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EISENSTEIN SERIES ON COVERING GROUPS, CRYSTAL GRAPHS, AND CANONICAL BASES

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ABSTRACT. The Whittaker coefficients of Eisenstein series on covering groups may be described by attaching number-theoretic quantities to objects that appear in the theory of quantum groups, namely crystal graphs and canonical bases. This description connects work by three mathematicians in apparently unrelated areas: T. Kubota (number theory/automorphic forms), M. Kashiwara (quantum groups) and T. Tokuyama (combinatorics).

1. INTRODUCTION

Let $n \geq 1$, F be a number field containing a full set μ_n of n-th roots of unity, and G be a split semisimple algebraic group defined over F. Then there is a central simple extension $\widetilde{G}$ of $G(\mathbb{A}_F)$ by μ_n,

$$1 \to \mu_n \to \widetilde{G} \to G(\mathbb{A}_F) \to 1.$$

The construction of such an extension goes back to Matsumoto [11]; generalizations to wider classes of groups G were given by Brylinski and Deligne [5]. Our object here is to describe Eisenstein series on these covering groups when G is the general linear group and to answer a basic question: what are the Whittaker-Fourier coefficients of such an Eisenstein series?

This question may be phrased in a concrete way, and indeed it is helpful do to so in order to carry out computations. Such a formulation goes back to Kubota. Let us suppose that $n > 1$ and that in fact F contains the $2n$-th roots of unity (so in particular F has no real embeddings). Let $\left(\frac{c}{d}\right)_n$ be the n-th power residue symbol. Let Γ be the principal congruence subgroup of $SL(2, \mathcal{O}_F)$ modulo n^2. Then Kubota [9] showed that the map $\kappa : \Gamma \to \mu_n$ given by

$$\kappa\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} \left(\frac{c}{d}\right)_n & \text{if } c \neq 0 \\ 1 & \text{if } c = 0 \end{cases}$$

is a homomorphism. The proof uses the n-th power reciprocity law. Note that this map is fundamentally different than sending a matrix to a Dirichlet character modulo d. Indeed, the kernel of κ is not a congruence subgroup.

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One may construct an Eisenstein series on SL_2 that incorporates κ:

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \kappa(\gamma)\, \Im(\gamma \circ z)^s.$$

Here z is in a product of r_2 copies of hyperbolic three space (where r_2 is the number of pairs of complex conjugate embeddings of F into $\mathbb{C}$), and $\Im$ is the natural analogue of the imaginary part function for this space. More generally one could take $\Im$ to be any function in a certain induced space. The study of these Eisenstein series is equivalent to the study of Eisenstein series on the n-fold cover of $GL_2(\mathbb{A}_F)$.

Kubota [10] analyzed the Fourier coefficients of $E(z,s)$ and showed that the m-th Fourier coefficient, $m \neq 0$, is a Dirichlet series in s whose coefficients are n-th order Gauss sums. These series are not (for $n > 2$) Langlands L-functions, but they have analytic continuation and functional equation in s! To describe these coefficients we change the notation slightly. Let S be a set of places containing all archimedean places and all finite places that are ramified over $\mathbb{Q}$ and that is sufficiently large that the ring of S-integers $\mathcal{O}_S$ has class number one. Let $\left(\frac{c}{d}\right)_n$ now be the n-th power residue symbol for $\mathcal{O}_S$. Then Brubaker and Bump [1] reformulated Kubota's result over $\mathcal{O}_S$ (they also gave an explicit scattering matrix for the functional equation). The m-th coefficient (for $\Re(s) \gg 0$) is of the form

$$\sum_{c \neq 0} \frac{g_n(m,c)}{Nc^{2s}} \Psi_m(c) \tag{1}$$

where Ψ_m ranges over a certain finite dimensional vector space of functions that will not concern us, N denotes the absolute norm, and $g_n(m,c)$ is the n-th order Gauss sum modulo c

$$g_n(m,c) = \sum_{\substack{d \bmod c \\ (d,c)=1}} \left(\frac{d}{c}\right)_n e(md/c)$$

where e is an additive character of conductor $\mathcal{O}_S$. The sum in (1) is over nonzero ideals in $\mathcal{O}_S$, and the function Ψ_m has the correct equivariance property so that each summand in (1) is independent of the choice of generator c for the ideal $c\mathcal{O}_S$.

The arithmetic piece of the coefficient, that is the Gauss sum $g_n(m,c)$, may be reconstructed by elementary means from the prime power coefficients of the form $g_n(p^a,p^b)$ with $a,b \geq 0$ and p ranging over all primes. So we focus on the coefficients $g_n(p^a,p^b)$. It is easy to see that

- If $b \geq a+2$: $g_n(p^a,p^b) = 0$ (because of the oscillation of the additive character);
- If $b \leq a$: $g_n(p^a,p^b)$ is $\phi(p^b)$ if $n \mid b$, and zero otherwise (because the additive character is identically 1). Here ϕ is the Euler phi-function for the ring $\mathcal{O}_S$. If $b \leq a$, we write $h_n(b)$ for this simple arithmetic function.

By contrast in the case $b = a+1$ the sum is always nonzero and gives a non-trivial n-th order Gauss sum when $(n,b) = 1$. We represent the situation with the graph

$$\underset{b=0}{\odot} - \underset{b=1}{\cdot} - \underset{b=2}{\cdot} - \cdots \underset{b=a}{\cdot} - \underset{b=a+1}{\boxdot} \tag{2}$$

Here the contributions when $b = 0$ and $b = a+1$ are special (being $(Np)^0$ and a non-trivial n-th order Gauss sum, resp.) and are so indicated in the picture with a circle and box, resp. For the remaining locations, the contribution is simply $h_n(b)$. We emphasize that while the functions h_n and g_n depend on n, the picture is essentially the same for any n.

A key point is that (2) represents the crystal graph attached to a representation of quantum gl_2! More precisely, the vertices and edges are the crystal graph of the irreducible representation of highest weight $(a+1)\epsilon$ where ϵ is the fundamental weight. These graphs were introduced by Kashiwara, and capture aspects of the representation theory of this algebraic object (the edges represent the Kashiwara operators). The two special locations marked with a box and a circle correspond to the maximal root string going to the lowest and to the highest weight vector, resp.

Remarkably, this description generalizes to GL_{r+1} for any $r \geq 1$. The analogue of $E(z,s)$ is the Borel Eisenstein series on GL_{r+1}, which is a function of r complex variables. The Whittaker coefficients are also indexed by r integral parameters (corresponding to the simple roots). To avoid a lot of notation, we shall state the result roughly.

Theorem 1 (Brubaker, Bump, Friedberg [2]). *Let $m_1, \dots, m_r \neq 0$. Then the $\mathbf{m} := (m_1, \dots, m_r)$-th Whittaker coefficient of the Borel Eisenstein series on an n-fold cover of GL_{r+1} is a multiple Dirichlet series of the form*

$$\sum_{c_1,\dots,c_r \neq 0} \frac{H_{\mathbf{m}}(c_1, \dots, c_r)}{Nc_1^{2s_1} \dots Nc_r^{2s_r}} \Psi_{\mathbf{m}}(c_1, \dots, c_r).$$

The arithmetic coefficients $H_{\mathbf{m}}(\mathbf{c})$ for general $\mathbf{m}, \mathbf{c}$ may be computed from the coefficients of the form $H_{p^{\mathbf{a}}}(p^{\mathbf{b}})$ with p prime and $\mathbf{a}$, $\mathbf{b} \in (\mathbb{Z}_{\geq 0})^r$. Moreover, these prime power coefficients may be expressed as sums of arithmetic quantities in terms of a crystal graph attached to quantum gl_{r+1}.

The description in terms of crystal graphs is a bit intricate. The highest weight of the underlying representation is determined from $\mathbf{a}$, and there is a shift by ρ, half the sum of the positive roots. For each vertex of the corresponding crystal graph, one attaches the path on this crystal graph from the vertex to the lowest weight vector which is obtained by applying the Kashiwara operators in the order determined by a certain factorization of the long element into simple reflections. One then records the lengths b_i corresponding to the pieces of the path which are the root strings for each Kashiwara operator, and decorates some of the b_i by boxes and some by circles corresponding to root strings which are extremal. The contribution from the segment of length b_i is $h_n(b_i)$ generically (that is, if it is neither boxed nor circled), Np^{b_i} if b_i is circled, and the Gauss sum $g_n(p^{b_i-1}, p^{b_i})$ if b_i is boxed. If b_i is both boxed and circled (which does not happen for GL_2 but does occur in higher rank situations), the contribution is zero. One then takes the product of these contributions to determine the arithmetic quantity attached to the given vertex. See [2] for details. There is also a dual version using paths to the highest weight vector.

Though we have specified that $n > 1$, in fact such a description applies when $n = 1$ as well. For the group itself (that is, for the 1-fold cover), the Whittaker coefficients at a prime p were shown by Shintani [12] to be Schur polynomials (this statement was generalized to other groups in the Casselman-Shalika formula). A formula of Tokuyama [13] expresses the Schur polynomial as a sum over semi-standard Young tableaux. Tokuyama's formula may be recast ([3], Chapter 5) as a formula for the Schur polynomial attached to a representation of $GL_{r+1}(\mathbb{C})$ of highest weight λ as a sum over the crystal graph attached to highest weight $\lambda+\rho$. This is exactly the expression of the above Theorem when $n=1$.

In closing we mention that these theorems generalize. Friedberg and Zhang [6] have established crystal graph descriptions of the Whittaker coefficients of Eisenstein series for covers of odd orthogonal groups (other root systems are in progress). They have also used Eisenstein series on symplectic groups to give new Tokuyama-type formulas for characters of the spin group $\mathrm{Spin}_{2r+1}(\mathbb{C})$ [7]. And Brubaker and Friedberg [4] have considered the Whittaker coefficients of maximal parabolic Eisenstein series on covering groups, establishing additional connections to the representation theory of quantum groups, and in particular to Lusztig's canonical bases.

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