数理解析研究所講究録
第2036巻 2017年 161-172



# On non-vanishing conditions for certain summands in Eisenstein cohomology
# アイゼンシュタインコホモロジーのある成分の非消滅条件について

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**Abstract**

Eisenstein cohomology is the non-cuspidal part of automorphic cohomology of a reductive group (over a number field). It decomposes into a direct sum arising from the decomposition of the space of automorphic forms along the cuspidal support. The non-vanishing of the summands, and their internal structure, is subject to a subtle combination of geometric (cohomological) and arithmetic (in terms of automorphic $L$-functions) conditions. In this expository paper we present necessary non-vanishing conditions for certain summands in (the square-integrable subspace of) Eisenstein cohomology, and their consequences. This is a joint work with Joachim Schwermer.

アイゼンシュタインコホモロジーとは(数体上の)簡約群の保型コホモロジーの非カスプ的成分である. 保型形式の空間はカスプ台に関する直和に分解されるので, アイゼンシュタインコホモロジーもカスプ台に関する直和に分解する. それら直和成分が0になるかどうか, そしてその内部構造には, 幾何(コホモロジー)的条件と(保型L関数に関する)数論的条件が関係している. この概説論文では, アイゼンシュタインコホモロジーの(平方可積分保型形式からの部分空間の)直和成分が0にならないための必要条件を与え, その応用を述べる. この研究はヨアヒム・シュベルマー氏との共同研究である.

## 1 Introduction

The cohomology of an arithmetic congruence subgroup $\Gamma$ of a reductive connected linear algebraic group $G$, defined over a totally real number field, is closely related to automorphic forms with respect to $\Gamma$. This relationship is best understood in the adèlic setting. It is captured in the object called the automorphic cohomology of $G$. The automorphic cohomology of $G$ is defined as the relative Lie algebra cohomology of the space of all automorphic forms on the group $G(\mathbb{A})$ of adèlic points of $G$.

*This work has been supported in part by Croatian Science Foundation under the project 9364 and by University of Rijeka research grant 13.14.1.2.02.

The natural decomposition of the space of automorphic forms along their cuspidal support gives rise to the corresponding decomposition in automorphic cohomology. The summands corresponding to cuspidal automorphic forms form the cuspidal cohomology. The natural complement of the cuspidal cohomology is called the Eisenstein cohomology. The summands in the Eisenstein cohomology correspond to the spaces of automorphic forms supported in the associate class of cuspidal automorphic representations of the Levi factors of an associate class of proper parabolic subgroups. These summands in Eisenstein cohomology, in particular, their non-vanishing, are the main object of concern in this paper.

The square-integrable cohomology is the subspace in cohomology represented by square-integrable automorphic forms. It is an important subspace in itself, but also serves as a starting point in the study of the internal structure of cohomology. The square-integrable Eisenstein cohomology is also called the residual Eisenstein cohomology, although the residues of Eisenstein series are not always square-integrable automorphic forms. In the decomposition along the cuspidal support, we may study the individual summands of the square-integrable cohomology.

Given a cuspidal automorphic representation $\pi$ of the Levi factor of a standard proper parabolic subgroup $P$ of $G$, there is a possibly trivial summand in Eisenstein cohomology supported in the associate class of $\pi$. There are certain necessary conditions on $\pi$, and appropriate automorphic $L$-functions associated to $\pi$, for non-vanishing of such a summand and its subspace in the square-integrable cohomology. These conditions are made explicit here for the case of the split symplectic group $G = Sp_n$, defined over $\mathbb{Q}$, and the summand in Eisenstein cohomology supported in the associate class of a cuspidal automorphic representation of the Levi factor of the Siegel parabolic subgroup.

In this expository paper we mainly present the results obtained in a joint work with J. Schwermer [10]. The example of $G = Sp_n$ and $P$ the Siegel parabolic subgroup is borrowed from that paper. There is a large body of our joint work [9], [12], [11], and the very recent preprint [13], which complements the results presented here, but is not covered at all. In another paper [8], we study the case of the split symplectic group of rank two over a totally real number field.

The paper is organized as follows. Sect. 2 provides the definition and classical motivation for the study of automorphic cohomology. In Sect. 3 the decomposition along the cuspidal support of the space of automorphic forms, and the corresponding decomposition in cohomology, as well as the square-integrable cohomology, are explained. The necessary non-vanishing conditions for the summands in the decomposition along the cuspidal support are presented in Sect. 4. The application to the case of the split symplectic group is given in Sect. 5.

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This paper follows the talk given by the author at the workshop *Automorphic Forms, Automorphic L-functions and Related Topics*, held in February 2016 at the Research Institute for Mathematical Sciences (RIMS), Kyoto, Japan. We express our gratitude to Shunsuke Yamana for his kind invitation to Kyoto at the time of the workshop. We are very grateful to Shunsuke Yamana and Atsushi Ichino for their hospitality and making the visit to Kyoto such a memorable experience. We would like to thank the organizers of the workshop, Shuichi Hayashida and Shoyu Nagaoka, for the opportunity to give a talk.

# 2 Automorphic cohomology

We begin with introducing the main objects considered in the paper, in particular, automorphic and Eisenstein cohomology, and relate them to the cohomology of arithmetic groups. This serves as a motivation for studying the Eisenstein cohomology of a reductive group.

## 2.1 Classical setting

Let $G$ be a connected semisimple linear algebraic group defined over the field $\mathbb{Q}$ of rational numbers. One could work with a reductive group over any totally real number field, but for simplicity we take $G$ to be semisimple over $\mathbb{Q}$.

Let $G(\mathbb{R})$ be the Lie group of real points of $G$, and $K_{\mathbb{R}}$ a fixed maximal compact subgroup of $G(\mathbb{R})$. Let $X = G(\mathbb{R})/K_{\mathbb{R}}$ be the corresponding symmetric space. Let $\Gamma$ be a torsion-free arithmetic subgroup of $G$, viewed as a discrete subgroup of the Lie group $G(\mathbb{R})$. We form the locally symmetric space $\Gamma\backslash X$.

Let $E$ be a finite-dimensional algebraic representation of $G$ in a complex vector space. We denote by $\mathfrak{g}$ the real Lie algebra of $G(\mathbb{R})$, and by $\mathcal{Z}$ the center of the universal enveloping algebra of the complexification of $\mathfrak{g}$.

We let $C^\infty(\Gamma\backslash G(\mathbb{R}))$ the space of smooth left $\Gamma$-invariant functions on $G(\mathbb{R})$, considered as a representation of $G(\mathbb{R})$ via the action by right translations.

Then we have the following sequence of isomorphisms, where the last isomorphism requires additional assumption that $\Gamma$ is a congruence subgroup,

$$\begin{aligned} H^*(\Gamma, E) \cong H^*(\Gamma\backslash X, E) &\cong H^*(\mathfrak{g}, K_{\mathbb{R}}; C^\infty(\Gamma\backslash G(\mathbb{R})) \otimes E) \\ &\cong H^*(\mathfrak{g}, K_{\mathbb{R}}; C^\infty_{\mathrm{umg}}(\Gamma\backslash G(\mathbb{R})) \otimes E) \\ &\cong H^*(\mathfrak{g}, K_{\mathbb{R}}; A(\Gamma\backslash G(\mathbb{R})) \otimes E). \end{aligned}$$

Here $H^*(\Gamma, E)$ is the Eilenberg–McLane cohomology of the arithmetic group $\Gamma$ with respect to $E$, $H^*(\Gamma\backslash X, E)$ is the de Rham cohomology of the locally symmetric space $\Gamma\backslash X$ with respect to the local system given by $E$, and $H^*(\mathfrak{g}, K_{\mathbb{R}}; V)$ is the relative Lie algebra cohomology of a $(\mathfrak{g}, K_{\mathbb{R}})$-module $V$. For all these notions see [4]. The space $C^\infty_{\mathrm{umg}}(\Gamma\backslash G(\mathbb{R}))$ is the subspace of $C^\infty(\Gamma\backslash G(\mathbb{R}))$ consisting of functions with uniform moderate growth, and $A(\Gamma\backslash G(\mathbb{R}))$ the space of automorphic forms on $G(\mathbb{R})$ with respect to $\Gamma$ (see [3]). The first two isomorphisms can be found in [4], the third one is proved in [1], and the last one in [5] (in the adèlic setting, hence the assumption that $\Gamma$ is a congruence subgroup).

This isomorphism provides a link between the geometry of a locally symmetric space and the arithmetic of automorphic forms. In that way, cohomological arguments may produce a flow of information in both directions, and moreover, in explicit calculations of cohomology of arithmetic groups both points of view should be combined.

## 2.2 Adèlic setting

For a prime $p$, finite or not, let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ at $p$. For $p = \infty$, we have $\mathbb{Q}_\infty = \mathbb{R}$. Let $\mathbb{A}$ be the ring of adèles of $\mathbb{Q}$, and $\mathbb{A}_f$ the subring of finite adèles. Let $G(\mathbb{A})$ be the group of adèlic points of $G$.

We fix, once for all, a maximal compact subgroup $K$ of $G(\mathbb{A})$ of the form $K = K_\mathbb{R} \times \prod_{p<\infty} K_p$, where $K_p$ is a fixed maximal compact subgroup of $G(\mathbb{Q}_p)$ for $p < \infty$, and $K_\mathbb{R}$ is as in Sect. 2.1. We may assume that $K_p$ is hyperspecial for almost all $p < \infty$.

Given an open compact subgroup $C$ of $G(\mathbb{A}_f)$, consider the space

$$X_C = G(\mathbb{Q})\backslash G(\mathbb{A})/K_\mathbb{R}C.$$

It is a finite disjoint union of locally symmetric spaces, and its cohomology $H^*(X_C, E)$ with respect to $E$ can be computed as de Rham cohomology. These cohomology spaces form a directed system with respect to inclusion of open compact subgroups, because for $C' \subset C$ open compact subgroups of $G(\mathbb{A}_f)$, we have a finite covering $X_{C'} \to X_C$, which gives rise to the inclusion $H^*(X_C, E) \to H^*(X_{C'}, E)$. The group $G(\mathbb{A}_f)$ acts on the directed system by conjugation. Then, the direct limit

$$H^*(G, E) = \varinjlim_{C} H^*(X_C, E)$$

is called the *automorphic cohomology* of $G$ with respect to $E$. It comes with a $G(\mathbb{A}_f)$ action, and the original spaces $H^*(X_C, E)$ may be recovered as $C$-invariants.

The name automorphic cohomology resembles the fact that, for the same reasons as in the classical setting of Sect. 2.1, we have the following isomorphism

$$H^*(G, E) \cong H^*(\mathfrak{g}, K_\mathbb{R}; \mathcal{A} \otimes E),$$

where $\mathcal{A} = \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is the space of all automorphic forms on $G(\mathbb{A})$ as in [3].

According to Wigner's lemma [4, Sect. I.4], only a subspace of $\mathcal{A}$ consisting of automorphic forms matching the infinitesimal character of $E$ may possibly contribute to $H^*(G, E)$. More precisely, let $\mathcal{J}$ be the annihilator in $\mathcal{Z}$ of the conjugate dual of $E$. It is an ideal of finite codimension in $\mathcal{Z}$. Then,

$$H^*(G, E) \cong H^*(\mathfrak{g}, K_\mathbb{R}; \mathcal{A}_\mathcal{J} \otimes E),$$

where $\mathcal{A}_\mathcal{J}$ is the subspace of automorphic forms annihilated by a power of $\mathcal{J}$.

# 3 Decomposition along the cuspidal support

As a first step in the study of automorphic cohomology $H^*(G, E)$, one should decompose this space according to the decomposition of the space of automorphic forms along the cuspidal support.

## 3.1 Decomposition of the space of automorphic forms

We fix, once for all, a minimal parabolic $\mathbb{Q}$-subgroup $P_0$ of $G$, with the Levi decomposition $P_0 = M_0 N_0$, which is in good position with respect to the fixed maximal compact subgroup $K$ of $G(\mathbb{A})$, as in [17, Sect. I.1.4]. Let $P = M_P N_P$ be a standard parabolic $\mathbb{Q}$-subgroup of $G$.

Let $\{P\}$ be the associate[1] class of parabolic $\mathbb{Q}$-subgroups of $G$ represented by $P$. Denote by $\mathcal{C}$ the set of all associate classes of parabolic $\mathbb{Q}$-subgroups of $G$.

Let $\pi$ be a (not necessarily unitary) cuspidal automorphic representation of the Levi factor $M_P(\mathbb{A})$. We may write $\pi \cong \pi^u \otimes \lambda$, where $\pi^u$ is conveniently normalized[2] unitary cuspidal automorphic representation of $M_P(\mathbb{A})$, and $\lambda$ is a character[3] of $M_P(\mathbb{A})$ given by an element $\lambda \in \check{\mathfrak{a}}_P = X^*(P) \otimes_{\mathbb{Z}} \mathbb{R}$, with $X^*(P)$ the $\mathbb{Z}$-module of $\mathbb{Q}$-rational characters of $P$.

Let $\phi_\pi = (\phi_{\pi,Q})_{Q\in\{P\}}$ be the associate class of cuspidal automorphic representations of Levi factors of parabolic subgroups in $\{P\}$, represented by $\pi$. Then $\phi_{\pi,Q}$ is a finite set, which consists of all conjugates of $\pi$ by elements of the Weyl group which conjugate $M_Q$ to $M_P$ (cf. [16, Sect. 1.3]). By replacing $\pi$ (and possibly $P$) by an associate representative, we may assume that $\lambda$ is in the closure of the positive Weyl chamber in $\check{\mathfrak{a}}_P$ determined by $P$.

In order to stay in $\mathcal{A}_{\mathcal{J}}$, there is a certain compatibility condition on $\phi_\pi$ (cf. [16, Sect. 1.3]). We denote by $\Phi_{\mathcal{J},\{P\}}$ the family of associate classes $\phi_\pi$ which are compatible with $\mathcal{J}$.

Then, there is a direct sum decomposition, referred to as the *decomposition along the cuspidal support*,

$$\begin{aligned}\mathcal{A}_{\mathcal{J}} &\cong \bigoplus_{\{P\}\in\mathcal{C}} \mathcal{A}_{\mathcal{J},\{P\}} \\ &\cong \bigoplus_{\{P\}\in\mathcal{C}} \bigoplus_{\phi_\pi\in\Phi_{\mathcal{J},\{P\}}} \mathcal{A}_{\mathcal{J},\{P\},\phi_\pi},\end{aligned}$$

where the spaces of automorphic forms $\mathcal{A}_{\mathcal{J},\{P\}}$, resp. $\mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}$, supported in $\{P\}$, resp. in $\phi_\pi$, appearing on the right-hand side, are introduced below.

## 3.2 Definition of the summands

We define the summands in the decomposition of $\mathcal{A}_{\mathcal{J}}$ along the cuspidal support using Eisenstein series.[4]

For $\nu \in \check{\mathfrak{a}}_{P,\mathbb{C}} = \check{\mathfrak{a}}_P \otimes \mathbb{C}$ and $\pi^u$ as in Sect. 3.1, consider the induced representation

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi^u \otimes \nu),$$

where, as above, we abuse the notation by writing $\nu$ for the character of $M_P(\mathbb{A})$ corresponding to $\nu \in \check{\mathfrak{a}}_{P,\mathbb{C}}$. Induction is normalized.

[1] Recall that two parabolic $\mathbb{Q}$-subgroups $P$ and $Q$ of $G$ are associate if their Levi factors $M_P$ and $M_Q$ are $G(\mathbb{Q})$-conjugate. Observe that it is sufficient to consider conjugation by elements of the Weyl group.

[2] The normalization is such that the poles of Eisenstein series associated to $\pi^u$ are real. This can always be achieved (cf. [14, Sect. 4.1] or the RIMS Kôkyûroku paper [7, Sect. 2.3]).

[3] We are deliberately imprecise here to avoid technicalities. See [21] or [17] for a precise statement.

[4] This is the definition as in [6, Sect. 1.3]. Another definition of these spaces is given in [17, Sect. III.2.6] (see also [6, Sect. 1.2]), but these are equivalent according to [6, Thm. 1.4].

Taking $f_\nu$, an appropriate section of these induced representations, we may define the Eisenstein series[5] $E(f_\nu, g)$ associated to $\pi^u$. The defining series is absolutely and locally uniformly convergent in a cone deep enough in the positive Weyl chamber in $\check{\mathfrak{a}}_{P,\mathbb{C}}$ determined by $P$. It may be analytically continued to a meromorphic function on all of $\check{\mathfrak{a}}_{P,\mathbb{C}}$. The singularities in the closure of the positive Weyl chamber are along a locally finite set of singular hyperplanes. See [17, Sect. IV.1] for these facts.

Recall that we may assume $\pi \cong \pi^u \otimes \lambda$, with $\lambda$ real and in the closure of the positive Weyl chamber of $\check{\mathfrak{a}}_{P,\mathbb{C}}$. Because of the local finiteness of singular hyperplanes, there is a polynomial $F(\nu)$ such that $F(\nu)E(f_\nu, g)$ is holomorphic around $\nu = \lambda$.

Now the space $\mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}$, of automorphic forms supported in $\phi_\pi$, is defined as the span of all coefficients in the Taylor expansion of $F(\nu)E(f_\nu, g)$ around $\nu = \lambda$. This is clearly independent of the choice of a polynomial $F(\nu)$. The space $\mathcal{A}_{\mathcal{J},\{P\}}$ is then simply defined as the direct sum of $\mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}$ over all $\phi_\pi \in \Phi_{\mathcal{J},\{P\}}$.

## 3.3 Decomposition in cohomology

The direct sum decomposition of $\mathcal{A}_{\mathcal{J}}$ along the cuspidal support, gives rise to the corresponding decomposition in cohomology

$$\begin{aligned} H^*(G,E) &\cong \bigoplus_{\{P\}\in\mathcal{C}} H^*(\mathfrak{g}, K_{\mathbb{R}}; \mathcal{A}_{\mathcal{J},\{P\}} \otimes E) \\ &\cong \bigoplus_{\{P\}\in\mathcal{C}} \bigoplus_{\phi_\pi\in\Phi_{\mathcal{J},\{P\}}} H^*(\mathfrak{g}, K_{\mathbb{R}}; \mathcal{A}_{\mathcal{J},\{P\},\phi_\pi} \otimes E). \end{aligned}$$

Since the summand $\mathcal{A}_{\mathcal{J},\{G\}}$, indexed by the full group $G$, consists precisely of all cuspidal automorphic forms compatible with $\mathcal{J}$, we define the *cuspidal cohomology* of $G$ with respect to $E$ as

$$H^*_{\mathrm{cusp}}(G,E) = H^*(\mathfrak{g}, K_{\mathbb{R}}; \mathcal{A}_{\mathcal{J},\{G\}} \otimes E) \cong \bigoplus_{\phi_\pi\in\Phi_{\mathcal{J},\{G\}}} H^*(\mathfrak{g}, K_{\mathbb{R}}; \mathcal{A}_{\mathcal{J},\{G\},\phi_\pi} \otimes E).$$

The natural complement of cuspidal cohomology, indexed by all $\{P\} \neq \{G\}$, forms the *Eisenstein cohomology*

$$H^*_{\mathrm{Eis}}(G,E) = \bigoplus_{\{P\}\neq\{G\}} H^*(\mathfrak{g}, K_{\mathbb{R}}; \mathcal{A}_{\mathcal{J},\{P\}} \otimes E) \cong \bigoplus_{\{P\}\neq\{G\}} \bigoplus_{\phi_\pi\in\Phi_{\mathcal{J},\{P\}}} H^*(\mathfrak{g}, K_{\mathbb{R}}; \mathcal{A}_{\mathcal{J},\{P\},\phi_\pi} \otimes E).$$

Our main goal is to understand the individual summands

$$H^*(\mathfrak{g}, K_{\mathbb{R}}; \mathcal{A}_{\mathcal{J},\{P\},\phi_\pi} \otimes E), \qquad \{P\} \neq \{G\}, \tag{$*$}$$

in the decomposition of the Eisenstein cohomology. In particular, we would like to find some kind of non-vanishing criterion, and, in the case of non-vanishing, investigate the internal structure of these summands.

[5] We are again skipping the definition. See [21] or [17].

### 3.4 Square-integrable cohomology

The first step in understanding the internal structure of the summand $(*)$ in the decomposition of the Eisenstein cohomology is to understand its square-integrable part.

Let $\mathcal{L}_{\mathcal{J},\{P\},\phi_\pi}$ be the (possibly trivial) subspace of $\mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}$, which consists of square-integrable automorphic forms with cuspidal support in $\phi_\pi$. By the Langlands spectral theory, roughly speaking, $\mathcal{L}_{\mathcal{J},\{P\},\phi_\pi}$ is spanned by the square-integrable iterated residues at $\nu = \lambda$ of the Eisenstein series $E(f_\nu, g)$ associated to $\pi^u$.

The inclusion $\mathcal{L}_{\mathcal{J},\{P\},\phi_\pi} \hookrightarrow \mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}$ gives rise to a map in cohomology

$$H^*(\mathfrak{g}, K_\mathbb{R}; \mathcal{L}_{\mathcal{J},\{P\},\phi_\pi} \otimes E) \to H^*(\mathfrak{g}, K_\mathbb{R}; \mathcal{A}_{\mathcal{J},\{P\},\phi_\pi} \otimes E),$$

which may not be injective any more. The image of this map is called the *square-integrable cohomology* with cuspidal support in $\phi_\pi$, or sometimes the *residual Eisenstein cohomology*, although the residues of Eisenstein series are not always square-integrable. It is denoted by

$$H^*_{(\mathrm{sq})}(\mathfrak{g}, K_\mathbb{R}; \mathcal{A}_{\mathcal{J},\{P\},\phi_\pi} \otimes E), \qquad (*_{\mathrm{sq}})$$

and may be thought of as a summand in the full square-integrable cohomology $H^*_{(\mathrm{sq})}(G, E)$, which is the image of the map induced in cohomology by the inclusion $\mathcal{L}_\mathcal{J} \hookrightarrow \mathcal{A}_\mathcal{J}$ of the space $\mathcal{L}_\mathcal{J}$ of all square integrable automorphic forms compatible with $\mathcal{J}$.

# 4 Necessary conditions for non-vanishing

The study of the summand $(*)$, for a given cuspidal support $\phi_\pi$, represented by a cuspidal automorphic representation $\pi \cong \pi^u \otimes \lambda$ of the Levi factor $M_P(\mathbb{A})$, is closely related to the structure of the space $\mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}$. The latter is determined by the analytic properties of the Eisenstein series $E(f_\nu, g)$ associated to $\pi^u$ at $\nu = \lambda$, and the Eisenstein series provide a link to the representation theoretic approach via the induced representation $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi^u \otimes \lambda)$. We are very vague at this point, but some sort of Frobenius reciprocity, combined with a result of Kostant about the Lie algebra cohomology of the unipotent radical [15, Th. 5.13], reduces the study of necessary conditions for non-vanishing of the summand $(*)$ to the non-vanishing of cuspidal cohomology for the Levi factor $M_P$ with respect to certain coefficient systems. The details are explained in [16, Sect. 3].

### 4.1 Geometric conditions

Before stating the necessary conditions for non-vanishing of the summand $(*)$, we need more notation. Let $W$, resp. $W_P$, be the absolute Weyl group of $G$, resp. $M_P$. The set of minimal coset representatives for the right cosets in $W_P \backslash W$, sometimes called the Kostant representatives, is denoted by $W^P$.

Let $\check{\mathfrak{a}}_0 = X^*(P_0) \otimes_\mathbb{Z} \mathbb{R}$ be the analogue of $\check{\mathfrak{a}}_P$ for the minimal parabolic subgroup $P_0$. Then, $\check{\mathfrak{a}}_P$ may be viewed as a subspace of $\check{\mathfrak{a}}_0$, and restriction of characters gives rise to a natural complement

$$\check{\mathfrak{a}}_0 = \check{\mathfrak{a}}_P \oplus \check{\mathfrak{a}}_0^P.$$

Let $\Lambda$ be the highest weight of $E$, viewed as an element of $\check{\mathfrak{a}}_0$, and let $\rho$ be the half-sum of positive absolute roots of $G$.

Then, the summand $(*)$ vanishes, except possibly if the following assertions are all simultaneously satisfied, with the same $w \in W^P$,

**G1.** $\lambda = -w(\Lambda + \rho)\big|_{\check{\mathfrak{a}}_P}$,

**G2.** the infinitesimal character of the archimedean component $\pi^u_\infty$ of $\pi^u$ is $-w(\Lambda+\rho)\big|_{\check{\mathfrak{a}}_0^P}$,

**G3.** $-w_{\mathrm{long},P}(\mu_w\big|_{\check{\mathfrak{a}}_0^P}) = \mu_w\big|_{\check{\mathfrak{a}}_0^P}$, where $\mu_w = w(\Lambda+\rho)-\rho$ and $w_{\mathrm{long},P}$ is the longest element in $W_P$,

**G4.** the archimedean component $\pi^u_\infty$ of $\pi^u$ is cohomological (with respect to some coefficient system),

where the vertical line stands for projections with respect to the above decomposition of $\check{\mathfrak{a}}_0$. The first two assertions, obtained in [19], follow from the compatibility with $\mathcal{J}$. The third assertion arises from the square-integrability on the level of Levi factors, as in [2]. The last assertion is obvious. We refer to these four assertions as the geometric conditions, because they are related to cohomological considerations.

## 4.2 Arithmetic conditions

A natural question to ask is when is the summand $(*_{\mathrm{sq}})$ non-trivial. Besides the geometric conditions of Sect. 4.1, there is an obvious arithmetic necessary condition for non-vanishing of the summand $(*_{\mathrm{sq}})$. It says that

**A.** $\mathcal{L}_{\mathcal{J},\{P\},\phi_\pi} \neq 0$,

or in other words, the Eisenstein series $E(f_\nu, g)$ associated to $\pi^u$ has a non-trivial square-integrable residue at $\nu = \lambda$. This condition cannot be stated explicitly in general, and is related to automorphic $L$-functions appearing in the constant term of the Eisenstein series. In some examples, we make it explicit in the theorems below.

# 5 Application for the case of the symplectic group

As an example of the application of the necessary conditions presented in Sect. 4, we consider the case of the split symplectic group $G = Sp_n$, of rank $n$, defined over $\mathbb{Q}$, and the cuspidal support in the Siegel maximal proper parabolic $\mathbb{Q}$-subgroup. This is one of the cases studied in [10].

More precisely, let $G = Sp_n$ be the split symplectic group defined over $\mathbb{Q}$, preserving the symplectic form on a $2n$-dimensional vector space over $\mathbb{Q}$ given, in some basis, by the matrix

$$\begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix},$$

where

$$J_n = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix},$$

with zeros outside the secondary diagonal. In this matrix realization of the split symplectic group $Sp_n$, we may choose for the Borel $\mathbb{Q}$-subgroup $P_0$ the group of all upper triangular matrices in $Sp_n$. The Levi factor $M_0$ of $P_0$ is a maximal $\mathbb{Q}$-split torus of $Sp_n$, which consists of all diagonal matrices in $Sp_n$, that is,

$$M_0(\mathbb{Q}) = \left\{ \mathrm{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}) \, : \, t_1, \ldots, t_n \in \mathbb{Q}^\times \right\}.$$

Let $e_i$, for $i = 1, \ldots, n$, denote the projection of $M_0$ to the $i$th coordinate,

$$e_i \left( \mathrm{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}) \right) = t_i.$$

We have $e_i \in X^*(P_0)$ is a $\mathbb{Q}$-rational character of $P_0$. We choose these projections $\{e_1, \ldots, e_n\}$ as a basis of $\check{\mathfrak{a}}_{0,\mathbb{C}}$. The choice of the Borel subgroup $P_0$ determines the set of positive and simple roots in the root system of $G$ with respect to $M_0$. The set $\Delta$ of simple roots consists of

$$\Delta = \left\{ e_1 - e_2, e_2 - e_3 \ldots, e_{n-1} - e_n, 2e_n \right\},$$

written in terms of $e_i$.

Let $P = M_P N_P$ be the Siegel maximal proper parabolic $\mathbb{Q}$-subgroup of $Sp_n$. It is the parabolic $\mathbb{Q}$-subgroup of $Sp_n$ corresponding to the subset $\Delta \setminus \{2e_n\}$ of the set $\Delta$ of simple roots. It is characterized by the fact that its Levi factor $M_P$ is isomorphic to $GL_n$.

**Theorem 5.1** (G., Schwermer [10]). *Let $G = Sp_n$ be the split symplectic group $Sp_n$, of rank $n$, defined over $\mathbb{Q}$. Let $P = M_P N_P$ be the Siegel standard parabolic subgroup, i.e., $M_P \cong GL_n$. Let $\pi^u$ be a unitary cuspidal automorphic representation of $M_P(\mathbb{A}) \cong GL_n(\mathbb{A})$, and let $\lambda \in \check{\mathfrak{a}}_P$ correspond to the character $|\det|^{s_0}$ of $M_P(\mathbb{A})$, where $s_0 \geq 0$. Let $\pi \cong \pi^u \otimes |\det|^{s_0}$. Let the highest weight $\Lambda = \sum_{i=1}^n \lambda_i e_i \in \check{\mathfrak{a}}_{0,\mathbb{C}}$, with $\lambda \in \mathbb{Z}$ and $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. Then, the space $(*_{\mathrm{sq}})$, that is,*

$$H^*_{(\mathrm{sq})}(\mathfrak{g}, K_\mathbb{R}; \mathcal{A}_{\mathcal{J},\{P\},\phi_\pi} \otimes E)$$

*is trivial, except possibly if the following assertions hold*

1. *$s_0 = 1/2$,*
2. *the exterior square L-function $L(s, \pi^u, \wedge^2)$ has a pole at $s = 1$,*
3. *the principal L-function $L(s, \pi^u)$ is non-zero at $s = 1/2$,*
4. *$n$ is even,*
5. *$\Lambda$ is such that $\lambda_{2j-1} = \lambda_{2j}$ for $j = 1, \ldots, n/2$,*
6. *the archimedean component*

$$\pi^u_\infty \cong \mathrm{Ind}_{Q(\mathbb{R})}^{GL_n(\mathbb{R})} \left( \otimes_{j=1}^{n/2} D(2\mu_j + 2n - 4j + 4) \right),$$

*where $\mu_j = \lambda_{2j-1} = \lambda_{2j}$, the parabolic subgroup $Q$ of $GL_n$ has the Levi factor $M_Q \cong GL_2 \times \cdots \times GL_2$ with $n/2$ copies of $GL_2$, and $D(k)$, for $k \geq 2$, is the discrete series representation of $GL_2(\mathbb{R})$ of lowest $O(2)$-type $k$.*

This theorem shows how the subtle interplay of geometric and arithmetic necessary conditions for non-vanishing of the summand $(*_{\mathrm{sq}})$, produces a quite restrictive set of necessary conditions in a given example. The conditions are not only arithmetic conditions on $\pi$, but also the rank $n$ of the group $G = Sp_n$ must be even, the highest weight $\Lambda$ must be of a special form, and the infinite component $\pi^u_\infty$ is a certain fixed tempered representation of $GL_n(\mathbb{R})$. It is an open problem to determine if there exists such $\pi$, for which all six assertions hold. To illustrate these six assertions in a low rank example, we take a special case in the following corollary.

**Corollary 5.2.** *In the notation of the previous theorem, let $G = Sp_2$, i.e., $n = 2$, and let $E = \mathbb{C}$ be trivial, i.e., $\lambda_i = 0$, for $i = 1, \dots, n$. Then, the space $(*_{\mathrm{sq}})$, that is,*

$$H^*_{(\mathrm{sq})}(\mathfrak{g}, K_{\mathbb{R}}; \mathcal{A}_{\mathcal{J},\{P\},\phi_\pi})$$

*is trivial, except possibly if the following assertions hold*

1. $s_0 = 1/2$,
2. *the central character $\omega_{\pi^u}$ of $\pi^u$ is trivial,*
3. $L(1/2, \pi^u) \neq 0$,
4. $\pi^u_\infty \cong D(4)$.

In a recent preprint [13], we study carefully such low rank cases. In the case of the corollary, we show in *loc. cit.* that $\pi^u$, satisfying necessary non-vanishing conditions of the corollary, really exist. It is a consequence of a non-trivial result of Trotabas [22].

However, the existence of $\pi^u$ satisfying all assertions of the corollary is still not sufficient to show that the summand $(*_{\mathrm{sq}})$ is non-trivial. We need an additional argument, showing that the image of the map in cohomology is non-trivial. This was pursued in the preprint [13] using [18].

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