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WEIGHTED COMPOSITION OPERATORS BETWEEN DIFFERENTIABLE FUNCTION ALGEBRAS

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1. Introduction

Let $C(X)$ be the Banach algebra of all continuous complex-valued functions on a compact Hausdorff space X with the uniform norm,

$$\|f\|_X = \sup_{x\in X} |f(x)|, \qquad f \in C(X).$$

A Banach function algebra is a subalgebra B of $C(X)$ which separates the points of X, contains the constants and is complete under an algebra norm. If the algebra norm on B is equivalent to the uniform norm, then the subalgebra B is called a uniform algebra.

A function algebra B on a compact Hausdorff space X is *natural* if every nonzero complex homomorphism on B is an evaluation homomorphism at any point of X [7, 4.1.3]. For each $x \in X$, the evaluation map δ_x is defined by $\delta_x(f) = f(x)$ for every function $f \in B$. In the case where B is a Banach function algebra on X, we say that B is natural if its maximal ideal space $\mathcal{M}(B)$ coincides with X.

Let A and B be linear spaces of functions on sets X and Y, respectively. Let u be a complex-valued function on Y, and φ be a map from Y to X. A linear operator uC_φ, defined by

$$uC_\varphi f = u(f\circ\varphi), \quad f \in A$$

is called a weighted composition operator from A to B, whenever $u(f\circ\varphi) \in B$ for each $f \in A$. The operator uC_φ can be regarded as a generalization of a multiplication operator and a composition operator. In the case where $u = 1$, the operator uC_φ reduces to the

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composition operator C_φ. In the case where $X = Y$ and $\varphi(x) = x$, it reduces to the multiplication operator M_u.

Using the closed graph theorem, every weighted composition operator from a Banach function algebra to another is automatically continuous and therefore a bounded linear operator between them.

A complex-valued function f defined on a perfect compact plane set X is complex-differentiable on X if at each point $z_0 \in X$ the limit

$$f'(z_0) = \lim_{\substack{z \to z_0 \\ z \in X}} \frac{f(z) - f(z_0)}{z - z_0},$$

exists. The n-th complex-derivative of f is denoted by $f^{(n)}$.

Suppose that $D^n(X)$ is the algebra of n-times continuously complex-differentiable functions on a perfect compact plane set X. This algebra with the norm

$$\|f\|_n = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_X}{k!} \qquad (f \in D^n(X)),$$

is a normed function algebra on X which is not necessarily complete, even for a fairly nice X. For example, Bland and Feinstein in [4, Theorem 2.3] showed that if a compact, perfect plane set X has infinitely many components then the algebra $D^n(X)$ is incomplete. By standard methods, the completeness of $D^1(X)$ implies the completeness of $D^n(X)$ for each $n \in \mathbb{N}$. As Bland and Feinstein showed in [4, Theorem 2.5], there exists an example of a set X which is the image of a rectifiable Jordan arc in the plane and yet $D^1(X)$ is incomplete. Therefor, the completeness of $D^1(X)$ is far from being a topological property of X. To provide a sufficient condition for the completeness of $D^1(X)$, let us recall the definition of pointwise regularity and uniform regularity for compact plane sets.

Definition 1.1. Let X be a compact plane set with more than one point.

(i) X is called pointwise regular if for each $z_0 \in X$ there exists a constant c_{z_0} such that, for every $z \in X$ there exists a rectifiable path $\gamma : [a, b] \to X$ with $\gamma(a) = z_0$, $\gamma(b) = z$ and $|\gamma| \leq c_{z_0}|z - z_0|$ where $|\gamma|$ is the length of the path γ.

(ii) X is called uniformly regular if there exists a constant c such that for all $z, w \in X$, there exists a rectifiable path $\gamma : [a, b] \to X$ with $\gamma(a) = z$, $\gamma(b) = w$ and $|\gamma| \leq c|z - w|$.

Clearly all pointwise and uniformly regular sets are perfect and path-connected. We note that every convex compact plane set is obviously uniformly regular. There are also non-convex uniformly regular sets, like the Swiss cheese defined in [14]. Clearly there are pointwise regular sets which are not uniformly regular. For example, the union of two closed discs tangent from outside is a pointwise regular set which is not uniformly regular. It is also interesting to note that if the boundary of a compact plane set X satisfies one of these two regularity conditions then it satisfies the same condition (of course this is not a necessary condition), see [4, Theorem 3.5].

We now provide sufficient conditions for the completeness of $D^1(X)$. Dales and Davie in [8, Theorem 1.6] showed that when X is a finite union of uniformly regular sets, for each $z_0 \in X$ there exists a constant c_{z_0} such that for all $f \in D^1(X)$ and each $z \in X$,

$$|f(z) - f(z_0)| \leq c_{z_0}|z - z_0|(\|f\|_X + \|f'\|_X). \tag{1.1}$$

Using this inequality, they obtained the following result.

Theorem 1.2. [8, Theorem 1.6] *If X is a compact plane set which is a finite union of uniformly regular sets, then $D^n(X)$ is a Banach function algebra on X.*

Later in [11], it was shown that the condition (1.1) is still valid when X is a finite union of pointwise regular sets. In fact, in [11], it was shown that the condition (1.1) is a necessary and sufficient condition for the completeness of $D^1(X)$.

Theorem 1.3. [11] *Let X be a compact plane set. Then $D^1(X)$ is complete if and only if for each $z_0 \in X$ there exists a constant c_{z_0} such that for all $f \in D^1(X)$ and each $z \in X$,*

$$|f(z) - f(z_0)| \leq c_{z_0}|z - z_0|(\|f\|_X + \|f'\|_X).$$

As a consequence of the above theorem, the following result was also established.

Theorem 1.4. [11] *If X is a finite union of pointwise regular sets, then $D^n(X)$ is a Banach function algebra on X.*

In general, it is not known whether or not the converse of this theorem holds true. However, as it was proved in [9], there are several classes of connected, compact plane sets X for which the completeness of $D^1(X)$ is equivalent to the pointwise regularity of

X. For example, this is true for all rectifiably connected, polynomially convex, compact plane sets with empty interior, for all star-shaped, compact plane sets, and for all Jordan arcs in $\mathbb{C}$. Note that in Theorem 1.3, X need not be connected.

As it was shown in [8], the algebra $D^n(X)$ is natural when X is uniformly regular. However, as mentioned in [12], one can show that the algebra $D^n(X)$ is natural for every perfect compact plane set X (see also [9, Theorem 4.1]).

In this article, we discuss the boundedness and compactness of weighted composition operators acting on algebras $D^n(X)$ when perfect compact plane sets X satisfy the condition (1.1). In the case that $u = 1$, we give a necessary and sufficient condition for the composition operators between two Banach algebras $D^n(X)$ and $D^m(Y)$ to be bounded and compact. As a consequence, we state certain results about power compact and quasicompact composition operators on these algebras. Then using these results, by giving examples we show that there exist quasicompact or Riesz operators on these algebras which are not power compact.

2. Boundedness and compactness of uC_φ on $D^n(X)$

It is known that if $u, \varphi \in D^n(X)$, then uC_φ is a weighted composition operator on $D^n(X)$. Conversely, if uC_φ is a weighted composition operator on $D^n(X)$, then $u, u\varphi \in D^n(X)$ since $D^n(X)$ contains the constant functions and the coordinate function z. Although, φ does not necessarily belong to $D^n(X)$ as it may not be even continuous on X. The following theorem gives a necessary and sufficient condition on u and φ for uC_φ to be a weighted composition operator on $D^1(X)$.

Theorem 2.1. [2, Theorem 2.1] *Let X be a perfect compact plane set. Let u be a complex-valued function on X, and φ be a self-map of X not necessarily continuous. Then uC_φ is a weighted composition operator on $D^1(X)$ if and only if u and $u\varphi$ belong to $D^1(X)$.*

In general, for a constant self-map φ of X, the weighted composition operator uC_φ on a normed function algebra B on X is a rank one operator, so it is compact. We now give a sufficient condition for compactness of uC_φ on $D^n(X)$ for those φ which are not constant self-maps of X.

Theorem 2.2. [2, Theorem 2.2] *Let X be a perfect compact plane set satisfying the condition* (1.1). *Let $u, \varphi \in D^n(X)$. If $\varphi(\operatorname{coz}(u)) \subseteq \operatorname{int}X$, then the weighted composition operator uC_φ is compact on $D^n(X)$, where* $\operatorname{coz}(u) = \{z \in X : u(z) \neq 0\}$.

The condition $\varphi(\operatorname{coz}(u)) \subseteq \operatorname{int}X$ is also necessary for compactness of weighted composition operators uC_φ on algebras $D^n(X)$ for certain compact plane sets X. This is indeed the motivation for the following definition.

Definition 2.3. A plane set X has an internal circular tangent at $\zeta \in \partial X$ if there exists an open disc U such that $\zeta \in \partial U$ and $\overline{U} \setminus \{\zeta\} \subseteq \operatorname{int}X$. A plane set X is strongly accessible from the interior if it has an internal circular tangent at each point of its boundary.

A compact plane set X is said to have a peak boundary with respect to a set $B \subseteq C(X)$ if for each $\zeta \in \partial X$ there exists a non-constant function $h \in B$ such that $\|h\|_X = h(\zeta) = 1$.

The closed unit disc $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $\overline{\Delta}(z_0, r) \setminus \cup_{k=1}^n \Delta(z_k, r_k)$ where closed discs $\overline{\Delta}(z_k, r_k)$ are mutually disjoint in $\Delta(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ are examples of plane sets which are strongly accessible from the interior. Moreover, if X is a compact plane set such that $\mathbb{C} \setminus X$ is strongly accessible from the interior, then X has a peak boundary with respect to every subset of $C(X)$ which contains the rational functions with poles off X, in particular, with respect to $D^n(X)$. To see this, take $\zeta \in \partial X$. Then there exists a disc $D = D(z_0, r)$ such that $\zeta \in \partial D$ and $\bar{D} \setminus \{\zeta\} \subseteq \mathbb{C} \setminus X$. The function $h(z) = \dfrac{\zeta - z_0}{z - z_0}$ satisfies the conditions in the definition of the peak boundary (see [3, 15]).

Theorem 2.4. [2, Theorem 2.5] *Let X be a perfect compact plane set with connected interior satisfy the condition* (1.1), *be strongly accessible from the interior and have a peak boundary with respect to $D^n(X)$. Let a complex function u and a self-map φ of X be in $D^n(X)$. If the weighted composition operator uC_φ on $D^n(X)$ is compact, then either φ is constant or $\varphi(\operatorname{coz}(u)) \subseteq \operatorname{int}X$.*

In the case where $u = 1$, the weighted composition operator uC_φ reduces to the composition operators C_φ. The following corollary can be concluded immediately from the above theorems for composition operators C_φ on $D^n(X)$.

Corollary 2.5. [2, Corollary 2.6] *Let X be a perfect compact plane set satisfying the condition* (1.1). *Let a self-map φ of X be in $D^n(X)$.*

(i) *If either* φ *is constant or* $\varphi(X) \subseteq \mathrm{int}X$, *Then* C_φ *is compact on* $D^n(X)$.

(ii) *Let* X *be strongly accessible from the interior, have a peak boundary with respect to* $D^n(X)$ *and let* $\mathrm{int}X$ *be connected. If* C_φ *is compact on* $D^n(X)$, *then either* φ *is constant or* $\varphi(X) \subseteq \mathrm{int}X$.

Corollary 2.6. *Let* C_φ *be a composition operator on* $D^n(\overline{\mathbb{D}})$ *induced by a self-map* φ *of* $\overline{\mathbb{D}}$. *Then* C_φ *is compact if and only if either* φ *is constant or* $\varphi(\overline{\mathbb{D}}) \subseteq \mathbb{D}$.

3. COMPOSITION OPERATORS BETWEEN THE ALGEBRAS $D^n(X)$ AND $D^m(Y)$

In this section, we discuss the composition operators between the algebras of continuously complex differentiable functions.

Let X, Y be two perfect compact plane sets and n, m be two positive integers with $m \leq n$. Then for a map $\varphi : Y \to X$, C_φ is a composition operator from $D^n(X)$ into $D^m(Y)$ if and only if $\varphi \in D^m(Y)$. If X satisfies the condition (1.1) and $m < n$, then by using the Arzela-Ascoli Theorem, one can show that the condition $\varphi \in D^m(Y)$ is a sufficient condition for compactness of composition operator C_φ. But in the case $n = m$, by Corollary 2.5, this condition is not sufficient for compactness of C_φ. Thus, we have the following results for composition operators.

Theorem 3.1. *Let* X, Y *be two perfect compact plane sets satisfying the condition (1.1) and* n, m *be two positive integers with* $m < n$. *Then the following conditions are equivalent.*

(i) $\varphi \in D^m(Y)$.

(ii) C_φ *is a bounded operator from* $D^n(X)$ *into* $D^m(Y)$.

(iii) C_φ *is a compact operator from* $D^n(X)$ *into* $D^m(Y)$.

Theorem 3.2. *Let* X, Y *be two perfect compact plane sets satisfying the condition (1.1),* n *be a positive integer and the map* $\varphi : Y \to X$ *be in* $D^n(Y)$.

(i) *If either* φ *is constant or* $\varphi(Y) \subseteq \mathrm{int}X$, *then* C_φ *is a compact operator from* $D^n(X)$ *into* $D^n(Y)$.

(ii) *Let* X *have a peak boundary with respect to* $D^n(X)$ *and let* Y *be strongly accessible from the interior. Assume that* $\mathrm{int}X$ *is connected. If* C_φ *is a compact operator from* $D^n(X)$ *into* $D^n(Y)$, *then either* φ *is constant or* $\varphi(Y) \subseteq \mathrm{int}X$.

For the case $n < m$, we need the following formula for higher derivatives of composite functions which is known as Faà di Bruno's formula [1, page 823].

Let $f : X \to \mathbb{C}$ and $\varphi : Y \to X$ be n-times continuously differentiable functions. Then

$$(f \circ \varphi)^{(n)} = \sum_{j=1}^{n} (f^{(j)} \circ \varphi) \cdot \psi_{j,n},$$

where

$$\psi_{j,n} = \sum_a \left(\frac{n!}{a_1! a_2! \cdots a_n!} \prod_{i=1}^{n} \left(\frac{\varphi^{(i)}}{i!} \right)^{a_i} \right),$$

the sum $\sum_a$ is taken over all non-negative integers $a_1, a_2, \ldots, a_n$ satisfying $a_1 + a_2 + \cdots + a_n = j$ and $a_1 + 2a_2 + \cdots + na_n = n$. For example, $\psi_{1,n} = \varphi^{(n)}$ and $\psi_{n,n} = (\varphi')^n$.

Theorem 3.3. *Let X, Y be two perfect compact plane sets. Let n, m be two positive integers with $n < m$. If $\varphi \in D^m(Y)$ and $\varphi(Y) \subseteq \operatorname{int}X$, then C_φ is a compact operator from $D^n(X)$ into $D^m(Y)$.*

Proof. First we show that C_φ is a bounded operator from $D^n(X)$ into $D^m(Y)$. Let $f \in D^n(X)$. Then f is analytic and so infinitely differentiable in $\operatorname{int}X$. In particular, f is m-times continuously differentiable on the compact subset $\varphi(Y) \subseteq \operatorname{int}X$. Thus, using Faà di Bruno's formulas, $C_\varphi(f) = f \circ \varphi \in D^m(Y)$. Hence C_φ is a composition operator from $D^n(X)$ into $D^m(Y)$.

We now prove the compactness of C_φ. To do this, let $\{f_k\}$ be a bounded sequence in $D^n(X)$ with $\|f_k\|_n = \sum_{r=0}^{n} \frac{\|f_k^{(r)}\|_X}{r!} \leq 1$. Then $\{f_k\}$ is a uniformly bounded sequence of analytic functions in $\operatorname{int}X$. Thus it is a normal family in the sense of Montel and by using a subsequence if necessary, we may assume that there exists a function f analytic in $\operatorname{int}X$ with $f_k \to f$ uniformly on compact subsets of $\operatorname{int}X$. Also, by [6, VII, Theorem 2.1], $f_k^{(r)} \to f^{(r)}$ uniformly on compact subsets of $\operatorname{int}X$ for each $r \geq 0$. By assumption, $\varphi(Y) \subseteq \operatorname{int}X$, so one can define a function F on Y by $F(y) = f(\varphi(y))$. Since f is an analytic function in $\operatorname{int}X$, it is infinitely differentiable function on $\operatorname{int}X$, in particular, it is m-times continuously differentiable on $\operatorname{int}X$. Also, note that $\varphi \in D^m(Y)$ and therefore $F = f \circ \varphi \in D^m(Y)$. Using Faà di Bruno's formulas, we show that $C_\varphi(f_k) \to F$ in $D^m(Y)$

as $k \to \infty$.

$$\begin{aligned}\|C_\varphi(f_k) - F\|_m =& \sum_{r=0}^{m} \frac{\|(f_k \circ \varphi - F)^{(r)}\|_Y}{r!} = \sum_{r=0}^{m} \frac{\|((f_k - f)\circ\varphi)^{(r)}\|_Y}{r!} \\ \leq& \|(f_k - f)\circ\varphi\|_Y + \sum_{r=1}^{m}\frac{1}{r!}\sum_{j=1}^{r}\|(f_k - f)^{(j)}\circ\varphi\|_Y \cdot \|\psi_{j,r}\|_Y \\ \leq& \|f_k - f\|_{\varphi(Y)} + \sum_{r=1}^{m}\frac{1}{r!}\sum_{j=1}^{r}\|f_k^{(j)} - f^{(j)}\|_{\varphi(Y)}\cdot\|\psi_{j,r}\|_Y.\end{aligned}$$

Therefore, $\|C_\varphi(f_k) - F\|_m \to 0$ as $k \to \infty$, since $\varphi(Y)$ is a compact subset of int$X$ and $f_k^{(r)} \to f^{(r)}$ uniformly on $\varphi(Y)$ for each $r \geq 0$. □

Using the same arguments as in the proof of the above theorem we obtain the following result.

Theorem 3.4. *Let m be a positive integer and X, Y be two perfect compact plane sets. If $\varphi \in D^m(Y)$ and $\varphi(Y) \subseteq$ int$X$, then C_φ is a compact operator from $A(X)$ into $D^m(Y)$.*

As usual, $A(X)$ denotes the uniform algebra of all continuous functions on a compact plane set X which are analytic on int$X$.

To prove the next theorem, we require the following lemma due to Julia [5, Chapter I of Part Six].

Lemma 3.5. *Let $\overline{\mathbb{D}}$ be the closed unit disc in $\mathbb{C}$ and let h be a continuously differentiable function on $\overline{\mathbb{D}}$. If $h(\zeta) = \|h\|_{\overline{\mathbb{D}}}$ for some $\zeta \in \overline{\mathbb{D}}$, then either h is constant or $h'(\zeta) \neq 0$.*

For convenience, for each $z_0 \in X$ and each function $f : X \to \mathbb{C}$ we define

$$p_{z_0}(f) := \sup_{\substack{z\in X \\ z\neq z_0}} \frac{|f(z) - f(z_0)|}{|z - z_0|}.$$

Then when X satisfies the condition (1.1), for each $z_0 \in X$ there exists a constant c_{z_0} such that

$$p_{z_0}(f) \leq c_{z_0}(\|f\|_X + \|f'\|_X) \qquad (f \in D^1(X)). \tag{3.1}$$

Theorem 3.6. *Let n, m be two positive integers and X, Y be two perfect compact plane sets satisfying the condition (1.1) such that X has a peak boundary with respect to $D^{n+1}(X)$ and int$X$ is connected. Let Y be strongly accessible from the interior. If $n < m$, then the following conditions are equivalent.*

(i) *$\varphi \in D^m(Y)$ and either φ is constant or $\varphi(Y) \subseteq \mathrm{int}X$.*

(ii) *C_φ is a bounded operator from $D^n(X)$ into $D^m(Y)$.*

(iii) *C_φ is a compact operator from $D^n(X)$ into $D^m(Y)$.*

Proof. (i)→(iii) has been proved in Theorem 3.3. (iii)→(ii) is obvious.

(ii)→(i). We know that $\varphi \in D^m(Y)$, since $D^n(X)$ contains the coordinate function z. Assume that $\varphi(\zeta) \in \partial X$ for some $\zeta \in Y$. Then by open mapping theorem for analytic functions we have that $\zeta \in \partial Y$. Since X has a peak boundary with respect to $D^{n+1}(X)$, there exists a non-constant function $h \in D^{n+1}(X)$ such that $h(\varphi(\zeta)) = \|h\|_X = 1$. Let

$$f_k(z) = \frac{h^k(z)}{k(k-1)\cdots(k-n)}, \quad (z \in X, k > n).$$

Then $\{f_k\}$ is a bounded sequence in $D^n(X)$ and $f_k^{(r)} \to 0$ uniformly on X for each $r = 0, 1, 2, \ldots, n$. Therefore $\|f_k\|_n \to 0$ and hence, by boundedness of C_φ,

$$\|f_k \circ \varphi\|_m = \|C_\varphi(f_k)\|_m \to 0 \quad \text{as} \quad k \to \infty.$$

Thus $\|(f_k \circ \varphi)^{(r)}\|_Y \to 0$ for each $r = 0, 1, 2, \ldots, m$ and consequently, using the inequality (3.1), $p_\zeta((f_k \circ \varphi)^{(r)}) \to 0$ for each $r = 0, 1, 2, \ldots, m-1$. In particular,

$$p_\zeta((f_k \circ \varphi)^{(n)}) \to 0 \quad \text{as} \quad k \to \infty. \tag{3.2}$$

Also by (3.1), it follows from the uniformly convergence $f_k^{(r)} \to 0$ on X for each $r = 0, 1, 2, \ldots, n$, that

$$p_{\varphi(\zeta)}(f_k^{(r)}) \to 0 \quad \text{as} \quad k \to \infty \quad (r = 0, 1, 2, \ldots, n-1). \tag{3.3}$$

Using Faà di Bruno's formula,

$$\begin{aligned}
p_\zeta((f_k^{(n)} \circ \varphi)(\varphi')^n) \leq & p_\zeta((f_k \circ \varphi)^{(n)}) + \sum_{j=1}^{n-1} p_\zeta((f_k^{(j)} \circ \varphi) \cdot \psi_{j,n}) \\
\leq & p_\zeta((f_k \circ \varphi)^{(n)}) + \sum_{j=1}^{n-1} \|f_k^{(j)} \circ \varphi\|_Y p_\zeta(\psi_{j,n}) + \sum_{j=1}^{n-1} p_\zeta(f_k^{(j)} \circ \varphi)\|\psi_{j,n}\|_Y \\
\leq & p_\zeta((f_k \circ \varphi)^{(n)}) + \sum_{j=1}^{n-1} \|f_k^{(j)}\|_X p_\zeta(\psi_{j,n}) + \sum_{j=1}^{n-1} p_{\varphi(\zeta)}(f_k^{(j)}) p_\zeta(\varphi)\|\psi_{j,n}\|_Y.
\end{aligned}$$

This inequality, along with the limits (3.2), (3.3) and the property of $\{f_k\}$ imply that

$$p_\zeta((f_k^{(n)} \circ \varphi)(\varphi')^n) \to 0 \quad \text{as} \quad k \to \infty. \tag{3.4}$$

By the definition of $f_k^{(n)}$,

$$\frac{1}{k-n}p_\zeta(((h\circ\varphi)')^n(h\circ\varphi)^{k-n}) \leq p_\zeta((f_k^{(n)}\circ\varphi)(\varphi')^n) + \frac{P(k)p_\zeta(\psi)}{k(k-1)\cdots(k-n)}, \tag{3.5}$$

where the function ψ is a combination of φ', h and the derivatives of h, and $P(k)$ is a polynomial in terms of k with degree less than $n+1$. Hence $\frac{P(k)}{k(k-1)\cdots(k-n)} \to 0$ as $k\to\infty$. Using this limit together with the limit (3.4) and the inequality (3.5), we obtain

$$\frac{1}{k-n}p_\zeta(((h\circ\varphi)')^n\cdot(h\circ\varphi)^{k-n}) \to 0 \quad \text{as} \quad k\to\infty. \tag{3.6}$$

On the other hand, we have

$$\begin{aligned}\sup_{\substack{z\in\overline{U}\\ z\neq\zeta}}&|(h\circ\varphi)'(z)|^n\frac{|h^{k-n}(\varphi(z))-h^{k-n}(\varphi(\zeta))|}{(k-n)|z-\zeta|}\\ &\leq\frac{1}{k-n}\{p_\zeta(((h\circ\varphi)')^n\cdot(h\circ\varphi)^{k-n})+p_\zeta(((h\circ\varphi)')^n)\|h\|_X^{k-n}\}.\end{aligned}$$

Using (3.6) and the fact that $\|h\|_X=1$, one can conclude from the above inequality that

$$\sup_{\substack{z\in\overline{U}\\ z\neq\zeta}}|(h\circ\varphi)'(z)|^n\frac{|h^{k-n}(\varphi(z))-h^{k-n}(\varphi(\zeta))|}{(k-n)|z-\zeta|}\to 0, \qquad \text{as} \quad k\to\infty.$$

Let $\varepsilon>0$. Then

$$|(h\circ\varphi)'(z)|^n\frac{|h^{k-n}(\varphi(z))-h^{k-n}(\varphi(\zeta))|}{(k-n)|z-\zeta|}<\varepsilon,$$

for some positive integer $k>n$ and for all $z\in\overline{U}$ with $z\neq\zeta$. Taking limit as $z\to\zeta$, we get $|(h\circ\varphi)'(\zeta)|^{n+1}\leq\varepsilon$, for each $\varepsilon>0$, since $h(\varphi(\zeta))=1$. Consequently, $|(h\circ\varphi)'(\zeta)|^{n+1}=0$, hence, $(h\circ\varphi)'(\zeta)=0$. By Julia's Lemma 3.5, $h\circ\varphi$ is constant on $\overline{U}$. Using the identity theorem [6, IV, Theorem 3.7], the analytic function $h\circ\varphi$ is constant on the connected set int$X$. The hypothesis, X is strongly accessible from the interior, implies that X has dense interior, so $h\circ\varphi$ is constant on X. But h is not constant, thus φ must be constant. □

The assumption, *X has a peak boundary with respect to $D^{n+1}(X)$*, in Theorem 3.6 is a mild restriction, since $D^{n+1}(X)$ contains all rational functions with poles off X. In particular, when $X=Y=\overline{\mathbb{D}}$ we have the following result.

Theorem 3.7. *Let n,m be two positive integers with $n<m$, then the following conditions are equivalent.*

(i) *$\varphi\in D^m(\overline{\mathbb{D}})$ and either φ is constant or $\varphi(\overline{\mathbb{D}})\subseteq\mathbb{D}$.*

(ii) *C_φ is a bounded operator from $D^n(\overline{\mathbb{D}})$ into $D^m(\overline{\mathbb{D}})$.*

(iii) C_φ *is a compact operator from* $D^n(\overline{\mathbb{D}})$ *into* $D^m(\overline{\mathbb{D}})$.

In the case that the underlying set X has empty interior, the situation is different. For example, as Kamowitz mentioned in [13], we have the following result when X is the unit interval $[0,1]$. As usual, in this case, we denote $D^n(X)$ by $C^n([0,1])$.

Theorem 3.8. *A non-zero composition operator* C_φ *on* $C^n([0,1])$ *is compact if and only if* φ *is a constact function.*

Thus every non-zero compact endomorphism T on $C^n([0,1])$ has the form $Tf = f(z_0)1$ for some $z_0 \in [0,1]$.

4. Quasicompact, Riesz and power compact operators on $D^n(X)$

Using the result of the previous section, we will prove some results about quasicompactness, Riesz and power compactness of C_φ on $D^n(X)$. For convenience, we first recall their definitions.

Let E be an infinite dimensional Banach space. We denote the Banach algebra of bounded linear operators on E by $\mathcal{B}(E)$ and the Banach algebra of compact linear operators on E by $\mathcal{K}(E)$. Then $\mathcal{K}(E)$ is a closed ideal in $\mathcal{B}(E)$. The operator $T \in \mathcal{B}(E)$ is a Fredholm operator if T has finite-dimensional kernel and cokernel. When E is an infinite dimensional Banach space, by Atkinson Theorem, $T \in \mathcal{B}(E)$ is Fredholm if and only if $T+\mathcal{K}(E)$ is invertible in the Calkin algebra $\mathcal{B}(E)/\mathcal{K}(E)$. The *essential spectrum* $\sigma_e(T)$ of an operator $T \in \mathcal{B}(E)$ is the set of complex numbers λ, such that $\lambda I - T$ is not Fredholm. This is also equal to the spectrum of $T + \mathcal{K}(E)$ in the Calkin algebra $\mathcal{B}(E)/\mathcal{K}(E)$. The *essential spectral radius* $r_e(T)$ of $T \in \mathcal{B}(E)$ is the spectral radius of $T+\mathcal{K}(E)$ in the Calkin algebra $\mathcal{B}(E)/\mathcal{K}(E)$, that is

$$r_e(T) = \lim_{n\to\infty} \|T^n + \mathcal{K}(E)\|^{\frac{1}{n}}.$$

An operator $T \in \mathcal{B}(E)$ is called *quasicompact* if $r_e(T) < 1$. This holds if and only if there is a natural number n such that the distance from T^n to $\mathcal{K}(E)$, $\|T^n + \mathcal{K}(E)\|$ is strictly less than 1. An operator $T \in \mathcal{B}(E)$ is called *Riesz* if $\lambda I - T$ is Fredholm for all non-zero complex numbers λ. Thus T is Riesz if and only if $r_e(T) = 0$. Also, an operator T is *power compact* if T^N is compact for some positive integer N. Obviously, every power compact operator is Riesz and hence quasicompact. The converse is not true in general.

Feinstein and Kamowitz proved in [10, Theorem 1.2 (iii)] that if φ induces a quasicompact endomorphism of a unital commutative semi-simple Banach algebra B with connected maximal ideal (character) space X, then $\bigcap \varphi_n(X) = \{x_0\}$ for some $x_0 \in X$, where φ_n denotes the n-th iterate of φ. By using this relation and the obtained condition for compactness of composition operators on algebras $D^n(X)$, we have the following result.

Theorem 4.1. [2, Theorem 2.7] *Let X be a perfect compact plane set satisfying the condition* (1.1). *Let a self-map φ of X be in $D^n(X)$.*

(i) *If $\bigcap \varphi_n(X) = \{z_0\}$ for some $z_0 \in \text{int}X$, then C_φ is power compact on $D^n(X)$.*
(ii) *Let X be strongly accessible from the interior, have a peak boundary with respect to $D^n(X)$ and let* int$X$ *be connected. If φ is non-constant and C_φ is power compact on $D^n(X)$, then $\bigcap \varphi_n(X) = \{z_0\}$ for some $z_0 \in \text{int}X$.*

Using the same argument as in the proof of [10, Lemma 2.1], one can obtain the following Theorem .

Theorem 4.2. *Let X be a connected perfect compact plane set, φ be a self-map of X with fixed point z_0. If C_φ is a quasicompact composition operator on $D^n(X)$, then $|\varphi'(z_0)| < 1$.*

It was also shown in [10, Theorem 3.2] that if $T = C_\varphi$ acts on $C^1([0,1])$, and $\bigcap \varphi_n([0,1]) = \{x_0\}$ for some $x_0 \in [0,1]$, then $r_e(T) = |\varphi'(x_0)|$. By the following example we show that this is not, in general, true for $D^1(X)$.

Example 4.3. Let $\varphi(z) = \frac{1-z}{2}$ for every $z \in \overline{\mathbb{D}}$. Note that $z_0 = \frac{1}{3}$ is the fixed point of φ in $\mathbb{D}$ and $|\varphi'(z_0)| = \frac{1}{2}$. On the other hand, $\varphi(-1) = 1$, so $\varphi(\overline{\mathbb{D}}) \not\subseteq \mathbb{D}$ and the composition operator C_φ on $D^1(\overline{\mathbb{D}})$ is not compact. However, $|\varphi_2(z)| \leq \frac{1}{2} < 1$ for all $z \in \overline{\mathbb{D}}$. Hence, C_φ is power compact on $D^1(\overline{\mathbb{D}})$ and then $\bigcap \varphi_n(\overline{\mathbb{D}}) = \{z_0\}$ and $r_e(C_\varphi) = 0$.

A question which may be asked is whether every quasicompact or Riesz operator on $D^n(X)$ is necessarily power compact. As proven by Feinstein and Kamowitz, there exists a quasicompact operator on $C^1([0,1])$ which is not Riesz and there exists a Riesz operator on $C^1([0,1])$ which is not power compact [10, Corollary 3.3].

Example 4.4. Let $\varphi(x) = \frac{x+x^2}{3}$. Then $\bigcap \varphi_n([0,1]) = \{0\}$ and $r_e(C_\varphi) = |\varphi'(0)| = \frac{1}{3}$. Therefore, C_φ is a quasicompact operator on $C^1([0,1])$ which is not Riesz.

Let now $\varphi(x) = \frac{x^2}{2}$. Then $\bigcap \varphi_n([0,1]) = \{0\}$ and $r_e(C_\varphi) = |\varphi'(0)| = 0$. Therefore, C_φ is a Riesz operator on $C^1([0,1])$ which is not power compact since non-iterate of φ is constant.

The following example shows that there exists a quasicompact operator on $D^n(X)$ which is not necessarily power compact.

Example 4.5. [2, Example 2.9] Let $c > 1$ and $\varphi(z) = \frac{z+(c-1)}{c}$ for every $z \in \overline{\mathbb{D}}$. Then C_φ is a composition operator on $D^n(\overline{\mathbb{D}})$ and $r_e(C_\varphi) < 1$. Hence C_φ is a quasicompact operator on $D^n(\overline{\mathbb{D}})$ which is not power compact since $\bigcap \varphi_m(\overline{\mathbb{D}}) = \{1\} \not\subseteq \mathbb{D}$.

However, as shown by Feinstein and Kamowitz, if Dales-Davie algebra $D(X, M)$ is a natural Banach function algebra on a connected perfect compact plane set X with a non-analytic weight sequence $M = \{M_n\}$, then every quasicompact endomorphism of $D(X, M)$ induced by an analytic self-map of X is power compact [10, Theorem 2.2].

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, U.S. Department of Commerce, Washington, D.C., 1964.

[2] S. Amiri, A. Golbaharan and H. Mahyar, Weighted composition operators on algebras of differentiable functions, *Bull. Belg. Math. Soc. Simon Stevin* 23 (2016) 595-608.

[3] F. Behrouzi and H. Mahyar, Compact endomorphisms of certain analytic Lipschitz algebras, *Bull. Belg. Math. Soc. Simon Stevin* 12 (2005) 301-312.

[4] W. J. Bland and J. F. Feinstein, Completions of normed algebras of differentiable functions, *Studia Math.* 170 (2005) 89-111.

[5] C. Caratheodory, *Theory of Functions of a Complex Variable*, Vol. II, Chelsea, New york, 1960.

[6] J. B. Conway, *Functions of One Complex Variable*, Spinger-Verlag New York, 1978.

[7] H. G. Dales, *Banach algebras and automatic continuity*, London Math. Soc. Monographs 24, The Clarendon Press, Oxford, 2000.

[8] H. G. Dales and A. M. Davie, Quasianalytic Banach function algebras, *J. Funct. Anal.* 13 (1973) 28-50.

[9] H. G. Dales and J. F. Feinstein, Normed algebras of differentiable functions on compact plane sets, *Indian J. Pure Appl. Math.* 41 (2010) 153-187.

[10] J. F. Feinstein and H. Kamowitz, Quasicompact and Riesz endomorphisms of Banach algebras, *J. Funct. Anal.* 225 (2005) 427-438.

[11] T. G. Honary and H. Mahyar, Approximation in Lipschitz algebras of infinitely differentiable functions, *Bull. Korean Math. Soc.* 36 (1999) 629-636.

[12] K. Jarosz, $Lip_{Hol}(X,\alpha)$, *Proc. American Math. Soc.* 125 (1997) 3129-3130.

[13] H. Kamowitz, S. Scheinberg and D. Wortman, Compact endomorphisms of Banach algebras II, *Proc. American Math. Soc.* 107 (1989) 417-421.

[14] G. M. Leibowitz, *Lectures on Complex Function Algebras*, Scott, Foresman and Co., Glenview, Illinois, 1970.

[15] H. Mahyar and A. H. Sanatpour, Compact and quasicompact homomorphisms between differentiable Lipschitz algebras, *Bull. Belg. Math. Soc. Simon-Stevin* 17 (2010) 485-497.

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