjective isometry is the composition of a standard and a purely nonstandard isometry (when we are not in the case $E$ and $F$ complete and $X$ or $Y$ not complete).

## 3. Bilinear isometries on spaces of continuous functions

In the setting of bilinear isometries, we do not find such an extensive literature. The first result that we can cite is a bilinear version of the Holsztyński theorem obtained by Moreno and Rodríguez [26, 2005]. They proved that if $X, Y, Z$ are compact Hausdorff spaces and $\Phi : C(X) \times C(Y) \to C(Z)$ is a bilinear mapping satisfying $\|\Phi(f, g)\| = \|f\|\,\|g\|$ for every $(f, g) \in C(X) \times C(Y)$, then there exists a closed subset $Z_0$ of $Z$, a continuous surjective mapping $\varphi\colon Z_0 \to X \times Y$, and a norm-one function $\tau \in C(Z)$ with $|\tau(z)| = 1$ such that

$$\Phi(f,g)(z) = \tau(z)\, f(\pi_X(\varphi(z)))\, g(\pi_Y(\varphi(z)))$$

for all $(f,g) \in C(X) \times C(Y)$ and $z \in Z_0$ (where $\pi_X, \pi_Y$ stand for the natural coordinate projections).

Moreno and Rodríguez's theorem was extended by Font and Sanchis. Firstly, to the case of certain subspaces of scalar-valued continuous functions [10, 2010]; and, secondly, to the case of vector-valued continuous functions [11, 2012]. Moreover Hosseini, Font and Sanchis got a multilinear version of that theorem [15, 2015].

## 4. Bilinear isometries on spaces of Lipschitz functions

We follow the ideas of Moreno and Rodríguez and get a description of the bilinear isometries of $\mathrm{Lip}(X)$-spaces. Concretely, our main theorem is

**Theorem 4.1.** *Let $X, Y, Z$ be compact metric spaces, and $\Phi\colon \mathrm{Lip}(X) \times \mathrm{Lip}(Y) \to \mathrm{Lip}(Z)$ be a bilinear mapping taking the pair of constant functions one $(1_X, 1_Y)$ into a contraction and satisfying $\|\Phi(f,g)\| = \|f\|\,\|g\|$ for every $(f,g) \in \mathrm{Lip}(X) \times \mathrm{Lip}(Y)$. Then there exist a closed subset $Z_0$ of $Z$, a surjective mapping $\varphi\colon Z_0 \to X \times Y$ and a function $\tau \in \mathrm{Lip}(Z)$ with $|\tau(z)| = 1$ for every $z \in Z_0$ such that*

$$\Phi(f,g)(z) = \tau(z)\, f(\varphi_1(z))\, g(\varphi_2(z)) \qquad ((f,g) \in \mathrm{Lip}(X) \times \mathrm{Lip}(Y), z \in Z_0).$$

*Here, $\varphi_1\colon Z_0 \to X$ and $\varphi_2\colon Z_0 \to Y$ denote the compositions of $\varphi$ with the natural coordinate projections. Moreover, if it is considered on $X \times Y$ the maximum distance $d_\infty$, it holds that $\varphi$ is Lipschitz with $L(\varphi) \leq \max\{1, \mathrm{diam}(X)/2, \mathrm{diam}(Y)/2\}$, and $d_\infty(\varphi(z), \varphi(z')) \leq d(z,z')$ for every $z, z' \in Z_0$ with $d(z,z') < 2$.*

By taking in this theorem the space $Y$ reduced to a point, we can get as a consequence the description of the into linear isometries given earlier in [18, Theorem 2.4].

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