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EISENSTEIN CONGRUENCES AND ENDOSCOPIC LIFTS

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1. Introduction

This paper is written in a somewhat informal style, lacking in precise definitions, based on my talk at a Kyoto RIMS workshop in February 2016. The reader can consult [Du] for a more careful account. In that paper we show how many of the congruences between Ikeda lifts and non-Ikeda lifts, proved by Katsurada [Ka], can be reduced to congruences involving only forms of genus 1 and 2, using various liftings constructed using Arthur's multiplicity formula. Similarly, we show that conjectured congruences between Ikeda-Miyawaki lifts and non-lifts [IKPY] can often be reduced to congruences involving only forms of genus 1, 2 and 3. Here we look only at a couple of illustrative examples, but we offer a different perspective, observing the parallels with Chenevier and Lannes's proof [CL] of the original mod 41 instance of Harder's conjecture [H].

I am indebted to O. Taïbi for answering emails I sent him during the week of the workshop, in particular for the computations in §4, and for comments on a previous version of this article. I thank also the organisers Profs. Hayashida and Nagaoka, and Prof. Ibukiyama whose grant funded my visit.

2. Background

Let $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ be a normalised Hecke eigenform. From f one produces, in a standard way, a function $\Phi_f : \mathrm{PGL}_2(\mathbb{A}) \to \mathbb{C}$, left-invariant under $\mathrm{PGL}_2(\mathbb{Q})$. Under right translation, it generates a cuspidal automorphic representation Π_f of $\mathrm{PGL}_2(\mathbb{A})$, which decomposes as a restricted tensor product $\prod_{p\leq\infty}\Pi_p$, Π_p an irreducible, admissible representation of $\mathrm{PGL}_2(\mathbb{Q}_p)$.

Each Π_p has a Langlands parameter. At $p=\infty$ this is $c(\Pi_\infty) : W_\mathbb{R} \to \mathrm{SL}_2(\mathbb{C})$. Note that $\mathrm{SL}_2(\mathbb{C})$ is the L-group of PGL_2. The Weil group $W_\mathbb{R}$ has an index 2 subgroup isomorphic to $\mathbb{C}^\times$, and for $z \in \mathbb{C}^\times$ we have

$$z \mapsto \mathrm{diag}((z/|z|)^{(k-1)/2}, (z/|z|)^{(1-k)/2}) =: c_\infty(\Pi).$$

It is convenient to write

$$\text{“}c_\infty(\Pi) \implies [(k-1)/2, (1-k)/2]\text{”}.$$

At a finite prime p we have $c(\Pi_p) : W_{\mathbb{Q}_p} \to \mathrm{SL}_2(\mathbb{C})$. This factors through the abelian quotient $\mathbb{Q}_p^\times$, in fact through $\mathbb{Q}_p^\times/\mathbb{Z}_p^\times$, with $p \mapsto \mathrm{diag}(\alpha_p, \alpha_p^{-1}) =: c_p(\Pi)$ (the Satake parameter), where $p^{(k-1)/2}(\alpha_p + \alpha_p^{-1}) = a_p(f)$, the eigenvalue of T_p on f, equivalently the coefficient of q^p in its Fourier expansion.

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Given $d \geq 1$, there is a discrete automorphic representation $\Pi_f[d]$ of $\mathrm{PGL}_{2d}(\mathbb{A})$, cuspidal only for $d=1$. Its Langlands parameters "are"

$$c_\infty(\Pi[d]) = c_\infty(\Pi) \otimes \mathrm{Sym}^{d-1}(\mathrm{diag}((z/|z|)^{1/2}, (z/|z|)^{-1/2}))$$

and

$$c_p(\Pi[d]) = c_p(\Pi) \otimes \mathrm{Sym}^{d-1}(\mathrm{diag}(p^{1/2}, p^{-1/2})).$$

To illustrate these ideas, consider the case $k=12$, $f=\Delta$, which we also denote Δ_{11}, since $c_\infty(\Pi) \implies [11/2, -11/2]$. If $L(s,\Pi) = \prod_p (1-\alpha_p p^{-s})^{-1}(1-\alpha_p^{-1}p^{-s})^{-1}$ is the automorphic L-function attached to Π (and the standard representation of $\mathrm{SL}_2(\mathbb{C})$) and $L(s,\Delta) = \prod_p (1 - a_p(\Delta)p^{-s} + p^{11-2s})^{-1}$ is the "motivic" L-function attached to Δ, then $L(s,\Pi) = L(s+(11/2),\Delta)$. Now

$$c_\infty(\Pi[2]) \implies [12/2, 10/2, -10/2, -12/2],$$

$$L(s,\Pi[2]) = L(s+(10/2),\Delta)L(s+(12/2),\Delta),$$

$$c_\infty(\Pi[3]) \implies [13/2, 11/2, 9/2, -9/2, -11/2, -13/2],$$

$$L(s,\Pi[3]) = L(s+(9/2),\Delta)L(s+(11/2),\Delta)L(s+(13/2),\Delta)$$

and

$$c_p(\Pi[3]) = \mathrm{diag}(p\alpha_p, p\alpha_p^{-1}, \alpha_p, \alpha_p^{-1}, p^{-1}\alpha, p^{-1}\alpha_p^{-1}).$$

Now suppose we have a genus 2 Hecke eigenform $F \in S_\rho(\mathrm{Sp}_2(\mathbb{Z}))$, in general vector-valued in the representation $\rho = \mathrm{Sym}^j \otimes \det^\kappa$ of the standard representation of $\mathrm{GL}_2(\mathbb{C})$. We have an associated automorphic representation Π_F of $\mathrm{PGSp}_2(\mathbb{A}) \simeq \mathrm{SO}(3,2)(\mathbb{A})$. (This is explained in detail in [AS].) The L-group of $\mathrm{PGSp}_2 \simeq \mathrm{SO}(3,2)$ is $\mathrm{Spin}(3,2)(\mathbb{C}) \simeq \mathrm{Sp}_2(\mathbb{C})$. There is a Langlands parameter at ∞

$$c(\Pi_\infty) : W_\mathbb{R} \to \mathrm{Spin}(3,2)(\mathbb{C}) \simeq \mathrm{Sp}_2(\mathbb{C}),$$

which can be composed with the natural inclusions to $\mathrm{SL}_5(\mathbb{C})$ and $\mathrm{SL}_4(\mathbb{C})$, the "standard" and "spin" representations respectively. These are the L-groups of PGL_5 and PGL_4 respectively, so there are liftings of Π_F to automorphic representations Π_F^{st} of $\mathrm{PGL}_5(\mathbb{A})$ and Π_F^{spin} of $\mathrm{PGL}_4(\mathbb{A})$. (That these actually exist is part of Arthur's work [A].) The degree-5 L function $L(s,\Pi_F^{\mathrm{st}})$ coincides with what is usually called the standard L-function, while the degree-4 L-function $L(s,\Pi_F^{\mathrm{spin}})$ is $L(s+(j+2\kappa-3)/2, F, \mathrm{spin})$, in terms of the usual spin L-function. We have

$$c_\infty(\Pi_F^{\mathrm{st}}) \implies [j+\kappa-1, \kappa-2, 0, 2-\kappa, 1-\kappa-j]$$

and

$$c_\infty(\Pi_F^{\mathrm{spin}}) \implies [(j+2\kappa-3)/2, (j+1)/2, -(j+1)/2, -(j+2\kappa-3)/2].$$

For example, when $(j,\kappa) = (4,10)$ (for which $S_\rho(\mathrm{Sp}_2(\mathbb{Z}))$ is 1-dimensional), we have $c_\infty(\Pi_F^{\mathrm{spin}}) \implies [21/2, 5/2, -5/2, -21/2]$, so we call Π_F^{spin} "$\Delta_{21,5}$".

In the special case $j=0$, $F \in S_\kappa(\mathrm{Sp}_2(\mathbb{Z}))$ is scalar-valued and

$$c_\infty(\Pi_F^{\mathrm{st}}) \implies [\kappa-1, \kappa-2, 0, 2-\kappa, 1-\kappa].$$

More generally, for a Hecke eigenform $G \in S_\kappa(\mathrm{Sp}_g(\mathbb{Z}))$ we have

$$c_\infty(\Pi_G^{\mathrm{st}}) \implies [\kappa-1, \kappa-2, \ldots, \kappa-g, 0, g-\kappa, \ldots, 1-\kappa],$$

which is now with reference to the standard representation of $\mathrm{Spin}(g+1,g)(\mathbb{C})$ (the L-group of PGSp_g) into $\mathrm{SL}_{2g+1}(\mathbb{C})$.

3. Chenevier and Lannes's proof

Let X_{24} be the set of isometry classes of even, unimodular lattices in Euclidean $\mathbb{R}^{24}$.

$$X_{24} \simeq O_{24}(\mathbb{Q})\backslash O_{24}(\mathbb{A}_f)/O_{24}(\hat{\mathbb{Z}}),$$

where O_{24} is the orthogonal group of any such lattice. On $\mathbb{C}[X_{24}]$, the space of scalar-valued automorphic forms for O_{24} (of "level one"), there is a natural action of the Hecke algebra $\mathbb{C}[O_{24}(\hat{\mathbb{Z}}))\backslash O_{24}(\mathbb{A}_f)/O_{24}(\hat{\mathbb{Z}})]$, which is a restricted direct product over finite primes of local Hecke algebras $\mathbb{C}[O_{24}(\mathbb{Z}_p)\backslash O_{24}(\mathbb{Q}_p)/O_{24}(\mathbb{Z}_p)]$. Let T_p denote the characteristic function of the double coset $O_{24}(\mathbb{Z}_p)\gamma_p O_{24}(\mathbb{Z}_p)$ where, under the isomorphism $O_{24}(\mathbb{Q}_p) \simeq O_{12,12}(\mathbb{Q}_p)$, $\gamma_p \mapsto \mathrm{diag}(p,1,\ldots,1,p^{-1},1,\ldots,1)$.

There exists a basis $\{v_1, v_2, \ldots, v_{24}\}$ for $\mathbb{C}[X_{24}]$, comprising Hecke eigenforms in $\mathbb{Z}[X_{24}]$, found by Nebe and Venkov [NV]. In fact, the eigenvalues for T_2 are distinct integers $\lambda_1(2) > \lambda_2(2) > \ldots > \lambda_{24}(2)$. Each v_i, viewed as a function on $O_{24}(\mathbb{A})$, left-invariant under $O_{24}(\mathbb{Q})$ and right-invariant under $O_{24}(\mathbb{R})O_{24}(\hat{\mathbb{Z}})$, generates a cuspidal automorphic representation of $O_{24}(\mathbb{A})$. This lifts to a discrete automorphic representation Π_i of $\mathrm{PGL}_{24}(\mathbb{A})$, whose Langlands parameters are certain direct sums of those of smaller $\mathrm{PGL}_n(\mathbb{A})$, making it an "endoscopic lift". These are listed in [CL, Table C5]. For example, associated to v_{18} is

$$\Delta_{21}[2] \oplus \Delta_{17}[2] \oplus \Delta_{11}[4] \oplus [1] \oplus [7],$$

while associated to v_{21} is

$$\Delta_{21,5}[2] \oplus \Delta_{17}[2] \oplus \Delta_{11}[4] \oplus [1] \oplus [3].$$

Note that always

$$c_\infty(\Pi_i) \implies [11, 10, \ldots, 2, 1, 0, 0, -1, -2, \ldots, -10, -11],$$

which is determined by the infinitesimal character of the trivial representation, these automorphic forms for O_{24} being scalar-valued. Writing out the Hecke eigenvalues at p, with $\tau_{21,5}(p)$ the eigenvalue of genus-2 $T(p)$ on $\Delta_{21,5}$, we have

$$\lambda_{18}(p) = \tau_{21}(p)(1+p) + \tau_{17}(p)(p^2+p^3) + \tau_{11}(p)(p^5+p^6+p^7+p^8) + p^{11} + (p^8+p^9+p^{10}+p^{11}+p^{12}+p^{13}+p^{14})$$

and

$$\lambda_{21}(p) = \tau_{21,5}(p)(1+p) + \tau_{17}(p)(p^2+p^3) + \tau_{11}(p)(p^5+p^6+p^7+p^8) + p^{11} + (p^{10}+p^{11}+p^{12}).$$

The v_i do not span the whole of $\mathbb{Z}[X_{24}]$ over $\mathbb{Z}$, only a $\mathbb{Z}$-submodule of finite index. Chenevier and Lannes proved [CL, X, Proposition 4.3] that there is a congruence mod 41 between v_{18} and v_{21}. This implies a congruence mod 41 of the corresponding Hecke eigenvalues, hence

$$\tau_{21,5}(p)(1+p) \equiv \tau_{21}(p)(1+p) + (p^8+p^{13})(1+p) \pmod{41}.$$

They were able to deal with the $(1+p)$ and show that

$$\tau_{21,5}(p) \equiv \tau_{21}(p) + p^8 + p^{13} \pmod{41}.$$

Recall that $\Delta_{21,5}$ comes from a genus 2 Hecke eigenform F, vector-valued of weight $(j,\kappa) = (4,10)$, Δ_{21} from a genus 1 Hecke eigenform f of weight 22. This congruence is an instance of Harder's conjecture, in fact it is the original example for which he gave experimental evidence [H] using Hecke eigenvalues computed by Faber and van der Geer [FvdG]. The significance of the modulus 41 is that it divides the "algebraic part" of the critical value $L(f, j+\kappa) = L(f, 14)$. This is an example of an "Eisenstein congruence", and is analogous to Ramanujan's congruence $\tau_{11}(p) \equiv 1 + p^{11} \pmod{691}$, where $691 \mid \frac{\zeta_{12}}{\pi^{12}}$. For Harder's conjecture as an instance of something very general, see [BD, §7].

4. Using congruences for Ikeda lifts

How might we see Harder's congruence via scalar-valued Siegel modular forms of high genus rather than scalar-valued automorphic forms for O_{24}? Since $21/2$ is half-way between 10 and 11,

$$c_\infty(\Delta_{21}[6] \oplus [1]) \implies [13, 12, 11, 10, 9, 8, 0, -8, -9, -10, -11, -12, -13].$$

This is $[k-1, \ldots, k-g, 0, g-k, \ldots, 1-k]$ with $k = 14, g = 6$. In fact there exists a Hecke eigenform $G \in S_{14}(\mathrm{Sp}_6(\mathbb{Z}))$ (a 9-dimensional space according to [T1, Table 3]), with $c(\Pi_G^{\mathrm{st}}) = \Delta_{21}[6] \oplus [1]$, where Π_G is a cuspidal automorphic representation of $\mathrm{Sp}_6(\mathbb{A})$ obtained from G, and Π_G^{st} its lifting to $\mathrm{PGL}_{13}(\mathbb{A})$ via the standard representation of the L-group $\mathrm{SO}_{7,6}(\mathbb{C})$. We have $L(s, G, \mathrm{St}) = \zeta(s)L(s+8, f) \ldots L(s+13, f)$, in fact $G = \mathrm{Ik}_6(f)$, the genus 6 Ikeda lift of f, whose existence was conjectured by Duke and Imamoglu and proved by Ikeda [Ik1].

Recalling that for F we have $(j, \kappa) = (4, 10)$, so $[\kappa-1, \kappa-2, 0, 2-\kappa, 1-\kappa] = [13, 8, 0, -8, -13]$, we see that also

$$c_\infty(\Delta_{21}[4] \oplus \Pi_F^{\mathrm{st}}) \implies [13, 12, 11, 10, 9, 8, 0, -8, -9, -10, -11, -12, -13].$$

A conjecture of Ibukiyama [Ib] in response to lifting puzzles of Poor, Ryan and Yuen [PRY] would imply the existence of a Hecke eigenform $H \in S_{14}(S_6(\mathrm{Sp}_6(\mathbb{Z}))$ with $c(\Pi_H^{\mathrm{st}}) = \Delta_{21}[4] \oplus \Pi_F^{\mathrm{st}}$, so $L(s, H, \mathrm{St}) = L(s, F, \mathrm{St})L(s+9, f) \ldots L(s+12, f)$.

I am indebted to O. Taïbi for computing the endoscopic types of all 9 automorphic representations arising from Hecke eigenforms in $S_{14}(\mathrm{Sp}_6(\mathbb{Z}))$. His list is the following, where $\Delta_{23}^{(2)}$ comes from either of the 2 normalised Hecke eigenforms of genus 1 and weight 24 (and level 1):

$$\Delta_{23}^{(2)}[2] \oplus \Delta_{19}[2] \oplus \Delta_F^{\mathrm{st}};$$
$$\Delta_{21}[4] \oplus \Delta_F^{\mathrm{st}};$$
$$\Delta_{25,17}[2] \oplus \Delta_{21}[2] \oplus [1];$$
$$\Delta_{25}[2] \oplus \Delta_{21}[2] \oplus \Delta_{17}[2] \oplus [1];$$
$$\Delta_{25}[2] \oplus \Delta_{19}[4] \oplus [1];$$
$$\Delta_{17}[2] \oplus \Delta_{23}^{(2)}[4] \oplus [1];$$
$$\Delta_{21}[6] \oplus [1].$$

The last line is the Ikeda lift, and the second confirms the existence of Ibukiyama's H. Taïbi used the trace formula to compute the dimension of the space, then he used Arthur's multiplicity formula to find the above parameters. As he pointed out, their correctness is no longer conditional, thanks to recent work of Arancibia, Moeglin and Renard [AMR]. The same remark applies to the applications of Arthur's multiplicity formula to the construction of various lifts in [Du].

If we could show that there is a mod 41 congruence of Hecke eigenvalues between $\mathrm{Ik}_6(f)$ and H then

$$p^{13} + a_p(f)(1+p+p^2+p^3+p^4+p^5) \equiv p^{13} + \lambda_{1,p^2}(F) + a_p(f)(p+p^2+p^3+p^4) \pmod{41},$$

so

$$\lambda_{1,p^2}(F) \equiv a_p(f)(1+p^5) \pmod{41},$$

which is Harder's conjecture applied to the eigenvalues of the Hecke operator usually denoted $T_1(p^2)$, associated to the double coset of $\mathrm{diag}(p,1,p^{-1},1)$ in $\mathrm{Sp}_2(\mathbb{Z}_p)\backslash\mathrm{Sp}_2(\mathbb{Q}_p)/\mathrm{Sp}_2(\mathbb{Z}_p)$ (or of $\mathrm{diag}(p^2,p,1,p)$ in $\mathrm{GSp}_2(\mathbb{Z}_p)\backslash\mathrm{GSp}_2(\mathbb{Q}_p)/\mathrm{GSp}_2(\mathbb{Z}_p)$). (For why the contribution of Δ_F^{st} is $p^{13}+\lambda_{1,p^2}(F)$ rather than just $\lambda_{1,p^2}(F)$, see [G, (5.4)].) We would not deduce the more familiar $\tau_{21,5}(p) \equiv \tau_{21}(p)+p^8+p^{13} \pmod{41}$, where $\tau_{21,5}(p)$ is an eigenvalue of $T(p)$ associated to the double coset of $\mathrm{diag}(p,p,1,1)$ in $\mathrm{GSp}_2(\mathbb{Z}_p)\backslash\mathrm{GSp}_2(\mathbb{Q}_p)/\mathrm{GSp}_2(\mathbb{Z}_p)$. This is because we are using Π_F^{st} rather than $\Pi_F^{\mathrm{spin}} = \Delta_{21,5}$. So from the congruence of Hecke eigenvalues between $\mathrm{Ik}_6(f)$ and H we would deduce a weak form of Harder's conjecture.

A theorem of Katsurada [Ka], applied to this case, might appear to give us exactly what we want. It says (in this instance) that if $q>2k$ is prime and divides $L_{\mathrm{alg}}(f,14)L_{\mathrm{alg}}(3,f,\mathrm{St})L_{\mathrm{alg}}(5,f,\mathrm{St})$, and if further weak conditions are satisfied, then there exists a Hecke eigenform $K \in S_{14}(\mathrm{Sp}_6(\mathbb{Z}))$, *not* an Ikeda lift, with a mod q congruence of Hecke eigenvalues between $\mathrm{Ik}_6(f)$ and K. In this case $k=14$ and $q=41$, so $q>2k$ is satisfied, and though we are vague about what exactly is meant by algebraic part, the divisibility is satisfied. Naturally we suspect that $K=H$, and it would be easy to check this by looking at a few Hecke eigenvalues to eliminate the other possibilities on Taïbi's list. But unfortunately in Katsurada's theorem there is a condition $k \geq 2g+4$, which just fails when $k=14$ and $g=6$. So we cannot use Katsurada's theorem to prove Harder's congruence. But we do at least see that although the K in Katsurada's theorem is not an Ikeda lift, we can expect it to be some other kind of lift, and Katsurada's congruence accounted for by Harder's. In fact, in this particular mod 41 instance, though Katsurada's congruence does not follow from his theorem, it does follow (in weak form for Sp_g Hecke operators) from Chenevier and Lannes's proof of Harder's congruence. (The congruence in their paper is for $T(p)$, but one can take the exterior square of the 4-dimensional Galois representation to deduce the congruence for $T_1(p^2)$.)

5. Congruences for Ikeda-Miyawaki lifts

In this section, take Hecke eigenforms $f \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ and $h \in S_{k+n+1}(\mathrm{SL}_2(\mathbb{Z}))$, with $k+n+1$ even. There exists a Hecke eigenform $G \in S_{k+n+1}(\mathrm{Sp}_{2n+1}(\mathbb{Z}))$ such that

$$c(\Pi_G^{\mathrm{st}}) = \Pi_h^{\mathrm{st}} \oplus \Pi_f[2n].$$

We write $G=\mathrm{IM}(f,h)$, the Ikeda-Miyawaki lift. Its existence was conjectured by Miyawaki and proved by Ikeda [Miy, Ik2].

Example. $2k=16, k+n+1=16, k=8, n=7, 2n+1=15$. Recalling that $c_\infty(\Pi_h^{\mathrm{st}}) \implies [15,0,-15]$ and $c_\infty(\Pi_f) \implies [15/2,-15/2]$, we see that

$$c_\infty(\Pi_G^{\mathrm{st}}) \implies [15,14,\ldots,2,1,0,-1,-2,\ldots,-14,-15].$$

Suppose that a prime $q>2k+2n-2$ divides $L_{\mathrm{alg}}(f\otimes\mathrm{Sym}^2h, 2k+2n)\prod_{i=1}^{n-1} L_{\mathrm{alg}}(2i+1,f,\mathrm{St})$. Then a conjecture of Ibukiyama, Katsurada, Poor and Yuen [IKPY] asserts the existence of a Hecke eigenform $K \in S_{k+n+1}(\mathrm{Sp}_{2n+1}(\mathbb{Z}))$, not an Ikeda-Miyawaki lift, with a congruence mod q of Hecke eigenvalues between $\mathrm{IM}(f,h)$ and K. In the current example, $q=37$ divides $L_{\mathrm{alg}}(f\otimes\mathrm{Sym}^2h,30)$ (according to [IKPY, Table 4]). According to computations (and subject to a conjecture) of Bergström, Faber and van der Geer, there is a unique (up to scaling) F for $\mathrm{Sp}_3(\mathbb{Z})$, vector-valued of type $(a,b,c) = (12,12,0)$, in the notation of [BFvdG] or [IKPY]. In fact, the existence of this F follows from recent work of Taïbi [T1]; see the line $[15,14,1]$ in the third table at [T2]. Since $(a+3,b+2,c+1) = (15,14,1)$, we have $c_\infty(\Pi_F^{\mathrm{st}}) \implies [15,14,1,0,-1,-14,-15]$. It follows that

$$c_\infty(\Pi_F^{\mathrm{st}} \oplus \Pi_f[2n-2]) \implies [15,14,\dots,2,1,0,-1,-2,\dots,-14,-15],$$

same as $c_\infty(\Pi_G^{\mathrm{st}})$. Arthur's multiplicity formula then implies the existence of a Hecke eigenform (for $T_i(p^2)$) $H \in S_{k+n+1}(\mathrm{Sp}_{2n+1}(\mathbb{Z}))$ such that

$$c(\Pi_H^{\mathrm{st}}) = \Pi_F^{\mathrm{st}} \oplus \Pi_f[12],$$

so

$$L(s,H,\mathrm{St}) = L(s,F,\mathrm{St})L(s+2,f)L(s+3,f)\dots L(s+13,f).$$

If the K in the conjectured congruence is this H, then for all p,

$$(a_p(h)^2-p^{15})+a_p(f)(p+p^2+\dots+p^{13}+p^{14}) \equiv \lambda_{1,p^2}(F)+a_p(f)(p^2+p^3+\dots+p^{12}+p^{13})$$

(mod 37), i.e.

$$\lambda_{1,p^2}(F) \equiv (a_p(h)^2 - p^{15}) + a_p(f)(p+p^{13}) \pmod{37}.$$

This is another example of an Eisenstein congruence [BD, §8]. These are Hecke eigenvalues for $T_1(p^2)$, but for $T(p)$ it looks like

$$\lambda_p(F) \equiv a_p(h)(a_p(f) + p^{14} + p) \pmod{37}.$$

This is one of seventeen instances of [BFvdG, Conjecture 10.8] (that given f,h and q as above, there exists F satisfying the congruence), checked by them for $p \leq 17$.

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