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メビウスジャイロベクトル空間における有限生成ジャイロベクトル部分空間
Finitely generated gyrovector subspaces in the Möbius gyrovector space

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Abstract. We show that any finitely generated gyrovector subspace in the Möbius gyrovector space coincides with the intersection of the linear subspace generated by the same generators and the Möbius ball. As an application, we present a notion of orthogonal gyrodecomposition and clarify the relationship with the orthogonal decomposition. In addition, an announce of the abstract of the results which were recently obtained by the second author will be made. One of the main results is the orthogonal gyroexpansion of an arbitrary element with respect to any orthogonal basis in the Möbius gyrovector space and its concrete procedure to calculate the gyrocoefficients.

1 Introduction

Let us recall the definitions of the (gyrocommutative) gyrogroups, abstract gyrovector spaces, the Einstein and the Möbius gyrovector spaces. Please refer [U1] for the precise statements and basic facts.

Definition. A magma $(G, \oplus)$ is a nonempty set G with a map $\oplus : G \times G \to G$. We use the notation $a \oplus b$ to denote $\oplus(a, b)$ for all $a, b \in G$. An automorphism ϕ of a magma $(G, \oplus)$ is a bijective self-map of G, $\phi : G \to G$, such that $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. The set of all automorphisms of $(G, \oplus)$ is denoted by $\mathrm{Aut}(G, \oplus)$.

Definition (Gyrocommutative Gyrogroups). [U1] A magma $(G, \oplus)$ is a gyrocommutative gyrogroup if

$$\begin{array}{ll} \text{(G1)} & \exists 0 \in G \text{ s.t. } 0 \oplus a = a \ \ (\forall a \in G) \\ \text{(G2)} & \forall a \in G \, \exists x \in G \text{ s.t. } x \oplus a = 0 \\ \text{(G3)} & \exists 1 \text{gyr}[a,b]c \in G \text{ s.t. } a \oplus (b \oplus c) = (a \oplus b) \oplus \text{gyr}[a,b]c \\ \text{(G4)} & \text{gyr}[a,b] \in \text{Aut}(G, \oplus) \\ \text{(G5)} & \text{gyr}[a,b] = \text{gyr}[a \oplus b, b] \\ \text{(G6)} & a \oplus b = \text{gyr}[a,b](b \oplus a) \end{array}$$

for all $a, b, c \in G$.

Definition (Gyrovector Spaces). [U1] $(G, \oplus, \otimes)$ is a real inner product gyrovector space (gyrovector space, in short), if $(G, \oplus)$ is a gyrocommutative gyrogroup and there exists a real inner product space $\mathbb{V}$ such that $G \subset \mathbb{V}$ and $\otimes : \mathbb{R} \times G \to G$ possesses the following properties:

$$\begin{array}{ll} \text{(V0)} & \text{gyr}[\boldsymbol{u},\boldsymbol{v}]\boldsymbol{a} \cdot \text{gyr}[\boldsymbol{u},\boldsymbol{v}]\boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b} \\ \text{(V1)} & 1 \otimes \boldsymbol{a} = \boldsymbol{a} \\ \text{(V2)} & (r_1 + r_2) \otimes \boldsymbol{a} = r_1 \otimes \boldsymbol{a} \oplus r_2 \otimes \boldsymbol{a} \\ \text{(V3)} & (r_1 r_2) \otimes \boldsymbol{a} = r_1 \otimes (r_2 \otimes \boldsymbol{a}) \\ \text{(V4)} & \dfrac{|r| \otimes \boldsymbol{a}}{||r \otimes \boldsymbol{a}||} = \dfrac{\boldsymbol{a}}{||\boldsymbol{a}||} \\ \text{(V5)} & \text{gyr}[\boldsymbol{u},\boldsymbol{v}]\,(r \otimes \boldsymbol{a}) = r \otimes \text{gyr}[\boldsymbol{u},\boldsymbol{v}]\boldsymbol{a} \\ \text{(V6)} & \text{gyr}[r_1 \otimes \boldsymbol{v}, r_2 \otimes \boldsymbol{v}] = I \\ \text{(VV)} & \text{(other) operations } \oplus, \otimes \text{ are defined on the set } ||G|| = \{\pm ||\boldsymbol{a}||;\ \boldsymbol{a} \in G\} \\ & \subset \mathbb{R} \text{ so that } (||G||, \oplus, \otimes) \text{ is a real, one-dimensional vector space satisfying} \\ \text{(V7)} & ||r \otimes \boldsymbol{a}|| = |r| \otimes ||\boldsymbol{a}|| \\ \text{(V8)} & ||\boldsymbol{a} \oplus \boldsymbol{b}|| \leq ||\boldsymbol{a}|| \oplus ||\boldsymbol{b}|| \end{array}$$

for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a}, \boldsymbol{b} \in G$, $r_1, r_2, r \in \mathbb{R}$.

Example (Einstein Gyrovector Spaces).[U1] Let c be the speed of right in the vacuum and let $\mathbb{R}^3_c = \{\boldsymbol{a} \in \mathbb{R}^3;\ ||\boldsymbol{a}|| < c\}$ be the all relativistically admissible velocities of material particles. The Einstein addition $\oplus_{\text{E}}$ in $\mathbb{R}^3_c$ and the scalar multiplication $\otimes_{\mathrm{E}}$ are given by the equations

$$a \oplus_{\mathrm{E}} b = \frac{1}{1+\frac{a\cdot b}{c^2}} \left\{ a + b + \frac{1}{c^2}\frac{\gamma_a}{1+\gamma_a}(a \times (a \times b)) \right\}$$

$$r \otimes_{\mathrm{E}} a = c \tanh\left(r \tanh^{-1} \frac{||a||}{c} \right) \frac{a}{||a||} \quad (\text{if } a \neq 0), \quad r \otimes_{\mathrm{E}} 0 = 0$$

for $a, b \in \mathbb{R}^3_c$, $r \in \mathbb{R}$, where $\gamma_a = \dfrac{1}{\sqrt{1 - \frac{||a||^2}{c^2}}}$.

The addition $\oplus_{\mathrm{E}}$ and the scalar multiplication $\otimes_{\mathrm{E}}$ for the set $||\mathbb{R}^3_c|| = (-c, c)$ in the axiom (VV) of gyrovector spaces are defined by the equations

$$a \oplus_{\mathrm{E}} b = \frac{a+b}{1+\frac{1}{c^2}ab}$$

$$r \otimes_{\mathrm{E}} a = c \tanh\left(r \tanh^{-1} \frac{a}{c} \right)$$

for any $a, b \in (-c, c)$, $r \in \mathbb{R}$. Then, $(\mathbb{R}^3_c, \oplus_{\mathrm{E}}, \otimes_{\mathrm{E}})$ is a gyrovector space.

Example (Möbius Gyrovector Spaces).[U1] Let $\mathbb{V}$ be an arbitrary real inner product space and $\mathbb{V}_s = \{a \in \mathbb{V};\ ||a|| < s\}$ for any fixed $s > 0$. The Möbius addition and the Möbius scalar multiplication are given by the equations

$$a \oplus_{\mathrm{M}} b = \frac{\left(1 + \frac{2}{s^2} a\cdot b + \frac{1}{s^2}||b||^2\right) a + \left(1 - \frac{1}{s^2}||a||^2\right) b}{1 + \frac{2}{s^2} a\cdot b + \frac{1}{s^4}||a||^2||b||^2}$$

$$r \otimes_{\mathrm{M}} a = s \tanh\left(r \tanh^{-1} \frac{||a||}{s} \right) \frac{a}{||a||} \quad (\text{if } a \neq 0), \qquad r \otimes_{\mathrm{M}} 0 = 0$$

for $a, b \in \mathbb{V}_s$, $r \in \mathbb{R}$. Note that each of the Möbius scalar multiplication and the operations on the set $||\mathbb{V}_s||$ is identical to the corresponding operation for the Einstein gyrovector spaces. Then, $(\mathbb{V}_s, \oplus_{\mathrm{M}}, \otimes_{\mathrm{M}})$ is a gyrovector space. We simply denote $\oplus_{\mathrm{M}}, \otimes_{\mathrm{M}}$ by $\oplus, \otimes$, respectively.

If several kinds of operations appear in a formula simultaneously, we always give priority by the following order (1) ordinary scalar multiplication (2) gyroscalar multiplication $\otimes$ (3) gyroaddition $\oplus$, that is,

$$r_1 \otimes w_1 a_1 \oplus r_2 \otimes w_2 a_2 = \{r_1 \otimes (w_1 a_1)\} \oplus \{r_2 \otimes (w_2 a_2)\},$$

and the parentheses are omitted in such cases. In general, we note that gyroaddition does not distribute with (both ordinary and gyro)scalar multiplications:

$$\boldsymbol{a} \oplus \boldsymbol{b} \neq \boldsymbol{b} \oplus \boldsymbol{a}$$
$$\boldsymbol{a} \oplus (\boldsymbol{b} \oplus \boldsymbol{c}) \neq (\boldsymbol{a} \oplus \boldsymbol{b}) \oplus \boldsymbol{c}$$
$$r \otimes (\boldsymbol{a} \oplus \boldsymbol{b}) \neq r \otimes \boldsymbol{a} \oplus r \otimes \boldsymbol{b}$$
$$t(\boldsymbol{a} \oplus \boldsymbol{b}) \neq t\boldsymbol{a} \oplus t\boldsymbol{b}.$$

They, however, are enjoying algebraic rules such as the left (and right) gyroassociative law(G3), the gyrocommutative law(G6), the scalar distributive law(V2) and the scalar associative law(V3), so there exist rich symmetrical structures which we should clarify precisely.

In the limit of large s, $s \to \infty$, the ball $\mathbb{V}_s$ expands to the whole space $\mathbb{V}$. The next proposition suggests that each result for inner product spaces can be restored from the counterpart in the Möbius gyrovector spaces.

Proposition.[U1] The Möbius addition (resp. Möbius scalar multiplication) reduces to the vector addition (resp. scalar multiplication) as $s \to \infty$, that is,

$$\boldsymbol{a} \oplus \boldsymbol{b} \to \boldsymbol{a} + \boldsymbol{b} \quad (s \to \infty)$$
$$r \otimes \boldsymbol{a} \to r\boldsymbol{a} \quad (s \to \infty).$$

Example. If we identify $\mathbb{R}^2$ with the complex plain $\mathbb{C}$, then the Möbius addition in $\mathbb{R}^2_1$ reduces to $a \oplus b = \dfrac{a+b}{1+\overline{a}b}$. If we take

$$a = \frac{i}{2}, \qquad b = -\frac{2}{5} - \frac{2}{5}i, \qquad c = \frac{1}{2},$$

then

$$a \oplus (b \oplus c) = 0$$
$$(a \oplus b) \oplus c = \frac{4+16i}{53-8i}$$
$$(a \oplus b) \oplus \frac{1+a\overline{b}}{1+\overline{a}b} c = 0$$
$$a \oplus \left(b \oplus \frac{1+b\overline{a}}{1+\overline{b}a} c \right) = \frac{4+16i}{53-8i}.$$

T. Abe raised the following question in his talk[A]:

Question. Let $(G, \oplus, \otimes)$ be a gyrovector space, or any kind of generalization of gyrovector space and $\boldsymbol{a}_1, \boldsymbol{a}_2 \in G$. Can we have the following:

$$\{r_1 \otimes \boldsymbol{a}_1 \oplus r_2 \otimes \boldsymbol{a}_2\,;\, r_1, r_2 \in \mathbb{R}\} = \{\lambda_2 \otimes \boldsymbol{a}_2 \oplus \lambda_1 \otimes \boldsymbol{a}_1\,;\, \lambda_1, \lambda_2 \in \mathbb{R}\}\,?$$

$$r \otimes (r_1 \otimes \boldsymbol{a}_1 \oplus r_2 \otimes \boldsymbol{a}_2) \in \{\lambda_1 \otimes \boldsymbol{a}_1 \oplus \lambda_2 \otimes \boldsymbol{a}_2\,;\, \lambda_1, \lambda_2 \in \mathbb{R}\}\,?$$

We gave an answer to this problem and its natural extension in the Möbius gyrovector space in the lecture which was made at this RIMS conference. In this paper, we present a survey of the lecture, and will announce an abstract of the results which were recently obtained by the second author.

2 Finitely generated gyrovector subspaces and orthogonal gyrodecomposition

We assume that $s = 1$ for simplicity.
In the Möbius gyrovector space, we can show

$$\{r_1 \otimes \boldsymbol{a}_1 \oplus r_2 \otimes \boldsymbol{a}_2\,;\, r_1, r_2 \in \mathbb{R}\} = \{\lambda_1 \boldsymbol{a}_1 + \lambda_2 \boldsymbol{a}_2;\, \lambda_1, \lambda_2 \in \mathbb{R}\} \cap \mathbb{V}_1$$

for $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathbb{V}_1$.

($\subset$) From the definitions of $\oplus, \otimes$, it follows that $r_1 \otimes \boldsymbol{a}_1 \oplus r_2 \otimes \boldsymbol{a}_2$ is a linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2$. The fact that $\mathbb{V}_1$ is a gyrovector space contains that $\mathbb{V}_1$ is closed under the operations $\oplus, \otimes$, therefore $r_1 \otimes \boldsymbol{a}_1 \oplus r_2 \otimes \boldsymbol{a}_2 \in \mathbb{V}_1$.
($\supset$) By the next Theorem.

Theorem 1.[AW] Let $(\mathbb{V}_1, \oplus, \otimes)$ be the Möbius gyrovector space and $\boldsymbol{0} \neq \boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathbb{V}_1$. Put $\alpha = \dfrac{\boldsymbol{a}_1}{||\boldsymbol{a}_1||} \cdot \dfrac{\boldsymbol{a}_2}{||\boldsymbol{a}_2||}$. Suppose that $0 \neq t_1, t_2 \in \mathbb{R}$ satisfy the condition

$$\left|\left| t_1 \frac{\boldsymbol{a}_1}{||\boldsymbol{a}_1||} + t_2 \frac{\boldsymbol{a}_2}{||\boldsymbol{a}_2||} \right|\right| < 1.$$

(I) If $2\alpha t_2 + t_1 \neq 0$, then we put

$$c_1 = \frac{{t_1}^2 + 2\alpha t_1 t_2 + {t_2}^2 + 1 - \sqrt{({t_1}^2 + 2\alpha t_1 t_2 + {t_2}^2 + 1)^2 - 8\alpha t_1 t_2 - 4{t_1}^2}}{2(2\alpha t_2 + t_1)}$$

$$c_2 = \frac{{t_1}^2 + 2\alpha t_1 t_2 + {t_2}^2 - 1 + \sqrt{({t_1}^2 + 2\alpha t_1 t_2 + {t_2}^2 + 1)^2 - 8\alpha t_1 t_2 - 4{t_1}^2}}{2t_2}.$$

(II) If $2\alpha t_2 + t_1 = 0$, then we put

$$c_1 = \frac{t_1}{{t_2}^2 + 1}$$
$$c_2 = t_1.$$

Then, we have $0 < |c_1|, |c_2| < 1$ and

$$t_1 \frac{\boldsymbol{a}_1}{||\boldsymbol{a}_1||} + t_2 \frac{\boldsymbol{a}_2}{||\boldsymbol{a}_2||} = r_1 \otimes \boldsymbol{a}_1 \oplus r_2 \otimes \boldsymbol{a}_2,$$

where

$$r_1 = \frac{\tanh^{-1} c_1}{\tanh^{-1} ||\boldsymbol{a}_1||} \quad \text{and} \quad r_2 = \frac{\tanh^{-1} c_2}{\tanh^{-1} ||\boldsymbol{a}_2||}.$$

Theorem 1 is deduced from **Theorem 2**. In our proof of **Theorem 2**, it is not difficult to derive the right-hand sides of x, y, however, we need some arguments to compare their absolute values to 1, which is one of the most crucial points in this study.

Theorem 2.[AW] Consider the following system of equations for real numbers:

$$\begin{cases} x^2y^2 + (\gamma x^2 + 2\alpha x - \gamma)y + 1 = 0 & (1) \\ xy^2 + ((2\alpha + \beta)x^2 - \beta)y + x = 0 & (2) \end{cases}$$

Suppose that $-1 \leq \alpha \leq 1$, $\beta \neq 0$ and $1 + \beta(2\alpha + \beta) < \gamma^2$.

(I) If $2\alpha + \beta \neq 0$, then

$$x = \frac{1 + \beta(2\alpha + \beta) + \gamma^2 - \sqrt{(1 + \beta(2\alpha + \beta) + \gamma^2)^2 - 4(2\alpha + \beta)\beta\gamma^2}}{2(2\alpha + \beta)\gamma}$$
$$y = \frac{1 + \beta(2\alpha + \beta) - \gamma^2 + \sqrt{(1 + \beta(2\alpha + \beta) + \gamma^2)^2 - 4(2\alpha + \beta)\beta\gamma^2}}{2\gamma}$$

is a unique pair as the solution to the system of equations (1), (2), which satisfies $0 < |x|, |y| < 1$. Moreover,

$$x = \frac{1 + \beta(2\alpha + \beta) + \gamma^2 + \sqrt{(1 + \beta(2\alpha + \beta) + \gamma^2)^2 - 4(2\alpha + \beta)\beta\gamma^2}}{2(2\alpha + \beta)\gamma}$$
$$y = \frac{1 + \beta(2\alpha + \beta) - \gamma^2 - \sqrt{(1 + \beta(2\alpha + \beta) + \gamma^2)^2 - 4(2\alpha + \beta)\beta\gamma^2}}{2\gamma}$$

is a unique pair as the solution to the system of equations (1), (2), which satisfies $|x|, |y| > 1$.

(II) If $2\alpha + \beta = 0$, then

$$x = \frac{\beta\gamma}{1+\gamma^2}$$

$$y = \frac{1}{\gamma}$$

is a unique pair as the solution to the system of equations (1), (2), which satisfies $0 < |x|, |y| < 1$.

Definition. A nonempty subset M of $\mathbb{V}_1$ is a gyrovector subspace if M is closed under gyrovector space addition and scalar multiplication, that is,

$$\boldsymbol{a}, \boldsymbol{b} \in M,\ r \in \mathbb{R} \quad \Rightarrow \quad \boldsymbol{a} \oplus \boldsymbol{b} \in M,\ r \otimes \boldsymbol{a} \in M.$$

For any nonempty subset A of $\mathbb{V}_1$, the intersection of all gyrovector subspaces of $\mathbb{V}_1$ which contain A is said to be the gyrovector subspace generated by A, and denoted by $\bigvee^g A$, that is,

$$\bigvee{}^g A = \bigcap \{M;\ A \subset M,\ M \text{ is a gyrovector subspace of } \mathbb{V}_1\}.$$

For example, let $n = 4$ and $(i_1, i_2, i_3, i_4) = (1, 4, 2, 3)$. If we add parentheses in the formula $\boldsymbol{c}_1 \oplus \boldsymbol{c}_4 \oplus \boldsymbol{c}_2 \oplus \boldsymbol{c}_3$ to specify the order of gyroaddition, there are 5 possibilities, as follows:

$$\boldsymbol{c}_1 \oplus \{\boldsymbol{c}_4 \oplus (\boldsymbol{c}_2 \oplus \boldsymbol{c}_3)\}$$

$$(\boldsymbol{c}_1 \oplus \boldsymbol{c}_4) \oplus (\boldsymbol{c}_2 \oplus \boldsymbol{c}_3)$$

$$\boldsymbol{c}_1 \oplus \{(\boldsymbol{c}_4 \oplus \boldsymbol{c}_2) \oplus \boldsymbol{c}_3\}$$

$$\{\boldsymbol{c}_1 \oplus (\boldsymbol{c}_4 \oplus \boldsymbol{c}_2)\} \oplus \boldsymbol{c}_3$$

$$\{(\boldsymbol{c}_1 \oplus \boldsymbol{c}_4) \oplus \boldsymbol{c}_2\} \oplus \boldsymbol{c}_3$$

Theorem 3.[AW] Let $(\mathbb{V}_1, \oplus, \otimes)$ be the Möbius gyrovector space, $\boldsymbol{0} \neq \boldsymbol{a}_1, \cdots, \boldsymbol{a}_n \in \mathbb{V}_1$ and let $(i_1, \cdots, i_n)$ be a permutation of $(1, \cdots, n)$. For an arbitrary given order for gyroaddition of $r_{i_1} \otimes \boldsymbol{a}_{i_1} \oplus \cdots \oplus r_{i_n} \otimes \boldsymbol{a}_{i_n}$, we have the following:

$$\bigvee{}^g \{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\}$$

$$= \{r_{i_1} \otimes \boldsymbol{a}_{i_1} \oplus \cdots \oplus r_{i_n} \otimes \boldsymbol{a}_{i_n};\ r_{i_1}, \cdots, r_{i_n} \in \mathbb{R}\}$$

$$= \left\{ t_1 \frac{\boldsymbol{a}_1}{||\boldsymbol{a}_1||} + \cdots + t_n \frac{\boldsymbol{a}_n}{||\boldsymbol{a}_n||};\ t_1, \cdots, t_n \in \mathbb{R} \right\} \cap \mathbb{V}_1.$$

Remark. We have the same result for finitely generated gyrovector subspaces in the Einstein gyrovector space.

Next, we state orthogonal gyrodecomposition with respect to relatively closed gyrovector subspaces. It can be obtained from the ordinary orthogonal decomposition with respect to closed linear subspaces. It is also easy to deduce the result for general $s > 0$ from the case $s = 1$.

Theorem 4.[AW] Let $\mathbb{V}$ be a real Hilbert space and let $(\mathbb{V}_1, \oplus, \otimes)$ be the Möbius gyrovector space, and let M be a gyrovector subspace of $\mathbb{V}_1$ that is topologically relatively closed. Suppose that

$$\boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{x}_2, \quad \boldsymbol{x}_1 \in \mathrm{clin}M, \ \ \boldsymbol{x}_2 \in M^{\perp}$$

is the (ordinary) orthogonal decomposition of an arbitrary element $\boldsymbol{x} \in \mathbb{V}_1$ with respect to $\mathrm{clin}M$, which is the closed linear subspace generated by M. Then, a unique pair $(\boldsymbol{y}, \boldsymbol{z})$ exists that satisfies

$$\boldsymbol{x} = \boldsymbol{y} \oplus \boldsymbol{z}, \quad \boldsymbol{y} \in M, \ \ \boldsymbol{z} \in M^{\perp} \cap \mathbb{V}_1.$$

Moreover, if $\boldsymbol{x}_1, \boldsymbol{x}_2 \neq \boldsymbol{0}$, then these elements $\boldsymbol{y}, \boldsymbol{z}$ are determined by

$$\boldsymbol{y} = \lambda_1 \boldsymbol{x}_1, \ \ \boldsymbol{z} = \lambda_2 \boldsymbol{x}_2,$$

where

$$\lambda_1 = \frac{||\boldsymbol{x}_1||^2 + ||\boldsymbol{x}_2||^2 + 1 - \sqrt{(||\boldsymbol{x}_1||^2 + ||\boldsymbol{x}_2||^2 + 1)^2 - 4||\boldsymbol{x}_1||^2}}{2||\boldsymbol{x}_1||^2}$$

$$\lambda_2 = \frac{||\boldsymbol{x}_1||^2 + ||\boldsymbol{x}_2||^2 - 1 + \sqrt{(||\boldsymbol{x}_1||^2 + ||\boldsymbol{x}_2||^2 + 1)^2 - 4||\boldsymbol{x}_1||^2}}{2||\boldsymbol{x}_2||^2}.$$

In addition, the inequalities $0 < \lambda_1 < 1$ and $\lambda_2 > 1$ hold.

Remark. If the gyrovector subspace M above is closed with respect to the Poincaré matric h which is introduced by Ungar, then M is relatively closed with respect to the norm topology, so the above theorem is applicable to M.

Remark. We can obtain a similar result for the Einstein gyrovector spaces.

3 Gyrolinear independency, Orthogonal gyroexpansion with respect to an orthogonal basis

We announce an abstract of the results which were recently obtained by the second author.

Definition. A finite subset $\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\} \subset \mathbb{V}_s$ is gyrolinearly independent if, for any permutation $(i_1, \cdots, i_n)$ of $\{1, \cdots, n\}$ and for any order of gyroaddition, the following implication holds:

$$r_{i_1} \otimes \boldsymbol{a}_{i_1} \oplus \cdots \oplus r_{i_n} \otimes \boldsymbol{a}_{i_n} = \boldsymbol{0} \quad \Rightarrow \quad r_1 = \cdots = r_n = 0.$$

Consider the triple $\{a, b, c\}$ in the open unit disc of the complex plain which is stated as an example in Section 1. Then, $\{a, b, c\}$ is not gyrolinearly independent.

Theorem (W). Let $\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\}$ be a linearly independent set in $\mathbb{V}_s$. Suppose that two gyrolinear combinations $r_1 \otimes \boldsymbol{a}_1 \oplus \cdots \oplus r_n \otimes \boldsymbol{a}_n$, $\lambda_1 \otimes \boldsymbol{a}_1 \oplus \cdots \oplus \lambda_n \otimes \boldsymbol{a}_n$ are given the same order of gyroaddition and

$$r_1 \otimes \boldsymbol{a}_1 \oplus \cdots \oplus r_n \otimes \boldsymbol{a}_n = \lambda_1 \otimes \boldsymbol{a}_1 \oplus \cdots \oplus \lambda_n \otimes \boldsymbol{a}_n.$$

Then we have $r_j = \lambda_j \ \ (j = 1, \cdots, n)$.

Theorem (W). For any finite subset in $\mathbb{V}_s$, two notions of linearly independent and gyrolinearly independent coincide.

Definition (Ungar).[U1] The functions d and h on each Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ are defined by the equations

$$d(\boldsymbol{a}, \boldsymbol{b}) = ||\boldsymbol{b} \ominus \boldsymbol{a}||$$

$$h(\boldsymbol{a}, \boldsymbol{b}) = \tanh^{-1} \frac{d(\boldsymbol{a}, \boldsymbol{b})}{s}$$

for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{V}_s$. Then $(\mathbb{V}_s, h)$ is a metric space. If, in addition, $(\mathbb{V}, ||\cdot||)$ is complete as a metric space, then $(\mathbb{V}_s, h)$ is also complete.

Theorem (W). Let M be an h-closed gyrovector subspace of $\mathbb{V}_s$ and $\boldsymbol{x} \in \mathbb{V}_s$.

(1) Let

$$\boldsymbol{x} = \boldsymbol{y} \oplus \boldsymbol{z}, \qquad \boldsymbol{y} \in M, \ \ \boldsymbol{z} \in M^{\perp} \cap \mathbb{V}_s$$

be the orthogonal gyrodecomposition with respect to M. Then $\boldsymbol{y}$ is the closest point to $\boldsymbol{x}$ in M. Thus $\boldsymbol{y}$ satisfies the identity

$$h(\boldsymbol{x}, \boldsymbol{y}) = \inf_{\boldsymbol{m} \in M} h(\boldsymbol{x}, \boldsymbol{m}). \tag{3}$$

(2) Conversely, let $\boldsymbol{y}$ be the closest point to $\boldsymbol{x}$ in M, namely, $\boldsymbol{y}$ is an element in M satisfying identity (3). Then

$$\boldsymbol{x} = \boldsymbol{y} \oplus (\ominus \boldsymbol{y} \oplus \boldsymbol{x})$$

is the orthogonal gyrodecomposition with respect to M. Thus $\ominus \boldsymbol{y} \oplus \boldsymbol{x} \in M^{\perp} \cap \mathbb{V}_s$.

Theorem (W). Let $\{\boldsymbol{e}_n\}_{n=1}^{\infty}$ be a complete orthonormal sequence in a real Hilbert space $\mathbb{V}$. Let $\{w_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ such that $0 < w_n < s$ for all n. Then, for any $\boldsymbol{x} \in \mathbb{V}_s$, we have the orthogonal gyroexpansion

$$\boldsymbol{x} = r_1 \otimes w_1 \boldsymbol{e}_1 \oplus r_2 \otimes w_2 \boldsymbol{e}_2 \oplus \cdots \oplus r_n \otimes w_n \boldsymbol{e}_n \oplus \cdots .$$

This means that the sequence of partial sums converges to $\boldsymbol{x}$ with respect to the metric h stated before, and the partial sums do not depend on their order of gyroaddition by the orthgonality of the terms, so we do not need parentheses. Moreover, we can calculate the gyrocoefficients $\{r_n\}_{n=1}^{\infty}$ by an explicit procedure.

Lemma. If $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is an orthogonal set in $\mathbb{V}_s$, then the associative law holds, i.e.,

$$\boldsymbol{u} \oplus (\boldsymbol{v} \oplus \boldsymbol{w}) = (\boldsymbol{u} \oplus \boldsymbol{v}) \oplus \boldsymbol{w}.$$

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