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Maass's converse theorem and a lifting construction of automorphic forms on real hyperbolic spaces

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Abstract

This short note is a write-up of the talk presented by the first named author in the RIMS workshop 2016 "Automorphic forms, Automorphic L-functions and Related topics". We report the recent progress about our lifting construction of real analytic automorphic forms on real hyperbolic spaces.

1 Introduction

An interesting construction of cusp forms on a reductive group or its corresponding symmetric space is a lifting from a group or a symmetric space of smaller dimension. As is well-known, Saito-Kurokawa lifting provided a construction of holomorphic Siegel cusp forms of genus two with a lifting from elliptic cusp forms on the complex upper half plane. What is surprising is that this lifting leads to counterexamples to the Ramanujan conjecture. Since the discovery of the Saito-Kurokawa lifting many specialists have been believing that such counterexamples for a higher-dimensional group or symmetric space would be given by lifting from smaller groups or symmetric spaces.

Our research reported here started from two works [11] and [8], which deal with lifting constructions of real analytic cusp forms on orthogonal groups of signature $(1,4)$ and $(1,5)$ or on real hyperbolic spaces of dimensions four or five. We are trying higher-dimensional generalization of them. Our result is a lifting construction of automorphic forms on $O(1,8n+1)$ with respect to discrete subgroups associated with even unimodular lattices. When $n=1$ the automorphic forms are proved to be cusp forms. The fundamental tool of our resarch is Maass's converse theorem (cf. [7]) for real analytic automorphic forms on real hyperbolic spaces. More precisely we slightly modify the converse theorem to show the automorphy of our lifts.

2 Maass's converse theorem with a slight modification

2.1 Basic notation

Let H_n be the real hyperbolic space of dimension $n+1$ realized as

$$H_n := \{(x,y) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \mathbb{R}^n,\ y > 0\}.$$

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This is a Riemannian symmetric space realized also as $O(1,n+1)/O(1,n+1)\cap O(n+2)$, where $O(1,l)$ (respectively $O(m)$) denotes the orthogonal group with signature $(1,n+1)$ (respectively signature $(m,0)$ or $(0,m)$). With this identification we view H_n as a $O(1,n+1)$-homogeneous space

Given a discrete subgroup Γ of $O(1,n+1)$ we now define real analytic automorphic forms on H_n, which we call Maass forms on H_n:

Definition 2.1 *A C^∞-function $F : H_m \to \mathbb{C}$ is defined to be a Maass form with respect to Γ if it satisfies the following conditions:*

1. $F(\gamma(z)) = F(z)\ \forall(\gamma, z) \in \Gamma \times H_n$.
2. $\Omega \cdot F = -\frac{1}{2n}(r^2 + \frac{n^2}{4})F$ *with the Casimir operator* Ω $(r \in \mathbb{C})$.
3. *F is of moderate growth.*

We denote by $M(\Gamma, r)$ the space of Maass forms on H_n as are defined above.

We deal with $M(\Gamma, r)$ for a specified discrete subgroup Γ. For that purpose we introduce an even lattice (L, S) of $\mathbb{Z}$-rank n with a positive definite symmetric matrix S, where $L \subset \mathbb{R}^n$. Let $Q := \begin{pmatrix} & & 1 \\ & -S & \\ 1 & & \end{pmatrix}$ and $O(Q)$ denote the orthogonal group defined by

$$O(Q) := \{g \in M_{n+2}(\mathbb{R}) \mid {}^t g Q g = Q\},$$

which can be denoted also by $O(1,n+1)$. We will need an Iwasawa decomposition of $O(Q)$. To describe it we introduce three subgroups of $O(Q)$ as follows:

$$N := \left\{ n(x) := \begin{pmatrix} 1 & {}^t xS & \frac{1}{2}{}^t xSx \\ & 1_n & x \\ & & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{R}^n \right\},$$

$$A := \left\{ a_y := \begin{pmatrix} y & & \\ & 1_n & \\ & & y^{-1} \end{pmatrix} \;\middle|\; y \in \mathbb{R}_{>0} \right\},$$

$$K := O(Q) \cap O(R),$$

where $O(R)$ denotes the orthogonal group defined by the positive definite symmetric matrix $R = \begin{pmatrix} 1 & & \\ & S & \\ & & 1 \end{pmatrix}$. With these subgroups an Iwasawa decomposition is described as

$$O(Q) = NAK.$$

From this we see the identification $H_n \simeq NA \simeq O(Q)/K$.

We next introduce the discrete subgroup Γ_S of $O(Q)$ by

$$\Gamma_S := \{\gamma \in O(Q) \mid \gamma(\mathbb{Z} \oplus L \oplus \mathbb{Z}) = \mathbb{Z} \oplus L \oplus \mathbb{Z}\}.$$

We let Γ'_S be the subgroup of Γ_S generated by

$$\left\{ \begin{pmatrix} 1 & {}^t\lambda S & \frac{1}{2}{}^t\lambda S\lambda \\ & 1_n & \lambda \\ & & 1 \end{pmatrix}, \begin{pmatrix} & & \epsilon \\ & 1_n & \\ \epsilon & & \end{pmatrix}, \begin{pmatrix} 1 & & \\ & M & \\ & & 1 \end{pmatrix} \;\middle|\; \lambda \in L,\ \epsilon \in \{\pm 1\},\ M \in \mathrm{Aut}(L,S) \right\}.$$

2.2 Converse theorem

We are going to formulate Maass's converse theorem. Let $q_S(x) := \frac{1}{2}{}^t xSx$ for $x \in \mathbb{R}^n$. To state the converse theorem let F be a smooth function on $H_n \simeq NA$ given by the Fourier series

$$F(n(x)a_y) := \sum_{\lambda \in L^\sharp \setminus \{0\}} C_\lambda y^{n/2} K_r(4\pi y \sqrt{q_S(\lambda)}) \exp(2\pi\sqrt{-1}{}^t\lambda Sx), \tag{2.1}$$

where $L^\sharp$ denotes the dual lattice of L and K_r denotes the K-Bessel function parametrized by $r \in \mathbb{C}$. For this function we remark that F satisfies the second condition of Definition 2.1. We are now able to state the converse theorem as follows:

Theorem 2.2 ("Modified" Maass's converse theorem) *Let F be as above and recall that Γ'_S has been introduced in Section 2.1. For $F \in M(\Gamma'_S, r)$ the following conditions are necessary and sufficient:*

1. $C_\lambda = C_{u\lambda}$ *for* $u \in \mathrm{Aut}(L,S)$,
2. $|C_\lambda| = O(q_S(\lambda)^\kappa)$ *with some* $\kappa > 0$,
3. *For any fixed non-negative integer l, let $\{P_{l,\nu}\}_\nu$ be a basis of harmonic polynomials on $\mathbb{R}^n$ of degree l. Then, for any (l,ν), the Dirichlet series*

$$\xi(s, P_{l,\nu}) := (2\pi)^{-2s}\Gamma(s + \frac{\sqrt{-1}r}{2})\Gamma(s - \frac{\sqrt{-1}r}{2}) \sum_{\lambda \in L^\sharp \setminus \{0\}} \frac{C_\lambda P_{l,\nu}(\lambda)}{q_S(\lambda)^s}$$

satisfies the following

- $\xi(s, P_{l,\nu})$ *is entire and bounded on any vertical stripes,*
- *the functional equation* $\xi(s, P_{l,\nu}) = \xi(\frac{n}{2} + l - s, P_{l,\nu})$ *holds*

We have remarks on this theorem. The original converse theorem by Maass [7] uses the coordinate of the Clifford algebra to realize the real hyperbolic spaces and is formulated for smooth functions on the hyperbolic spaces given by the Fourier series *with the constant term.* We further remark that Maass's original formulation does not contain the first condition on C_λs as above, which is the modification we have made.

A convenient situation for us is that $\Gamma_S = \Gamma'_S$ holds. However, it looks difficult to prove this in general. We therefore provide such a convenient situation stated as follows:

Proposition 2.3 *(1) Suppose that an even lattice (L,S) satisfies the following condition on "covering radius":*

For any $x \in \mathbb{R}^n$ there is $\lambda \in L$ such that $q_S(x+\lambda) < 1$.

Then we have $\Gamma_S = \Gamma'_S$.
(2) For (L, S) as above we have $M(\Gamma_S, r) = M(\Gamma'_S, r)$. Namely the converse theorem holds for $M(\Gamma_S, r)$.

The first assertion is what was inspired by I. Kröcker [4]. We next come across the problem of how many even lattices with the condition on the covering radius we have. Such even lattices are totally classified by G. Nebe [10].

Proposition 2.4 (Nebe) *There are 69 even lattices with the condition on covering radius. The $\mathbb{Z}$-rank of such lattices are at most eight.*

What we should now note is that the table of even lattices with the condition on covering radius includes only one even unimodular lattice , which is the E_8-lattice. We consequently state the following:

Proposition 2.5 *Let $(\mathbb{Z}^8, S)$ be the E_8-lattice. Let F be a C^∞-function on H_9 given by the Fourier series (2.1) without the constant term. If F satisfies the three conditions in Theorem 2.2, $F \in M(\Gamma_S, r)$ and F is a cusp form.*

Let S be just as above. We remark that the number of Γ_S-cusps is verified to be exactly one for this S in a manner similar to [8, Lemma 2.3]. The class number of the $\mathbb{Q}$-orthogonal group defined by Q does not exceed that of the orthogonal group defined by S, and if the former class number is proved to one, we can show that the number of Γ_S-cusps coincides with the latter class number. It is well-known that the class numbers for S is exactly one, and we therefore see that both of the class number for Q and the number of the Γ_S-cusps are one. We consequently know that F is cuspidal since its Fourier expansion has no constant term.

3 Lifting construction and the main theorem

3.1 Statement of the main theorem

Let $\mathfrak{h}$ be the complex upper half plane $\{x + \sqrt{-1}y \in \mathbb{C} \mid y > 0\}$, which has the $SL_2(\mathbb{Z})$-action by the linear fractional transformation. By Δ we denote the hyperbolic Laplacian $y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ on $\mathfrak{h}$. We then introduce the notion of Maass cusp forms on $\mathfrak{h}$.

Definition 3.1 *A C^∞-function $f : \mathfrak{h} \to \mathbb{C}$ is called a Maass cusp form if it satisfies the following:*

1. $f(\gamma(z)) = f(z)\ \forall\gamma \in SL_2(\mathbb{Z})$,
2. $\Delta f = -(\frac{1}{4} + \frac{r^2}{4})f\ (r \in \mathbb{R})$,
3. *f vanishes at ∞, which means that the Fourier expansion of f is given by*

$$f(z) = \sum_{n\neq 0} c_f(n) W_{0,\frac{\sqrt{-1}r}{2}}(4\pi|n|y)\exp(2\pi\sqrt{-1}nx),$$

where $W_{0,\frac{\sqrt{-1}r}{2}}$ is the Whittaker function parametrized by $(0,\frac{\sqrt{-1}r}{2})$. We denote the space of these Maass cusp forms by $S(SL_2(\mathbb{Z}),r)$.

For this definition we remark that "$r\in\mathbb{R}$" in the eigenvalue condition for Ω is due to the validity of the Selberg conjecture for the Maass cusp forms for $SL_2(\mathbb{Z})$ (cf. [3, Corollary 11.5]).

Now let $(\mathbb{Z}^{8n},S)$ be an even umimodular lattice defined by a positive definite symmetric matrix S. In what follows, we often denote $\sqrt{\frac{1}{2}{}^t xSx}$ by $|x|$. Given $f\in S(SL_2(\mathbb{Z}),r)$ we define a function on H_{8n+1} by

$$F_f(n(x)a_y)=\sum_{\lambda\in\mathbb{Z}^{8n}\setminus\{0\}}A(\lambda)y^{4n}K_{\sqrt{-1}r}(4\pi|\lambda|y)\exp(2\pi\sqrt{-1}{}^t\lambda Sx),$$

where $A(\lambda)$ is defined as

$$A(\lambda):=|\lambda|\sum_{h|d_\lambda}c_f(-\frac{|\lambda|^2}{h^2})h^{4n-2}$$

with the greatest common divisor d_Λ of entries of λ. We are able to state our result on the lifting construction of Maass forms on a real hyperbolic space H_{8n+1}.

Theorem 3.2 (Main theorem) *(1) The mapping $f\mapsto F_f$ leads to the lifting as follows:*

$$S(SL_2(\mathbb{Z}),r)\ni f\mapsto F_f\in M(\Gamma'_S,r).$$

In particular, when $(\mathbb{Z}^{8n},S)$ is the E_8-lattice (namely $n=1$), $F_f\in M(\Gamma_S,r)$ and F_f is a cusp form.
(2) If $f\not\equiv 0$, $F_f\not\equiv 0$.

As an immediate consequence from Weyl's law for $SL_2(\mathbb{Z})$ by Selberg (cf. [3, Section 11.1]) we see the following:

Corollary 3.3 *There exists a non-zero F_f.*

3.2 Sketch of the proof for the main theorem

Our method of proof is to follow the argument in [11] and [8]. The first assertion is proved by the slightly modified converse theorem (cf. Theorem 2.2) and the proof of the second assertion is similar to [8, Theorem 4.4]. The result for the case of E_8-lattice then follows from Proposition 2.5.

Among the several points of the proof, the main difficulty is to study the analytic properties of the Dirichlet series attached to F_f as in the third condition of the converse theorem. The Dirichlet series for F_f has an integral expression of Rankin-Selberg type. To explain it we introduce theta series attached to harmonic polynomials $\{P_{l,\nu}\}$ (cf. Theorem 2.2) and a normalized Eisenstein series as follows:

$$\Theta_{l,\nu}(z):=\sum_{\beta\in\mathbb{Z}^{8n}}P_{l,\nu}(\beta)e^{2\pi\sqrt{-1}|\beta|^2z},$$

$$E(z,s):=\pi^{\frac{l}{2}+2n-\frac{1}{2}}\frac{\Gamma(s+2n+\frac{l}{2})}{\Gamma(s)}(\pi^{-s}\Gamma(s)\zeta(2s))\frac{1}{2}\sum_{\gamma\in\Gamma_\infty\setminus SL_2(\mathbb{Z})}(\frac{cz+d}{|cz+d|})^{l+4n}(\frac{\mathrm{Im}(z)}{|cz+d|^2})^s.$$

Here $\Gamma_\infty = \{\gamma \in SL_2(\mathbb{Z}) \mid \gamma(\infty) = \infty\}$. Let us introduce the Raknin-Selberg zeta integral

$$I(s) := \int_{SL_2(\mathbb{Z})\backslash\mathfrak{h}} f(z)\Theta_{l,\nu}(z)E(z,s)y^{\frac{l+4n}{2}}\frac{dxdy}{y^2},$$

and we then state the following:

Proposition 3.4 *(1) The zeta integral $I(s)$ is entire and bounded on any critical stripes, and satisfies the functional equation $I(s) = I(1-s)$.*
(2) We have

$$\xi(s+\frac{l}{2}+2n-\frac{1}{2}, P_{l,\nu}) = \begin{cases} I(s) & (l\text{:}even) \\ 0 & (l\text{:}odd) \end{cases},$$

which implies the desired functional equation

$$\xi(s, P_{l,\nu}) = \xi(\frac{l}{2}+4n-s, P_{l,\nu}).$$

4 Remaining problems

Around the end of the talk by the first author several problems in future were proposed. We now write down the two of them.

(1) Study of the automorphic representation generated by our lifts

If we successfully construct cusp forms by the lifting, an important problem is to study the "Ramanujan property" of the cusp forms, namely to know whether the cuspidal representations generated by the lifts have tempered local components at all places or not. Our previous work [8] provides a lifting construction for the case of the five dimensional hyperbolic space and shows that it lifts Hecke-eigen Maass cusp forms to Hecke-eigen cusp forms, namely the lifting is Hecke-equivariant. For this, note that there is an accidental isomorphism between $GSpin(1,5)$ and $GL(2)$ over a division quaternion algebra. In view of the global multiplicity one theorem for a general linear group over a division algebra by Badulescu-Renard [1] and [2], the images of the lifting [8] from Hecke eigen Maass forms generates irreducible cuspidal representations, and thus decompose into the restricted tensor products of local representations. The explicit calculation of the Hecke eigenvalues of the lifts carried out by [8] leads to the result that such a cuspidal representation has non-tempered local components for all non-archimedean primes.

However, we have no global multiplicity one theorem for orthogonal groups in general. Instead we think that the work [9] is useful to study the problem for our situation. It implies that the study on the Ramanujan properties of our lifting is reduced to that of Hecke-equivariance and Hecke eigenvalues for our lifting if the archimedean local representation is proved to be irreducible and tempered similarly as in [8, Theorem 6.8].

(2) Lifting from holomorphic modular forms

It is quite natural to consider the lifting of (holomorphic) elliptic cusp forms instead of that of (non-holomorphic) Maass cusp forms. For this we note that $(SL(2), O(p,q))$ forms a dual pair and that we can thus consider the theta lifting construction of the cusp forms on real hyperbolic spaces. The theta lifting construction from elliptic cusp forms (more generally, cusp forms generating discrete series representations at archimedean places) has been studied by Li [5]. In Li-Tan-Zhu [6] the archimedean representations of the theta lifts for the case of $O(1,n)$ are verified to be degenerate principal series representations which are cohomological representations "$A_{\mathfrak{q}}(\lambda)$". From these works we know that the cusp forms on $O(1,n)$ obtained by theta lifting from elliptic cusp forms contribute to the cohomologies of arithmetic groups and that the archimedean representation types of such cusp forms are explicitly given. We can thus say that such lifting construction would have arithmetic significance. Let us now note that the work by Li [5] is given in the framework of automorphic representations. We should then remark that the explicit lifting construction of the cusp forms on $O(1,n)$ with explicit Fourier coefficients are still open and significant problem to be investigated.

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