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CONTRACTIVE PROJECTIONS ON SUBSPACES OF CONTINUOUS FUNCTIONS

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ABSTRACT. This paper deals with the structure of contractive and bi-contractive projections on spaces of continuous functions defined on a compact and Hausdorff topological space.

1. INTRODUCTION

This paper deals with contractive and bi-contractive projections on subspaces of continuous functions. More precisely, the underlying spaces are closed subspaces of $C(\Omega)$, with Ω a compact Hausdorff space, endowed with the standard infinite norm. A generic closed subspace of $C(\Omega)$ is denoted by A. The operators under investigation are projections which are idempotent bounded operators on A. Each projection P determines a new projection $P^{\perp} = I - P$, called the complement of P. Within the class of projections, we are interested in those that are contractive, meaning $\|P\| = 1$, and also those that are bi-contractive, i.e. $\|P\| = \|P^{\perp}\| = 1$.

Friedman and Russo in [10] showed that contractive projections in $C(\Omega)$ can be described by its essential part. This is represented by an operator Q, taking values in the space of continuous and bounded functions defined on a special Borel subset of Ω, $C_b(S)$ also endowed with the infinite norm. The operator $Q : C(\Omega) \to C_b(S)$ simply restricts the action of P on f to S while preserving the norm $\|Q(f)\|_{\infty} = \|P(f)\|_{\infty}$. Furthermore, P is then retrieved from Q via an isometric simultaneous extension from the range of Q to the entire $C(\Omega)$.

As for contractive projections on $C(\Omega)$, a contractive projection on A can be represented by its essential part followed by an isometric simultaneous extension. The proof follows steps presented in [10] that are outlined in the section 2 of this paper.

The Friedman-Russo decomposition of contractive projections on $C(\Omega)$ has very powerful corollaries, one of which is the representation for the bi-contractive projections. Proposition 1.19 in [10] formulates that bi-contractive projections on $C(\Omega)$ are given as the average of the identity with an isometric reflection. This is a very interesting result since the bi-contractive projections on $C(\Omega)$ and the generalized bi-circular projections

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have exactly the same form. In this paper we explore this feature for some subspaces of continuous functions.

Bi-circular projections were introduced in 2004 by Stachó and Zalar. Bi-circular projections appeared as a characterization for Hilbert spaces among JB* triples, see [20]. For the structure of these projections on spaces of operator algebras we refer the reader to [19]. This notion was generalized by Fošner, Ilišević and Li to the so-called generalized bi-circular projections and, in [9], they found a representation of these projections on spaces of matrices.

Generalized bi-circular projections have been characterized on several Banach spaces, and often they can be represented as the average of the identity with an isometric reflection. These new settings include, spaces of continuous functions, Lipschitz functions and spaces of analytic functions, see [1, 5, 7] and many references therein.

It is known that generalized bi-circular projections are contractive, see [12] and [14]. It is also easy to see that generalized bi-circular projections are bi-contractive. It is not clear when the bi-contractive projections of a Banach space are exactly the generalized bi-circular projections of that space. There are many spaces where these two classes of projections coincide, as for example Hilbert spaces, $C(\Omega)$ and some vector valued spaces of continuous functions, to list a few examples. When this happens we say that the Banach space has GBPs=BCPs for short. In this paper we discuss some spaces of continuous functions with this property and also pose some questions.

In section 2, we followed the Friedman-Russo approach for a decomposition of a contractive projection for closed subspaces of $C(\Omega)$ and from this, we draw some observations about the existence of bi-contractive projections.

In section 3, we consider a class of spaces of continously differentiable functions defined on $[0, 1]$, endowed with a variety of norms (KKM spaces). These spaces can be viewed as subspaces of $C(\Omega)$. We give conditions under which KKM spaces are Banach algebras. The Gelfand theory provides powerful tools for the study of these algebras but in this case the Banach algebras are not self-adjoint and then the Gelfand transform is not an isometry. This leaves the problem of finding the bi-contractive projections supported by this new class of spaces. It is interesting to mention that the form of the generalized bi-circular projections supported by a given space is directly linked to the form of the respective surjective isometries. The form of the surjective isometries supported by the KKM spaces was derived by Kawamura, Koshimizu and Miura in [13]. This opens a pathway for a characterization of a class of bi-contractive projections on these new spaces, to be presented in a forthcoming work [8].

In section 4, we give an overview of some results of bi-contractive projections supported by spaces of vector valued continuous functions, details shall be available soon in [4].

2. Results on Contractive Projections on subspaces of $C(\Omega)$

We review the characterization of the contractive projections on $C(\Omega)$ due to Friedman and Russo adapted to closed subspaces of $C(\Omega)$, cf. [10].

Throughout this paper Ω denotes a compact Hausdorff space and $C(\Omega)$ denotes the space of all continuous functions endowed with the standard $\|\cdot\|_\infty$ norm. A contractive projection $P : C(\Omega) \to C(\Omega)$ is an idempotent bounded operator of norm 1.

We first observe that a contractive projection P induces projections of the same norm on the dual and double dual spaces, P^* and P^{**} respectively.

The Riesz-Fisher-Markov Theorem identifies the dual space $C(\Omega)^*$ with the space of all regular Borel measures of bounded variation, defined on the σ-algebra of the Borel subsets of Ω, for details we refer the reader to [16, 17].

Given a closed subspace of $C(\Omega)$, A, and an element τ in A^*, we denote by $\tilde{\tau}$ any Hahn-Banach extension of τ to $C(\Omega)^*$ such that $\|\tau\| = \|\tilde{\tau}\|$. We associate to $\tilde{\tau}$ the unique regular Borel measure μ representing $\tilde{\tau}$

$$\tilde{\tau}(f) = \int_\Omega f d\mu,$$

for every $f \in C(\Omega)$. We observe that all measures representing some Hahn-Banach extension of τ yield the same value when restricted to the functions in A. Hence for $\tau \in A^*$, when we say that a measure represents τ we refer to any measure representing some Hahn-Banach extension of τ.

We pursue by reviewing some additional definitions and by setting notation to be followed throughout this paper. Given a subspace of Ω, say Ω_0, we denote by $A_{\Omega_0} = \{g : \Omega_0 \to \mathbb{C} : g = f|_{\Omega_0}, \text{ for some } f \in A\}$. We also define support of a Borel measure ν as a Borel subset of Ω, S_ν, such that $x \in S_\nu$ if and only if $|\nu|(U) > 0$ for every open neighborhood U of x, where $|\nu|$ denotes the total variation measure of ν.

We now prove a result that, following the approach in [10], also describes the form of a contractive projection on A, with A a closed subspace of $C(\Omega)$. The next proposition follows an argument due to Atalla applied to the extreme points of $P^*(A_1^*)$, see [2].

Definition 2.1. *Let A be a subspace of $C(\Omega)$ and let P be a contractive projection on A. Then a family of extreme points of $P^*(A_1^*)$ is said to have the maximal support property if and only if any two distinct elements in the family have disjoint supports and the support of any given extreme point of $P^*(A_1^*)$ is equal to the support of some element in the family.*

The next proposition ensures that a family with the maximal support property associated with a contractive projection exists and it determines in a natural way the form of the elements in the range of the projection restricted to points in the support of any measure belonging to the family.

Proposition 2.2. *(cf. [10]) Let A be a closed subspace of $C(\Omega)$ and let P be a contractive projection on A. Then there exists a family of extreme points of $P^*(A_1^*)$, $\{\mu_i\}_{i\in I}$ which satisfies the maximal support property and there exist functions $\phi_i \in A$ such that, for every $f \in A$, $P(f) \cdot \overline{\phi_i}$ is constant on S_μ.*

Proof. We observe that A_1^* is a convex and closed subset of A^*, and since P^* is a contractive projection, then $P^*(A_1^*)$ is also a convex and closed subset of A_1^*.

The Krein-Milman Theorem implies the existence of an extreme point μ of $P^*(A_1^*)$, cf. [15]. We denote the support of μ by S_μ. The measure μ represents the functional on A given by $\tau_\mu(f) = \int_\Omega f d\mu$.

This measure can be decomposed as $\mu = |\mu| \cdot \varphi$, with $|\mu|$ denoting the variation of μ and φ the Radon-Nikodym derivative of μ with respect to $|\mu|$. As such, φ is a function in $L_1(|\mu|)$ with values in $\mathbb{S}^1$. Therefore, for every integrable function h, in particular all functions in A, we have $\int_\Omega h\, d\mu = \int_\Omega h \cdot \varphi\, d|\mu|$. For details on this decomposition we refer the reader to [16] or [17].

We claim that $P(f) \cdot \varphi$ is $|\mu|$-a.e. constant. Suppose otherwise, this means that there exists $f \in A$ such that $P(f) \cdot \varphi$ is not $|\mu|$-a.e. constant on S_μ. Therefore, there should exist a real number a such that either

$$|\mu|(\{x \in \Omega : \mathrm{Re}\,((P(f) \cdot \varphi)(x)) \geq a\}) > 0 \quad \text{and} \quad |\mu|(\{x \in \Omega : \mathrm{Re}\,((P(f) \cdot \varphi)(x)) < a\}) > 0$$

or

$$|\mu|(\{x \in \Omega : \mathrm{Im}\,((P(f) \cdot \varphi)(x)) \geq a\}) > 0 \quad \text{and} \quad |\mu|(\{x \in \Omega : \mathrm{Im}\,((P(f) \cdot \varphi)(x)) < a\}) > 0,$$

where Re and Im represent the real and imaginary parts of a complex number.

Without loss of generality, we assume that $|\mu|(\{x \in \Omega : \mathrm{Re}\,((P(f) \cdot \varphi)(x)) \geq a\}) > 0$ and $|\mu|(\{x \in \Omega : \mathrm{Re}\,((P(f) \cdot \varphi)(x)) < a\}) > 0$.

We set $\Omega_1 = \{x \in \Omega : \mathrm{Re}\,((P(f) \cdot \varphi)(x)) \geq a\}$ and $\Omega_2 = \{x \in \Omega : \mathrm{Re}\,((P(f) \cdot \varphi)(x)) < a\}$ then

$$|\mu|(\Omega_1) = t > 0 \quad \text{and} \quad |\mu|(\Omega_2) = 1 - t > 0,$$

since the total variation of μ is equal to 1.

We use these sets to define the following two measures:

$$\mu_1 = \frac{1}{t}\mu|_{\Omega_1} \quad \text{and} \quad \mu_2 = \frac{1}{1-t}\mu|_{\Omega_2}.$$

Therefore $\mu = t\mu_1 + (1-t)\mu_2$. Since P^* is a projection and μ is in the image of P^* then $P^*(\mu) = \mu$. Thus $\mu = tP^*(\mu_1) + (1-t)P^*(\mu_2)$ and

$$\mu = P^*(\mu_1) = P^*(\mu_2). \tag{1}$$

On the other hand, we have

$$P^*(\mu_1)(f) = \frac{1}{t}\int_{\Omega_1} P(f)\, d\mu = \frac{1}{t}\int_{\Omega_1} P(f) \cdot \varphi\, d|\mu|$$

and

$$P^*(\mu_2)(f) = \frac{1}{1-t}\int_{\Omega_2} P(f) \cdot \varphi\, d|\mu|,$$

thus $\mathrm{Re}(P^*(\mu_1)(f)) \geq a$ and $\mathrm{Re}(P^*(\mu_2)(f)) < a$, contradicting the equation displayed in (1). This proves that

$$P(f) \cdot \varphi = c_f \quad |\mu| - \text{a.e.}.$$

Integrating this last equation with respect to $|\mu|$ we have

$$c_f = \int_\Omega P(f) \cdot \varphi \, d|\mu| = \int_\Omega P(f)\, d\mu = \int_\Omega f\, d(P^*\mu) = \int_\Omega f\, d\mu.$$

Since μ is an extreme point of $P^*(A_1^*)$, there exists $g \in A$ such that $\int_\Omega g d\mu \neq 0$ and $P(g) = \left(\int_\Omega g d\mu\right) \overline{\varphi}$, $|\mu|$-a.e. on Ω. Therefore, setting $\overline{\varphi} = \frac{P(g)}{\int_\Omega g d\mu}$, we may assume that $\overline{\varphi} \in A$. Now we prove that $P(f) = \left(\int_\Omega f\, d\mu\right) \overline{\varphi}$ on S_μ for every $f \in A$. Suppose that $P(f)(x) \neq \left(\int_\Omega f\, d\mu\right) \overline{\varphi}(x)$ for some $f \in A$ and $x \in S_\mu$. By the continuity of $P(f)$ and $\overline{\varphi}$, there exists an open set U of Ω, containing x, such that $P(f) \neq \left(\int_\Omega f\, d\mu\right) \overline{\varphi}$ on U. Since $x \in S_\mu$, we have $|\mu|(U) > 0$. On the other hand, $P(f) = \left(\int_\Omega f\, d\mu\right) \overline{\varphi}$, $|\mu|$-a.e. as proved above. By the choice of U, $|\mu|(U) = 0$, which is a contradiction. We have proved that $P(f) = \left(\int_\Omega f\, d\mu\right) \overline{\varphi}$ on S_μ for every $f \in A$.

It remains to be shown the existence of a family of extreme points of $P^*(A_1^*)$ that satisfies the maximal support property. Towards this, we show that given two different extreme points of $P^*(A_1^*)$, μ and ν, with intersecting supports and decompositions $|\mu| \cdot \varphi$ and $|\nu| \cdot \psi$ respectively, must have equal supports. We observe that $\overline{\varphi} \in A_{S_\mu}$ and $\overline{\psi} \in A_{S_\nu}$. Let $x \in S_\mu \cap S_\nu$, the image of the Dirac measure concentrated on x, $P^*(\delta_x)$, applied to a function f yields

$$P^*(\delta_x)(f) = \int_\Omega P(f) d\delta_x = \overline{\varphi}(x) \int_\Omega f d\mu = \overline{\psi}(x) \int_\Omega f d\nu.$$

Therefore $\mu = \lambda\nu$, with $\lambda = \varphi(x)\overline{\psi(x)}$, a modulus 1 complex number. This implies that any two extreme points of $P^*(A_1^*)$ have either equal supports or disjoint supports.

We define a partial order on the collection of all families $\mathcal{F}_J = \{(\mu_i, S_{\mu_i})\}_{i \in J}$ such that, for each $i \in J$, μ_i is an extreme point of $P^*(A_1^*)$, with S_{μ_i} denoting the support of μ_i, and for $i \neq j$, we have that S_{μ_i} and S_{μ_j} are disjoint. We say $\mathcal{F}_{J_0} \leq \mathcal{F}_{J_1}$ if and only if $J_0 \subset J_1$. An application of Zorn's lemma ensures the existence of a maximal family $\mathcal{F}_I$ with the desired property. This completes the proof. □

Remark 2.3. *It is a consequence of the Krein-Milman Theorem that every element in $P^*(A_1^*)$ is the limit of a net of convex combinations of extreme points. For every $\nu \in P^*(A_1^*)$, $\nu = \lim_\alpha \sum_{i=1}^{n_\alpha} \lambda_i^\alpha \mu_{\alpha_i}$, with $\mu_{\alpha_i} \in extP^*(A_1^*)$, $0 \leq \lambda_i^\alpha \leq 1$ and $\sum_{i=1}^{n_\alpha} \lambda_i^\alpha = 1$. Therefore, the support of any measure representing functionals in $P^*(A_1^*)$ is contained in the union of the supports of the extreme points of $P^*(A_1^*)$. We denote by S the union of the supports of the measures in $\mathcal{F}_I$.*

We set $Q(f)$ equal to the restriction of $P(f)$ to S. We shall prove that $\sup_{x\in S} |Q(f)(x)| = \max_{x\in\Omega}|P(f)(x)|$. The operator $Q : A \to P(A)|_S$ is given by $Q(f)(x) = P(f)(x)$, for every $x \in S$. We observe that $Q(A)$ is a subspace of the space of all continuous and bounded functions defined on S. Moreover, there exists an operator $T : Q(A) \to A$ given by $T(Q(f)) = P(f)$, under some additional conditions on A.

We summarize these considerations in the next result. We denote by $P(A)|_S$ the space of all functions in the range of P restricted to S. The existence of the family $\mathcal{F}$ is established in Proposition 2.2. We first introduce a definition.

Definition 2.4. *Let W be a Borel subset of Ω. The space A has the W-norming property if and only if for every continuous function $f : W \to \mathbb{C}$ with a continuous extension to the closure of W, we have that $\|f\|_\infty = \sup_{\{\mu:\ \mu \in A_1^*\}} |\int_W f\, d\mu|$.*

Theorem 2.5. *(cf. [10]) Let A be a closed subspace of $C(\Omega)$ and let P be a contractive projection on A. Then there exist:*

(1) *A family $\mathcal{F} = \{\mu_i : i \in I\}$ of extreme points of $P^*(A_1^*)$ with the maximal support property,*
(2) *A function $\phi_i : \Omega \to \mathbb{S}^1$ such that, for every $i \in I$,*
$$\phi_i \in A_{S_{\mu_i}},$$
with S_{μ_i} denoting the support of μ_i, and an operator $Q : A \to P(A)|_S$, with $S = \cup_{i \in I} S_{\mu_i}$, such that, for every $x \in S_{\mu_i}$ and $f \in A$,
$$Q(f)(x) = \left(\int_\Omega f\, d\mu_i\right) \phi_i(x),$$
(3) *An operator $T : P(A)|_S \to A$ such that $\|T(Q(f))\|_\infty = \|Q(f)\|_\infty = \|P(f)\|_\infty$ and $P(f) = T(P(f)|_S)$, if A has the S^c-norming property.*

Proof. The proof provided for the Proposition 2.2 and follow-up considerations show the existence of a family of measures $\{\mu_i\}_{i \in I}$ which are extreme points of $P^*(A_1^*)$ with the maximal support property, as formulated in (1). For this collection of measures, and taking $\phi_i = \frac{P(g)}{\int_\Omega g\, d\mu_i}$, for a given $g \in A$ such that $\int_\Omega g\, d\mu_i \neq 0$, we have
$$P(f)(x) = \left(\int_\Omega f\, d\mu_i\right) \phi_i(x),$$
for every $f \in A$ and $x \in S_{\mu_i}$. By the definition of ϕ_i we infer that $\phi_i \in A_{S_{\mu_i}}$. We set $Q : A \to P(A)|_S$ defined by $Q(f) = P(f)|_S$. This proves (2).

The space A is isometrically embedded in A^{**}, via the canonical embedding J. For a function f in A we denote its image in A^{**} by $\tilde{f}$. We observe that $P^{**}(\tilde{f}) = \widetilde{P(f)}$, for every $f \in A$. This observation can be shown as follows: If $\tau \in A^*$, then
$$P^{**}(\tilde{f})(\tau) = \tilde{f}[P^*(\tau)] = P^*(\tau)(f) = \tau(P(f)) = \widetilde{P(f)}(\tau).$$

We recall the Goldstine Theorem: The closed unit ball of $J(A)$ is weak-* dense in the closed unit ball of A^{**}.

We set $P(f)|_S = \chi_S \cdot P(f)$, with χ_S denoting the characteristic function on S. This function is continuous on S but not necessarily on the topological boundary of S, this leads to considering the operator $\tilde{Q}$ on A^{**} defined by
$$\tilde{Q}(\xi) = \chi_S \cdot (P^{**}(\xi)),$$

for every $\xi \in A^{**}$. For $\mu \in A^*$, we set $\chi_S \cdot (P^{**}(\xi))(\mu) = \xi(P^*(\mu)|_S)$, where $P^*(\mu)|_S(f) = \int_S f\, dP^*(\mu)$ $(f \in A)$. In particular, for $f \in A$, $\tilde{Q}(\tilde{f}) = \chi_S \cdot \widetilde{Pf} = \widetilde{Pf}|_S$. Let $\tilde{R}$ be defined as follows:

$$\tilde{R}(\xi) = P^{**}(\xi) - \tilde{Q}(\xi), \;\; \xi \in A^{**}.$$

Hence, for $f \in A$,

$$\tilde{R}(\tilde{f}) = P^{**}(\tilde{f}) - \chi_S \cdot P^{**}(\tilde{f}) = \chi_{S^c} \cdot P^{**}(\tilde{f}).$$

We show that

$$\tilde{R}\tilde{Q} = \tilde{R}. \tag{2}$$

Remark 2.3 implies that $P^{**}(\chi_S \cdot \widetilde{f}) = P^{**}(\widetilde{f})$, since the support of any measure in A^* is contained in S. The weak-* density of $J(A)_1$ in A_1^{**} implies that $P^{**}(\chi_S \cdot \xi) = P^{**}(\xi)$, for every $\xi \in A^{**}$. Furthermore, $P^{**}(\chi_S \cdot P^{**}(\xi)) = P^{**}(\xi)$. We should recall that $\chi_S \cdot \widetilde{f}$ is given by $\chi_S \cdot \widetilde{f}(\mu) = \widetilde{f}(\mu|_S)$ with $\mu \in A^*$.

Towards the proof of the equation displayed in (2) we have

$$\tilde{R}\tilde{Q} = (P^{**} - \tilde{Q})\tilde{Q} = P^{**}\tilde{Q} - \tilde{Q} = P^{**} - \tilde{Q} = \tilde{R}.$$

Therefore $\tilde{R}\tilde{Q} = \tilde{R}$ and, for every $f \in A$, we have

$$\begin{aligned} \|\tilde{R}(\tilde{f})\| = \|\tilde{R}\tilde{Q}(\tilde{f})\| \leq \|\tilde{Q}(\tilde{f})\| &= \sup_{\mu \in A_1^*} |\chi_S \cdot (P^{**}(\tilde{f}))(\mu)| \\ &= \sup_{\mu \in A_1^*} \left| \int_S P(f) d\mu \right| \leq \|Q(f)\|_\infty. \end{aligned} \tag{3}$$

We now define the operator $T : Q(A) \to A$ given by $T(Q(f)) = P(f)$. First, we show that T is well defined. If f_0 and f_1, functions in A, are such that $Q(f_0) = Q(f_1)$ then $\tilde{Q}(\tilde{f}_0) = \tilde{Q}(\tilde{f}_1)$ and $\tilde{R}[\tilde{Q}(\tilde{f}_0) - \tilde{Q}(\tilde{f}_1)] = 0$. This implies that $\tilde{R}(\tilde{f}_0) = \tilde{R}(\tilde{f}_1)$. Hence $P^{**}(\tilde{f}_0) = P^{**}(\tilde{f}_1)$ or $P(f_0) = P(f_1)$.

Now, we prove that, for every $f \in A$,

$$\|P(f)\|_\infty = \|Q(f)\|_\infty.$$

For each function f we extend $Q(f)$ to the entire Ω by assigning zero to those points in $\Omega \setminus S$. We denote this new function by $Q(f)$ for simplicity of notation. Since $Q(f)$ and $(P - Q)(f)$ have disjoint supports then $\|P(f)\|_\infty = \max\{\|Q(f)\|_\infty, \|(P-Q)(f)\|_\infty\}$. We have shown that $\|\tilde{R}(\tilde{f})\| \leq \|Q(f)\|_\infty = \|\chi_S \cdot P(f)\|_\infty$ and we also have

$$\|P(f)\|_\infty = \|P^{**}(\tilde{f})\| = \max\{\|\chi_S \cdot P(f)\|_\infty, \|\chi_{S^c} \cdot P(f)\|_\infty\}.$$

The space A has the S^c-norming property, then applying this property to the function $(P-Q)f$ we have

$$\begin{aligned}\|\chi_{S^c}\cdot P(f)\|_\infty = \|(P-Q)(f)\|_\infty &= \sup_{\{\mu:\,\mu\in A_1^*\}}\left|\int_\Omega (P-Q)(f)\,d\mu\right| \\ &= \sup_{\mu\in A^*;|\mu|=1}\left|\int_{S^c} P(f)d\mu\right| \\ &= \sup_{\mu\in A^*;|\mu|=1}\left|\int_{S^c} fd(P^*\mu)\right| \\ &= \|\chi_{S^c}\cdot(P^{**}(\tilde{f}))\| = \|P^{**}(\tilde{f}) - \tilde{Q}(\tilde{f})\| \\ &= \|\tilde{R}(\tilde{f})\|.\end{aligned}$$

Thus

$$\|P(f)\|_\infty = \|Q(f)\|_\infty.$$

Then T is an isometric simultaneous extension and completes the proof. □

We now derive some results for bi-contractive projections on a closed subspace of $C(\Omega)$. We start with a definition.

Definition 2.6. *Given a contractive projection P on A, let $\mathcal{F}_I$ be a maximal family as defined in Theorem 2.5-1. Then A has the support extension property iff for every Borel subset W of S, the union of the supports of the measures in $\mathcal{F}_I$, every point $x\notin \overline{W}$, $\lambda\in\mathbb{S}^1$ and every $f\in A|_S$ there exists a function $g\in A$ such that $g|_W = f|_W$ and* $g(x)=\|g\|_\infty = 1$.

Proposition 2.7. *Let A be a closed subspace of $C(\Omega)$ with the support extension property. Let P be a bi-contractive projection on A and μ an extreme point of $P^*(A_1^*)$. Then the support of μ has at most two points.*

Proof. Let W be an open subset of S_μ. We claim that $|\mu|(W)\geq \frac{1}{2}$. Suppose that $0<|\mu|(W)<1/2$. Then, for every open subset W_0 of W such that $W_0\subset\overline{W_0}\subset W$ we have that $0<|\mu|(W_0)<\frac{1}{2}$. Theorem 2.5 implies that for every $f\in A$, $P(f)(x) = \left(\int_\Omega f d\mu\right)\phi(x)$, for every $x\in S_\mu$. We recall that $\phi\in A_{S_\mu}$ and $\mu=\overline{\phi}\cdot|\mu|$. We select $z\in S_\mu\setminus W_0$ such that $P(f)(z) = \left(\int_\Omega f d\mu\right)\phi(z)$.

The support extension property implies the existence of $f\in A$ such that

$$f(x) = -\phi(x)\cdot\overline{\phi(z)},\quad \text{for } x\in S_\mu\setminus W_0,\ \text{ and }\ \|f\|_\infty = f(z) = 1.$$

Since $P(f)(z) = \left(\int_{S_\mu} fd\mu\right)\phi(z)$, we have

$$P(f)(z) = \phi(z)\int_{W_0} f\,d\mu + \int_{S_\mu\setminus W_0} -\phi\,d\mu.$$

We observe that $|\phi(z) \cdot \int_{W_0} f d\mu| \leq |\mu|(W_0) < \frac{1}{2}$, which implies that Re $\left(\int_{W_0} f \cdot \phi(z) d\mu\right) < \frac{1}{2}$. On the other hand,

$$\int_{S_\mu \setminus W_0} -\phi \, d\mu = \int_{S_\mu \setminus W_0} -\phi \cdot \overline{\phi} \, d|\mu| = -|\mu|(S_\mu \setminus W_0) < -\frac{1}{2}.$$

Then $\mathrm{Re}(P(f)(z)) < 0$ and

$$|(I-P)(f)(z)| \geq 1 - \mathrm{Re}(P(f)(z)) > 1,$$

which contradicts the assumption that $I - P$ is contractive. This proves that, for every W, an open subset of S_μ, $|\mu|(W) \geq \frac{1}{2}$. Hence $S_\mu = \{x\}$ or $S_\mu = \{x, y\}$. In the first case S_μ is a singleton and the measure is the Dirac measure concentrated on x. In the second case, $|\mu|(\{x\}) = |\mu|(\{y\}) = \frac{1}{2}$. This completes the proof. □

The next result shows that under the same hypotheses of the Proposition 2.7, we have $(P - Q)(f)(x) = 0$, for every $f \in A$ and $x \notin S$.

Proposition 2.8. *Let A be a closed subspace of $C(\Omega)$ with the support extension property. Let P be a bi-contractive projection on A. Then for every $f \in A$, the support of $P(f)$ is contained in S.*

Proof. Suppose $(P - Q)(f)(x) \neq 0$, for some $f \in A$ and some point $x \notin \overline{S}$. We may assume that $\|f\|_\infty = 1$. Since A has the support extension property there exists g such that $g|_S = f|_S$ and $g(x) = 1 = \|g\|_\infty$.

If the real part of $(P - Q)(f)(x)$ is negative then we shall prove that the real part of $(I - P)(g)(x)$ is greater than 1. We observe that

$$(4) \qquad (I-P)(g)(x) = 1 - P(g)(x) = 1 - [Q + (P-Q)](g)(x) = 1 - (P-Q)(f)(x).$$

We claim that $(P-Q)(g) = (P-Q)(f)$ on S^c. To justify this claim we revisit the operator $\tilde{R}$ defined for the proof of Theorem 2.5. Since $\tilde{R} = P^{**} - \tilde{Q}$, then

$$\tilde{R}(\tilde{g}) = P^{**}(\tilde{g}) - \tilde{Q}(\tilde{g}) = \chi_{S^c} \cdot P^{**}(\tilde{g}).$$

On the other hand, we also have

$$\tilde{R}(\tilde{g}) = \tilde{R}\tilde{Q}(\tilde{g}) = \tilde{R}\tilde{Q}(\tilde{f}) = \tilde{R}(\tilde{f}) = \chi_{S^c} \cdot P^{**}(\tilde{f}).$$

Since $Q(g)$ and $Q(f)$ at any point in S^c are equal to zero then we have $(P - Q)(g) = (P-Q)(f)$ on S^c. Hence, $Q(g)(x) = 0$ and $(P-Q)(f)(x) = (P-Q)(g)(x)$. This explains the equalities displayed in (4).

Therefore, $\mathrm{Re}\left((I - P)(g)(x)\right) = \mathrm{Re}\left((1 - (P - Q)(f)(x))\right) > 1$. This contradicts the assumption that $I - P$ is contractive. If $\mathrm{Re}\left((P - Q)(f)(x)\right) > 0$ then we consider g such that $g|_S = -f|_S$ and $g(x) = 1 = \|g\|_\infty$ to get a contradiction. A similar reasoning applies if the imaginary part of $(P - Q)(f)(x)$ is nonzero. This completes the proof. □

Remark 2.9. *If P is a bi-contractive projection on a subspace of $C(\Omega)$, satisfying the hypotheses of Proposition 2.7 then P is given as the average of the identity with an isometric reflection. It is not clear which subspaces of $C(\Omega)$ satisfy the support extension property.*

3. Some remarks on the GBPs=BCPs

A generalized bi-circular projection P on a Banach space is an idempotent bounded operator P for which there exists a modulus 1 complex number λ, different from 1, such that $P + \lambda(I - P)$ is an isometry. If we set $T = P + \lambda(I - P)$, then T is a surjective isometry since

$$(P + \lambda(I - P))\,(P + \overline{\lambda}(I - P)) = I.$$

It is a known result that generalized bi-circular projections are bi-contractive, see [14]. For completeness of exposition we include a proof of this fact. For every $n \in \mathbb{N}$, we have

$$T^n = P + \lambda^n(I - P).$$

If the sequence $\{\lambda^n\}$ is dense and by considering a subsequence that converges to -1 we conclude that $2P - I$ is an isometry. Therefore $2\|P\| - 1 \leq 1$ or P is contractive. Moreover, we also have that $2\|I - P\| - 1 \leq \|2(P - I) + I\| = 1$, which implies that P is bi-contractive. If there exists n (the smallest positive integer) such that $\lambda^n = 1$, then

$$nP + \sum_{i=1}^{n} \lambda^i \,(I - P) = \sum_{i=1}^{n} T^i.$$

The sum $\sum_{i=1}^n \lambda^i = 0$ and $n\|P\| = \|\sum_{i=1}^n T^i\| \leq \sum_{i=1}^n \|T^i\| = n$, hence P is contractive. A similar proof applied to the complement projection $I - P$ implies that P is bi-contractive.

Generalized bi-circular projections on a Hilbert space are the hermitian projections, see Proposition 3.1 in [6]. Hermitian projections on a Hilbert space are the orthogonal projections, see [11]. Therefore the bi-contractive projections on a Hilbert space are the generalized bi-circular projections. Hilbert spaces have GBPs=BCPs.

We now recall Kawamura-Koshimizu-Miura spaces of continuously differentiable functions defined on the unit interval $[0, 1]$ endowed with any of the norms defined as follows:

$$\|\cdot\|_{\langle D\rangle},$$

where D is a connected and compact subset of $[0, 1]^2$ such that the union of the two canonical projections $\pi_1(D) \cup \pi_2(D) = [0, 1]$, then

$$\|f\|_{\langle D\rangle} = \sup_{(t,s)\in D} |f(t)| + |f'(s)|.$$

These spaces can be isometrically embedded in $C(D \times \mathbb{S}^1)$. Each such space can be identified to a subspace of $C(\Omega)$ with $\Omega = D \times \mathbb{S}^1$.

We observe that for those sets D such that $\pi_1(D) = \pi_2(D) = [0, 1]$, the corresponding KKM space is a commutative Banach algebra, then the Gelfand transform is a contraction. These spaces are not closed under conjugacy. Towards this claim we observe that for

F such that $F(s,t,z) = f(s) + zf'(t)$ with $f \in C^1[0,1]$, the complex conjugate of F, $\overline{F}(s,t,z) = \overline{f(s)} + \overline{z}\overline{f'(t)}$. If we assume that $\overline{F}$ is a function in the subspace of $C(\Omega)$ isometric to $C^1[0,1]$, then there exists $g \in C^1[0,1]$ such that for every $(s,t,z) \in \Omega$ we have

$$\overline{F}(s,t,z) = \overline{f(s)} + \overline{z}\overline{f'(t)} = g(s) + zg'(t).$$

In particular, for $z = \pm 1$ we conclude that $g(s) = \overline{f(s)}$ for every s, hence $g'(s) = \overline{f'(s)}$. Now setting $z = i$ we have $-i\overline{f'(t)} = ig'(t) = i\overline{f'(t)}$. This leads to contradiction.

Surjective linear isometries on KKM spaces were characterized in [13]. From this characterization we can describe the generalized bi-circular projections. As mentioned before generalized bi-circular projections are bi-contractive but it is not clear if those are the bi-contractive projections on these settings.

4. Bi-Contractive Projections on vector valued spaces of continuous functions

In this section we give a brief outline on how to extend the methods and results presented before to spaces of vector valued continuous functions. As before, Ω is a compact Hausdorff space and E is a uniformly convex Banach space with norm $\|\cdot\|_E$. Under these conditions we can extend the techniques of the scalar case to this new setting. We give a characterization for the bi-contractive projections and conditions under which the class of the generalized bi-circular projections coincide with the class of the bi-contractive projections, the details are available in a forthcoming paper, see [4].

We observe that for the space of all continuous functions $f : \Omega \to E$ endowed with the infinite norm, i.e. $\|f\|_\infty = \sup_{x\in\Omega} \|f(x)\|_E$ with E a selfadjoint commutative Banach algebra, the space $C(\Omega, E)$ is also a selfadjoint commutative Banach algebra. Under this condition the Gelfand theory applies and $C(X,E)$ is isometrically isomorphic to the space of continuous functions on the carrier space of $C(\Omega, E)$. It is known that the carrier space of $C(\Omega, E)$ or the space of nontrivial multiplicative functionals on $C(\Omega, E)$ is homeomorphic to $\Omega \times \triangle(E)$, where $\triangle(E)$ is the carrier space of E. This space endowed with the weak-* topology is a compact Hausdorff space. Contractive and bi-contractive projections can transfer to projections of the same type on a space of continuous functions on a compact Hausdorff space. Then we conclude that GBPs=BCPs.

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