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EFFECTIVE COMPUTATION OF DIMENSION FORMULAS FOR MODULAR FORMS TAKING VALUES IN WEIL REPRESENTATIONS

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ABSTRACT. Though explicit dimension formulas for vector valued modular forms are well-known they are not efficiently computable as soon as the dimension of the underlying $\mathrm{SL}(2,\mathbb{Z})$-module grows. For the case of Weil representations, we describe the tools for simplifying the critical terms in the corresponding dimension formulas in order to obtain formulas which can be rapidly computed.

CONTENTS

1. INTRODUCTION

For an integer or half integer k, an integral lattice $\underline{L}$ over the rational integers and a character ε^h of the nontrivial central extension $\mathrm{Mp}(2,\mathbb{Z})$ of $\mathrm{SL}(2,\mathbb{Z})$ by $\{\pm 1\}$, let $J_{k,\underline{L}}(\varepsilon^h)$ denote the space of Jacobi forms of weight k, index $\underline{L}$, and on the full modular group with character[1] ε^h (see [BS] or [Sko08] for the basic theory of these forms). There is a natural isomorphism of $J_{k,\underline{L}}(\varepsilon^h)$ with the space of vector valued modular forms of weight $k-n/2$, where $n=\mathrm{rank}(\underline{L})$, with values in the Weil representations attached to the discriminant module of $\underline{L}$ rescaled by -1 and twisted[2] by ε^h. This implies in particular that $J_{k,\underline{L}}(\varepsilon^h)=0$ for $k<\frac{n}{2}$, and that for $k\in\{\frac{n}{2},\frac{n+1}{2}\}$, the spaces $J_{k,\underline{L}}(\varepsilon^h)$ are naturally isomorphic to spaces of invariants of twisted Weil representation derived from the discriminant module of $\underline{L}$. An explicit dimension formula for vector valued modular forms are well-known; it was first given in [Sko85, Satz 5.1], and restated later by various authors. A straightforward application of this dimension formula ε^h to the vector valued modular forms corresponding to $J_{k,\underline{L}}(\varepsilon^h)$ gives the following.

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[1]Here h is an integer and ε the linear character of $\mathrm{Mp}(2,\mathbb{Z})$ afforded by Dedekind's eta-function.

[2]By the *twist of an* $\mathrm{Mp}(2,\mathbb{Z})$*-module* V *by* ε^h we mean the product $V\otimes\mathbb{C}(\varepsilon^h)$ of V with the 1-dimensional $\mathrm{Mp}(2,\mathbb{Z})$-module with character ε^h.

Theorem 1.1 ([BS]). *For every k in $\frac{1}{2}\mathbb{Z}$, every integral positive definite lattice $\underline{L} = (L, \beta)$ of rank n, and every integer h, one has $J_{k,\underline{L}}(\varepsilon^h) = 0$ if $p := k - h/2$ is not an integer. Otherwise one has*

$$\begin{aligned}
&\dim J_{k,\underline{L}}(\varepsilon^h) - \dim J^{skew,\, cusp}_{n+2-k,\underline{L}}(\varepsilon^h) \\
&\quad = \tfrac{1}{24}\left(k - \tfrac{n}{2} - 1\right)\left(\det(\underline{L}) + (-1)^{p+n_2}\, 2^{n-n_2}\right) \\
&\quad + \tfrac{1}{4}\operatorname{Re}\left(e_4(p)\,\chi_{\underline{L}}(2)\right) + \tfrac{1}{6}\left(\tfrac{12}{2p+2n+1}\right) \\
&\quad + \tfrac{(-1)^p}{3\sqrt{3}}\operatorname{Re}\left(e_6(p)e_{24}(n+2)\chi_{\underline{L}}(-3)\right) \\
&\quad + P_{\underline{L}}(h),
\end{aligned}$$

where

$$P_{\underline{L}}(h) = -\tfrac{1}{2}\sum_{x\in L^\bullet/L}\left\langle \tfrac{h}{24} - \beta(x)\right\rangle - \tfrac{(-1)^{p+n_2}}{2}\sum_{\substack{x\in L^\bullet/L\\ 2x\in L}}\left\langle \tfrac{h}{24} - \beta(x)\right\rangle.$$

Here n_2 is the rank of the unimodular constituent of the Jordan decomposition of $\underline{L}$ over $\mathbb{Z}_2$, and $\langle x\rangle = x - \lfloor x\rfloor - 1/2$. Moreover, for any integer t, we use

$$\chi_{\underline{L}}(t) = \frac{1}{\sqrt{\operatorname{card}(L^\bullet/L)}}\sum_{x\in L^\bullet/L} e\left(t\beta(x)\right).$$

Recall that a lattice $\underline{L}$ is a pair (L, β) of a free $\mathbb{Z}$-module of finite rank and of a symmetric nondegenerate bilinear form $\beta : L \otimes L \to \mathbb{Z}$. We use here and in the following

$$\beta(x) = \frac{1}{2}\beta(x, x).$$

In the formula of the theorem $L^\bullet$ denotes the shadow of the lattice $\underline{L}$ (whose definition is recalled in §2) and in the sums defining $P_{\underline{L}}(h)$ and $\chi_{\underline{L}}(t)$ the variable x runs over a complete set of representatives for the orbits $L^\bullet/L$.

A discussion of this formula and its consequences can be found in [BS] and, with slightly different notations, in [Sko08]. In particular, it can be shown that the term $J^{\mathrm{skew,cusp}}_{n+2-k,\underline{L}}(\varepsilon^h)$ is zero for $k > \frac{n}{2} + 2$, and equals a space of invariants of certain twisted Weil representations for $k = \frac{n}{2} + 2$ and $k = \frac{n}{2} + 3/2$.

Summarizing, we see that the problem of finding explicit and ready to compute formulas for the dimension of $J_{k,\underline{L}}(\varepsilon^h)$ reduces to three very different problems depending on the weight. For $k \in \{\frac{n}{2}, \frac{n}{2} + \frac{1}{2}, \frac{n}{2} + \frac{3}{2}, \frac{n}{2} + 2\}$ it is equivalent to the problem of computing dimensions of spaces of invariants of twisted Weil representations. For $k = \frac{n}{2} + 1$ is is open, no general method is known (though computationally, for lattices of small discriminant, one can sometimes successfully try to determine the subspace $\eta^l J_{\frac{n}{2}+1,\underline{L}}(\varepsilon^h)$ of $J_{\frac{n}{2}+1+\frac{l}{2},\underline{L}}(\varepsilon^{h+l})$).

For $k > \frac{n}{2} + 2$ the above theorem provides an explicit formula. However, for computations it is still not satisfactory, since a straightforward implementation which simply copies the formula literally into any computer algebra system would easily run into problems when $\operatorname{card}(L^\bullet/L) = \det(\underline{L})$ becomes large. The problem is caused by the functions $\chi_{\underline{L}}$ and the *parabolic contribution* $P_L(h)$, which in a naive implementation require to sum $\det(L)$ many terms. In this note we concentrate on this problem and sketch how to effectively solve it. The main results are summarized in Propositions 2.3, 3.2 and 3.3 for the calculation of $\chi_{\underline{L}}(t)$, and in Theorem 4.1 for the calculation of $P_{\underline{L}}(h)$.

2. The characteristic function of a lattice

For a given lattice $\underline{L}(L,\beta)$ the set

$$L^\bullet := \{r \in \mathbb{Q}\otimes L : \beta(r,x) \equiv \beta(x) \bmod \mathbb{Z} \text{ for all } x \in L\}$$

is called the *shadow of* $\underline{L}$ and its elements the *shadow vectors of* $\underline{L}$. If $\underline{L}$ is even (i.e. if $\beta(x)$ is integral for all x) the shadow of $\underline{L}$ equals its dual $L^\sharp$. If $\underline{L}$ is *odd* (i.e. integral but not even), then $L^\bullet$ equals the nontrivial coset of ${L_{\mathrm{ev}}}^\sharp/L^\sharp$, where L_{ev} denotes the maximal sublattice on which β is even, i.e.

$$L_{\mathrm{ev}} = \ker\left(L \to \mathbb{Z}/2\mathbb{Z},\ x \mapsto \beta(x) + 2\mathbb{Z}\right).$$

From this we deduce in particular $\operatorname{card}(L^\bullet/L) = \det(L)$ (where the right hand side is the determinant of any Gram matrix of L. The first observation for computing the functions $\chi_{\underline{L}}$ is

Proposition 2.1 ([BS]). *For any integral non-degenerate lattice $\underline{L}$ of signature s_∞, one has*

$$\chi_{\underline{L}}(1) = \chi_{\underline{L}_{\mathrm{ev}}}(1) = e_8(s).$$

The second identity is sometimes called Milgram's identity [MH73, p. 127]. The first one, in contrast, is easy and we refer to [BS].

For odd $\underline{L}$ we need, of course, to compute first of all $\underline{L}_{\mathrm{ev}}$. However this is rapidly done by the following proposition whose easy proof is left to the reader.

Proposition 2.2. *Let e_i be a basis of L, and let S be the set of indices i such that the $\beta(e_i)$ is not in $\mathbb{Z}$. A basis for $\underline{L}_{\mathrm{ev}}$ is then given by e_i ($i$not $\in S$) and $e_i + e_j$ ($i \in S$, $i \neq j$) and $2e_j$, where j is any fixed index in S.*

We shall see in a moment (Proposition 2.3) that the determination of $\chi_{\underline{L}}(t)$ for arbitrary t can be reduced to the computation of $\chi_{\underline{L}'}(1)$ for even lattices $\underline{L}'$ which depend on $\underline{L}$ and t. However it is inconvenient to determine the corresponding $\underline{L}'$ at the level of lattices. A more convenient treatment can be achieved by introducing finite quadratic modules, which provides also a more uniform treatment of the functions $\chi_{\underline{L}}$ for even latices.

To an even lattice we can associate its discriminant module $\operatorname{disc}(\underline{L})$. This is the *finite quadratic module* (FQM) with underlying abelian group $L^\sharp/L$ and quadratic form $\underline{\beta} : L^\sharp/L \to \mathbb{Q}/\mathbb{Z}$ given by $\underline{\beta}(x+L) = \beta(x) + \mathbb{Z}$. More generally, a finite quadratic module $\mathfrak{M}$ is a pair (M,Q), where M is a finite abelian group and $Q : M \to \mathbb{Q}/\mathbb{Z}$ a quadratic form. The latter means that $Q(ax) = a^2Q(x)$ for all integers a and x in M, that $Q(x,y) := Q(x+y) - Q(x) - Q(y)$ is bilinear and $x \mapsto Q(x,\_)$ defines an isomorphism of M with $\operatorname{Hom}(M,\mathbb{Q}/\mathbb{Z})$.

For any finite quadratic module, we set

$$\chi_{\mathfrak{M}}(t) = \frac{1}{\sqrt{\operatorname{card}(M)}} \sum_{x\in M} e\big(tQ(x)\big).$$

Since $\chi_{\underline{L}} = \chi_{\operatorname{disc}(\underline{L})}$ it suffices to discuss how to compute $\chi_{\mathfrak{M}}(t)$ for any given finite quadratic module $\mathfrak{M}$. Note that $\chi_{\underline{L}}(t)$ depends only on t modulo ℓ, where ℓ denote the *level of* $\mathfrak{M}$, i.e. the smallest positive integer such that $\ell Q = 0$.

It can be shown [Wal63, Theorem (6)] that every finite quadratic module is in fact isomorphic to he discriminant module of a lattice. In particular,

$$\chi_{\mathfrak{M}}(1) = e_8(s),$$

where s is the signature of any even $\underline{L}$ with disc($\underline{L}$) isomorphic to $\mathfrak{M}$. We call

$$s_\infty(\mathfrak{M}) := s \bmod 8$$

the *signature of* $\mathfrak{M}$ *at infinity.*

Using FQM we can now describe how to compute $\chi_{\underline{L}}(t)$ for odd $\underline{L}$ in terms of $\chi_{\mathfrak{M}}(t)$.

Proposition 2.3. *Let* $\underline{L} = (L, \beta)$ *be an odd lattice,* t *an integer, and let* $L_t := \frac{1}{t}L \cap L^\sharp$.

(1) *If* $t\beta$ *takes on integral values on* L_t*, then* $\mathfrak{M} := \big(L^\sharp/L_t, x + L_t \mapsto t\beta(x) + \mathbb{Z}\big)$ *defines an FQM, and one has* $\chi_{\underline{L}}(t) = \chi_{\underline{L}_{\mathrm{ev}}}(t) - \sqrt{\operatorname{card}(L_t/L)}\,\chi_{\mathfrak{M}}(1)$.
(2) *If* $t\beta(x)$ *is not integral for at least one* x *in* L_t*, one has* $\chi_{\underline{L}}(t) = \chi_{\underline{L}_{\mathrm{ev}}}(t)$.

We leave the easy proof of the proposition to the reader (for the proof the reader might wish to observe that $x \mapsto t\beta(x) + \mathbb{Z}$ defines a homomorphism of groups on L_t).

3. Formulas for the characteristic function of an FQM

The following proposition reduces the computation of $\chi_{\mathfrak{M}}(t)$ to the computation of $\chi_{\mathfrak{M}'}(1)$ for a suitable $\mathfrak{M}'$. Note that, for any integer t the maps $x \mapsto tQ(t)$ defines a n homomorphism of groups of $M[t]$ into $\mathbb{Q}/Z$. Her $M[t]$ is the submodule of x in M such that $tx = 0$.

Proposition 3.1. *Let* s *and* t *be integers.*

(1) *If the homomorphism* $M[t] \to \mathbb{Q}/\mathbb{Z}$, $x \mapsto tQ(x)$ *is non-trivial then one has* $\chi_{\mathfrak{M}}(t) = 0$.
(2) *Otherwise,* $\mathfrak{M}(t) := (M/M[t], x + M[t] \mapsto tQ(x))$ *is a finite quadratic module, and*

$$\chi_{\mathfrak{M}}(st) = \sqrt{\operatorname{card}(M[t])}\,\chi_{\mathfrak{M}(t)}(s).$$

For calculating $\chi_{\mathfrak{M}}(1)$ we recall that very finite quadratic p-module can be decomposed as a direct sum of modules of the form

$$\mathfrak{A}_q(a) := \left(\mathbb{Z}/q\mathbb{Z}, \frac{ax^2}{q}\right),$$

where q is a power of an odd prime and a an integer which is relatively prime to q, and of modules of the form

$$\mathfrak{A}_{2^t}(a) := \left(\mathbb{Z}/2^t\mathbb{Z}, \frac{ax^2}{2^{t+1}}\right),$$

$$\mathfrak{B}_{2^s} := \left(\mathbb{Z}/2^s\mathbb{Z} \times \mathbb{Z}/2^s\mathbb{Z}, \frac{x^2 + xy + y^2}{2^s}\right), \quad \mathfrak{C}_{2^s} := \left(\mathbb{Z}/2^s\mathbb{Z} \times \mathbb{Z}/2^s\mathbb{Z}, \frac{xy}{2^s}\right),$$

where s, t are positive integers and a is odd. Indeed, for a prime p, let $M[p^\infty]$ the p-part of M, which is the submodule of elements annihilated by a sufficiently large p-power. Then $\mathfrak{M}(p^\infty) := (M[p^\infty], Q|_M[p^\infty])$ is a finite quadratic module and one verifies that $\mathfrak{M}$ equals the direct sum of the $\mathfrak{M}(p^\infty)$. Moreover, if $\mathfrak{M}$ is a p-module, i.e. the exponent of $\mathfrak{M}$ is a p-power, then choosing any isomorphism of M with a direct sum of modules $\mathbb{Z}/p^{s_j}\mathbb{Z}$ we see that $\mathfrak{M}$ is isomorphic to a module of the form $\left(\prod_{j=1}^{n}\mathbb{Z}/p^{s_j}\mathbb{Z}, \frac{Q(x_1,\ldots,x_n)}{2^t}\right)$ with $t = 1 + \max s_j$ and a quadratic form Q in $\mathbb{Z}[X_1,\ldots,X_n]$ (in fact, if p is odd and sometimes also for $p = 2$, on can choose $t = \max s_j$). But Q can then be decomposed over $\mathbb{Z}_p$ (or $\mathbb{Z}/2^t\mathbb{Z}$) as a direct sum of unary and, for $p = 2$. possible binary forms.

Note that the characteristic function of the direct sum of finite quadratic modules equals the product of the characteristic functions of the components of the direct sum. The characteristic functions of the $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ can be easily computed using the well-known formulas for ordinary Gauss-sums as recalled in Table 1[3] (For the

$\mathfrak{M}$		$\chi_{\mathfrak{M}}(1)$
$\mathfrak{A}_q(a)$	(q odd)	$e_8(1-q)\left(\frac{2a}{q}\right)$
$\mathfrak{A}_{2^t}(a)$		$e_8(a)\left(\frac{a}{2^t}\right)$
$\mathfrak{B}_{2^s}$		$(-1)^s$
$\mathfrak{C}_{2^s}$		1

TABLE 1. The values of $\chi_{\mathfrak{M}}(1)$ for $\mathfrak{M}$ in the $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ series.

verification of the formula for $\mathfrak{M} = \mathfrak{B}_{2^s}$ the reader might want to verify first of all that $\chi_{\mathfrak{B}_{2^s}}(1) = \chi_{\mathfrak{B}_{2^{s-2}}}(1)$ for $s \geq 2$.)

For a (finite) prime p we define the *signature of* $\mathfrak{M}$ *at* p as the integer $s_p(\mathfrak{M})$ such that

$$\chi_{\mathfrak{M}(p^\infty)}(1) = e_8\left(-s_p(\mathfrak{M})\right).$$

We then have the product formula

$$\prod_{p \text{ prime or } \infty} e_8\left(s_p(\mathfrak{M})\right) = 1.$$

We note three immediate consequences of the decomposition into modules of the $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ series.

Proposition 3.2. *Let p be a prime number.*

(1) *If p is odd, one has*

$$e_8\left(-s_p(\mathfrak{M})\right) = e_8(k - q_1 - \cdots - q_k)\left(\frac{2a_1}{q_1}\right)\cdots\left(\frac{2a_k}{q_k}\right),$$

for any decomposition of $\mathfrak{M}(p^\infty)$ as direct sum of $\mathfrak{A}_{q_i}(a_i)$ $(i = 1,\ldots,k)$.

(2) *If $p = 2$ one has*

$$e_8\left(-s_2(\mathfrak{M})\right) = e_8(a_1 + \cdots + a_k + 4(s_1 + \cdots + s_l))\left(\frac{a_1}{q_1}\right)\cdots\left(\frac{a_k}{q_k}\right)$$

for any decomposition of $\mathfrak{M}(2^\infty)$ as direct sum of $\mathfrak{A}_{q_i}(a_i)$ $(i = 1,\ldots,k)$, $\mathfrak{B}_{q_j}$ $(j = 1,\ldots,l)$ and possibly some more modules from the $\mathfrak{C}$-series.

In particular, we obtain

[3] For integers a and $b > 0$, we use $\left(\frac{a}{b}\right)$ for the generalized Legendre symbol, i.e. the symbol which is multiplicative in a and in b, which equals the usual Legendre symbol if b is an odd prime, and which, for $b = 2$, equals 1, -1 or 0 accordingly as $a \equiv \pm 1 \bmod 8$, $a \equiv \pm 3 \bmod 8$ or a is even, respectively.

Proposition 3.3. *Let $\mathfrak{M}$ be a finite quadratic module. For any integer a which is relatively prime to* card (M), *one has*

$$\chi_{\mathfrak{M}}(a) = \left(\tfrac{a}{\operatorname{card}(M)}\right) e_8\left((a-1)T_{\mathfrak{M}}\right) \chi_{\mathfrak{M}}(1).$$

Here $T_{\mathfrak{M}}$ denotes the sum of all a_i $(i = 1, \ldots, k)$ in any decomposition of $\mathfrak{M}(2^\infty)$ as direct sum of $\mathfrak{A}_{q_i}(a_i)$ $(i = 1, \ldots, k)$ and possibly additional pieces $\mathfrak{B}_{2^g}$, $\mathfrak{C}_{2^h}$.

Alternatively, one can deduce a similar formula from the obvious identity

$$\chi_{\mathfrak{M}}(a) = \sigma_{a'}(\chi_{\mathfrak{M}(1)}) \frac{\sigma_{a'}(w)}{w},$$

where $a' \equiv a \bmod \ell$ is any integer relatively prime to 2 and the level ℓ of $\mathfrak{M}$, where $w = \sqrt{\operatorname{card}(M)}$, and $\sigma_{a'}$ is the Galois substitution of the 8ℓth cyclotomic field which maps a root of unity ζ to $\zeta^{a'}$.

Proposition 3.4. *Let 2^u be the exact 2-power dividing the integer t. One has $\chi_{\mathfrak{M}}(t) = 0$ if and only if $\mathfrak{A}_{2^u}(a)$, for some a, is a direct summand of $\mathfrak{M}$.*

4. A class number formula

For calculating the parabolic contribution $P_{\underline{L}}(h)$ we have to study expressions like

$$H := \sum_{x \in M} \mathbb{B}\left(Q(x) - \frac{h}{24}\right).$$

Here, as in the previous section, $\mathfrak{M} = (M, Q)$ denotes a finite quadratic module, and we use $\mathbb{B}(x)$ for the periodically continued first Bernoulli polynomial. In other words $\mathbb{B}(x) = x - \lfloor x \rfloor - \frac{1}{2}$ for $x \notin \mathbb{Z}$ and $\mathbb{B}(x) = 0$ for integral x. The Fourier expansion of $\mathbb{B}(x)$ is given by

$$\mathbb{B}(x) = -\frac{1}{\pi} \sum_{n \geq 1} \frac{\Im\left(e(nx)\right)}{n}.$$

Therefore

$$\sum_{x \in M} \mathbb{B}\left(Q(x) - \tfrac{h}{24}\right) = -\frac{\sqrt{\operatorname{card}(M)}}{\pi} \lim_{s \downarrow 0} \Im\left(D(\mathfrak{M}, h, s)\right)$$

where we use

$$D(\mathfrak{M}, h, s) = \sum_{n \geq 1} \frac{\chi_{\mathfrak{M}}(n) e\left(\frac{-hn}{24}\right)}{n^s}.$$

Note that the Dirichlet series $D(\mathfrak{M}, h, s)$ is absolutely convergent for $\Re(s) > 1$ (since its coefficients are periodic in n). In fact, it is a linear combination of Hurwitz zeta functions, and can therefore be holomorphically continued to the whole complex plane with the exception of $s = 1$, where it might have a simple pole.

Let ℓ denote the level of $\mathfrak{M}$ (i.e. the smallest positive integer such that $\ell Q = 0$). We decompose the Dirichlet series in the form

$$D(\mathfrak{M}, h, s) = \sum_{t \in D_{\mathfrak{M}}} \sqrt{\operatorname{card}(M[t])}\, \chi_{\mathfrak{M}(t)}(1)\, t^{-s} D_{\mathrm{pr.}}(\mathfrak{M}(t), ht, s),$$

where $D_{\mathfrak{M}}$ is the set of divisors of ℓ such that $\chi_{\mathfrak{M}}(t) \neq 0$ (cf. Proposition 3.4), and where, for any $\mathfrak{M}$, we set

$$D_{\mathrm{pr.}}(\mathfrak{M}, h, s) = \sum_{n\geq 1} e_{24}(-hn) \left(\tfrac{n}{\mathrm{card}(M)}\right) e_8\left((n-1)T_{\mathfrak{M}}\right) n^{-s}$$

For simplicity we assume from now on that $h = 0$ and that card (M) is odd (so that $T_{\mathfrak{M}} = 0$), and we drop the h in the above notations. We then have

$$D_{\mathrm{pr.}}(\mathfrak{M}, s) = \sum_{n\geq 1} \left(\tfrac{n}{f}\right) n^{-s} \prod_{p\mid N} \left(1 - \left(\tfrac{p}{f}\right) p^{-s}\right),$$

where $N = \mathrm{card}\,(M)$ and f denotes the squarefree part of N. Inserting this into the last formula for $D(\mathfrak{M}, s)$ yields

Proposition 4.1. *Assume that* card (M) *is odd. Then*

$$D(\mathfrak{M}, s) = \sum_{t\mid \ell} \sqrt{N_1/N_t}\, \chi_{\mathfrak{M}(t)}(1)\, t^{-s} L(\left(\tfrac{*}{f_t}\right), s) \prod_{p\mid N_t} \left(1 - \left(\tfrac{p}{f_t}\right) p^{-s}\right),$$

where $N_t = \mathrm{card}\,(M/M[t])$ *and* f_t *is the squarefree part of* N_t, *and where we use* $L((\left(\tfrac{*}{f_t}\right), s)$ *for the L-series associated to the Dirichlet character* $\left(\tfrac{*}{f_t}\right)$.

We now let $s > 1$ be real and consider the imaginary part of $D(\mathfrak{M}, s)$. Clearly the tth term is nonzero only if $\chi_{\mathfrak{M}(t)}(1)$ has an nonzero imaginary part, which be Prop. 3.2 holds true if and only if $f_t \equiv 3 \bmod 4$. But in this case

$$L(\left(\tfrac{*}{f_t}\right), 1) = \frac{\pi h(-f_t)}{w(-f_t)\sqrt{f_t}},$$

where, for a negative discriminant d we use $w(D) = 3, 2, 1$ accordingly as $D = -3, -4$ or $D < -4$, and where $h(-f_t)$ is the class number of $\mathbb{Q}(\sqrt{-f})$. Using the well-known formula (see e.g. [Lan73, Ch. 8, 1, Thm. 7] for a proof).

$$\frac{h(-N_t)}{w(-N_t)} = \frac{h(-f_t)}{w(-f_t)} g_t \prod_{p\mid N_t} \left(1 - \left(\tfrac{p}{f_t}\right) \tfrac{1}{p}\right),$$

where as before $N_t = \mathrm{card}\,(M/M[t]) = f_t g_t^2$ for a suitable positive integer g_t, we finally find

Theorem 4.1. *Let* $\mathfrak{M} = (M, Q)$ *be a finite quadratic module with level* ℓ. *Assume that* card (M) *is odd. Then*

$$\sum_{x\in M} \mathbb{B}\left(Q(x)\right) = - \sum_{\substack{t\mid \ell \\ N_t\equiv 3 \bmod 4}} \frac{\mathrm{card}\,(M[t])}{t} \Im\left(\chi_{\mathfrak{M}(t)}(1)\right) \frac{h(-N_t)}{w(-N_t)},$$

where we use $N_t = \mathrm{card}\,(M/M[t])$.

5. Conclusion

Though the dimension formula of Theorem 1.1 can be easily implemented in any existing computer algebra package its computation becomes slow or even unfeasible if the size of $L^\bullet/L$ grows. This is due to the sums $\chi_{\underline{L}}(2)$ and $\chi_{\underline{L}}(-3)$ and the parabolic contribution, which require in a naive implementation the summation over $L^\bullet/L$. The considerations in Section 2 reduce the calculation of $\chi_{\underline{L}}$ to the problem of calculation of the characteristic functions $\chi_{\mathfrak{M}}$ of suitable finite quadratic modules $\mathfrak{M}$. Section 3 reduces the calculation of $\chi_{\mathfrak{M}}(t)$ for a given $\mathfrak{M}$ to the diagonalization of a quadratic form (which depends on $\mathfrak{M}$ but not on t) in k variables over the localisation of $\mathbb{Z}$ at the ideal generated by the level of $\mathfrak{M}$, where k is the number of elementary divisors of $\mathfrak{M}$, and then for each t, the calculation of k generalized Legendre symbols.

The Theorem of Section 4 shows that the calculation of the parabolic contribution amounts essentially to the calculation of a class number and a value of $\chi_{\mathfrak{M}}$ for each divisor of the level. This theorem is not complete in that it does not include the case of lattices with even determinant and not the case of a nontrivial character of $\mathrm{Mp}(2,\mathbb{Z})$. A more complete version will eventually appear elsewhere.

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