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# HDG METHODS FOR SECOND-ORDER ELLIPTIC PROBLEMS

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ABSTRACT. In this article, we review the hybrid (hybridized or hybridizable) discontinuous Galerkin (HDG) method based on a classical hybrid finite element method for second-order elliptic problems. Our HDG method was firstly obtained by stabilizing the simplified hybrid displacement method. Optimal error estimates in the energy and $L^2$ norms were proved for the Poisson equation. The method was extended to convection-diffusion problems by introducing a kind of an upwind term. It was verified mathematically and numerically that the method is robust even in the convection-dominated case, where the standard finite element method fails due to its numerical instability.

## 1. INTRODUCTION

In recent years, *hybrid (hybridized or hybridizable)* discontinuous Galerkin (HDG) methods have been investigated and applied to various problems. The usual discontinuous Galerkin (DG) method utilizes two types of numerical fluxes to deal with the discontinuity of an approximate solution $u_h$ on inter-element boundaries. In the HDG method, a *numerical trace* $\widehat{u}_h$ is introduced to approximate the trace of a solution besides $u_h$, which is a new unknown and may be called the hybrid unknown.

The number of degrees of freedom (DOF) of the DG method is much larger than that of the standard finite element method. By the static condensation, that is, eliminating the hybrid unknown $\widehat{u}_h$ by $u_h$, we obtain a discretized equation in terms of only on $\widehat{u}_h$. As a result, the number of DOF of the HDG method can be considerably reduced, which is the main advantage of the HDG method over the DG method. We note that the HDG method has remarkable features besides the above advantage, such as superconvergence properties and various connections with other numerical methods (nonconforming and mixed finite element methods, etc.).

The HDG method was firstly introduced by Cockburn et al. [10], in which the hybridization of the local discontinuous Galerkin (LDG) method (cf. [3]) is successful to unify the formulations of various hybrid methods. An overview of the HDG methods was already provided in [10], and we refer the readers to it as a survey paper.

In this article, we revisit and review a different hybridization of the DG method based on a classical hybrid finite element method. Hybridization of the finite element method was early proposed by Pian in 1964 (cf. [36]) and by de Veubeque in 1965 [11, 43]. Later, the *hybrid displacement method* was proposed by Tong in 1970 [41], in which the hybrid displacement and Lagrange multiplier are introduced as new unknowns on inter-element boundaries. The *simplified* hybrid displacement method, where the Lagrange multiplier is taken to be the normal gradient of $u_h$, was also investigated by Kikuchi and Ando [19, 21, 22, 20, 23, 18, 24, 25]. Those methods were partially successful, however, they suffered from numerical instability. Decades later, in [26, 30, 35, 31], stabilized methods were developed for linear elasticity problems and the Poisson equation. The instability was overcome by introducing the stabilization technique of the interior penalty method [2], which is described in Section 2. Numerical results are not shown in this article, see, e.g. [26, 30, 35, 31].

For stationary convection-diffusion problems, the HDG method has been developed and there have been many published papers, for example, see [27, 13, 7, 8, 32, 14, 6, 38]. Here we focus on the present author's work [32] because the resulting HDG methods listed above are not so different from each other. In the HDG method, a convection-diffusion equation is decomposed into diffusive and convective parts, and they are discretized separately. The diffusive part can be discretized in the same way as the Poisson equation. The convective part is discretized by newly introducing a kind of an upwind term. The key idea of devising the upwind term is to switch $u_h$ and $\widehat{u}_h$ according to the outflow and inflow inter-element boundaries. In Section 3, we are going to state the upwind scheme proposed by the present author [32] for convection-diffusion problems. Numerical results will be presented to validate the stability of the scheme.

## 2. HDG METHOD FOR DIFFUSION PROBLEMS

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a bounded polygonal or polyhedral domain and $f \in L^2(\Omega)$ be a given function. We consider the Poisson equation with homogeneous boundary condition:

$$-\Delta u = f \text{ in } \Omega, \tag{2.1a}$$

$$u = 0 \text{ on } \partial\Omega. \tag{2.1b}$$

The HDG method can also be applied to the problems with non-homogeneous Dirichlet and Neumann boundary conditions, but we here consider only the homogeneous Dirichlet boundary condition for simplicity.

### 2.1. Notation.

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of meshes of the domain $\Omega$. The subscript $h$ stands for the mesh size $h := \max_{K\in\mathcal{T}_h} \operatorname{diam}(K)$. We assume that $\{\mathcal{T}_h\}_h$ satisfies the chunkiness condition [5, 17], which is equivalent to the shape-regular condition if all meshes consist of only triangles or tetrahedrons. We also assume that $\{\mathcal{T}_h\}_h$ satisfies the local quasi-uniformity [17],i.e., there exists a constant $C$ such that $\mathrm{diam}(K)/\mathrm{diam}(e) \leq C$ for any $K \in \mathcal{T}_h$ and edge $e \subset \partial K$. Throughout the article, the symbol $C$ denotes a generic constant independent of $h$. The set of all edges of $\mathcal{T}_h$ is denoted by $\mathcal{E}_h = \{e \subset \partial K : K \in \mathcal{T}_h\}$. The skeleton of $\mathcal{T}_h$, defined by $\bigcup_{K\in\mathcal{T}_h} \partial K$, is denoted by the same symbol $\mathcal{E}_h$. We define $L^2_D(\mathcal{E}_h) = \{\widehat{v} \in L^2(\mathcal{E}_h) : \widehat{v} = 0 \text{ on } \partial\Omega\}$. We will use the standard notation of the Sobolev spaces [1], such as $H^m(D)$, $W^{m,p}(D)$, $\|\cdot\|_{m,D} = \|\cdot\|_{H^m(D)}$, $|\cdot|_{m,D} = |\cdot|_{H^m(D)}$ for an integer $m$ and a domain $D$. The piecewise or broken Sobolev spaces are introduced: $H^m(\mathcal{T}_h) := \{v \in L^2(\Omega) : v|_K \in H^m(K)\ \forall K \in \mathcal{T}_h\}$. The inner products are denoted by

$$(u,v)_{\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h} \int_K uv dx, \quad \langle u,v\rangle_{\partial\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h} \int_{\partial K} uv ds.$$

The finite element spaces for approximating $u$ and its trace $\widehat{u}$ are denoted by $W_h$ and $M_h$, respectively. We impose the homogeneous Dirichlet boundary condition on $M_h$, i.e., assume $M_h \subset L^2_D(\mathcal{E}_h)$. In usual cases, the finite element spaces are set to be piecewise polynomial spaces of same degree; $W_h = P_k(\mathcal{T}_h)$ and $M_h = P_k(\mathcal{E}_h)$. Recently, it turned out that optimal convergences can be achieved by setting $W_h = P_{k+1}(\mathcal{T}_h)$, $M_h = P_k(\mathcal{E}_h)$ and taking an $L^2$-projection in the stabilization term, see [28, 33, 34, 39, 40, 29, 9, 37].

2.2. **The scheme.** We give the formulation of the HDG method proposed in [30, 35]. The method is equivalent to the IP-H method defined in [10], and we will here call it so. The IP-H method is as follows: find $u_h \in W_h$ and $\widehat{u}_h \in M_h$ such that

$$a^d_h(u_h, \widehat{u}_h; v_h, \widehat{v}_h) = (f, v_h)_\Omega \quad \forall v_h \in W_h, \widehat{v}_h \in M_h, \tag{2.2}$$

where

$$\begin{aligned} a^d_h(u_h, \widehat{u}_h; v_h, \widehat{v}_h) &:= (\nabla u_h, \nabla v_h)_{\mathcal{T}_h} + \langle \boldsymbol{n}\cdot\nabla u_h, \widehat{v}_h - v_h\rangle_{\partial\mathcal{T}_h} \\ &\quad + \langle \boldsymbol{n}\cdot\nabla v_h, \widehat{u}_h - u_h\rangle_{\partial\mathcal{T}_h} + S_h(u_h, \widehat{u}_h; v_h, \widehat{v}_h), \\ S_h(u_h, \widehat{u}_h; v_h, \widehat{v}_h) &:= \langle \tau(\widehat{u}_h - u_h), \widehat{v}_h - v_h\rangle_{\partial\mathcal{T}_h}. \end{aligned}$$

Here $\tau$ is a stabilization parameter, which is usually set to be $\tau = \tau_0/h_e$, where $\tau_0$ is a positive constant, $h_e := \mathrm{diam(e)}$ for an edge $e$ and $\boldsymbol{n}$ is the unit outward normal vector to $\partial K$. It can be proved that the scheme is coercive if $\tau_0$ is set to be sufficiently large. In general, too large $\tau$ is likely to spoil the discontinuity of the approximate solutions. Therefore, we should select a moderate value for $\tau_0$. The HDG method coercive for any positive $\tau_0$ was already obtained; the LDG-H method [10] and the HDG method using a lifting operator [31]. As will be shown later, both the methods are essentially equivalent to each other.

In the next section, we are going to describe how the method is derived.

2.3. **Derivation.** Let $K \in \mathcal{T}_h$ and $u \in H^2(\Omega)$. Multiplying (2.1a) by a test function $v \in H^2(K)$ and integrating it by parts over $K$, we get

$$(\nabla u, \nabla v)_K - \langle \boldsymbol{n} \cdot \nabla u, v \rangle_{\partial K} = (f, v)_K.$$

Summing the above over $K \in \mathcal{T}_h$ yields

$$(\nabla u, \nabla v)_{\mathcal{T}_h} - \langle \boldsymbol{n} \cdot \nabla u, v \rangle_{\partial \mathcal{T}_h} = (f, v)_\Omega. \tag{2.3}$$

We now introduce a hybrid function $\widehat{v} \in L^2_D(\mathcal{E}_h)$ as a test function. Since $\boldsymbol{n} \cdot \nabla u$ and $\widehat{v}$ are both single-valued on $\mathcal{E}_h$ and $\widehat{v}$ vanishes on $\partial\Omega$, the transmission condition follows:

$$\langle \boldsymbol{n} \cdot \nabla u, \widehat{v} \rangle_{\partial \mathcal{T}_h} = 0. \tag{2.4}$$

Adding this into (2.3) and symmetrizing it, we obtain

$$(\nabla u, \nabla v)_{\mathcal{T}_h} + \langle \boldsymbol{n} \cdot \nabla u, \widehat{v} - v \rangle_{\partial \mathcal{T}_h} + \langle \boldsymbol{n} \cdot \nabla v, \widehat{u} - u \rangle_{\partial \mathcal{T}_h} = (f, v)_\Omega, \tag{2.5}$$

where $\widehat{u}$ is the trace of $u$. Since the scheme is still not stable in general, we add the stabilization or penalty term $\tau \langle \widehat{u} - u, \widehat{v} - v \rangle_{\partial \mathcal{T}_h}$. Thus we obtain (2.2).

*Remark.* In (2.2), taking $\widehat{v}_h \equiv 0$ on $\mathcal{E}_h$ and $v_h \equiv 0$ on $\Omega \setminus K$ for some $K \in \mathcal{T}_h$, we have

$$\begin{aligned}(\nabla u_h, \nabla v_h)_K &- \langle \boldsymbol{n} \cdot \nabla u_h, v_h \rangle_{\partial K} - \langle \boldsymbol{n} \cdot \nabla v_h, u_h \rangle_{\partial K} + \langle \tau u_h, v_h \rangle_{\partial K} \\ &= (f, v_h)_K + \langle \widehat{u}_h, \tau v_h - \boldsymbol{n} \cdot \nabla v_h \rangle_{\partial K},\end{aligned}$$

which implies that $u_h|_K$ can be determined by only $\widehat{u}_h|_{\partial K}$. There is no direct connection between $u_h|_K$ and $u_h|_{K'}$ for distinct elements $K, K' \in \mathcal{T}_h$, and they are linked only through the numerical trace $\widehat{u}_h|_{\partial K \cap \partial K'}$. It enables us to do the so-called static condensation, i.e., the construction of a linear system in terms of only $\widehat{u}_h$ by element-by-element elimination of $u_h$.

2.4. **Error estimates.** We present the outline of error analysis for the IP-H method. The energy norm is defined by

$$|\!|\!|(v_h, \widehat{v}_h)|\!|\!|_d^2 := \|\nabla v_h\|^2_{\mathcal{T}_h} + \|h_e^{-1/2}(\widehat{v}_h - v_h)\|^2_{\partial \mathcal{T}_h},$$

where

$$\|\nabla v_h\|^2_{\mathcal{T}_h} := (\nabla v_h, \nabla v_h)_{\mathcal{T}_h},$$

$$\|h_e^{-1/2}(\widehat{v}_h - v_h)\|^2_{\partial \mathcal{T}_h} := \langle h_e^{-1}(\widehat{v}_h - v_h), \widehat{v}_h - v_h \rangle_{\partial \mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \sum_{e \subset \partial K} \int_e h_e^{-1} |\widehat{v}_h - v_h|^2 ds.$$

We assume the approximation property: for any $v \in H^{k+1}(\Omega)$ and its trace $\widehat{v}$, there exists a constant $C$ independent of $h$ such that

$$\inf_{v_h \in W_h, \widehat{v}_h \in M_h} |||(v - v_h, \widehat{v} - \widehat{v}_h)|||_d \leq Ch^k |v|_{k+1,\Omega}.$$

The following fundamental properties on the bilinear form $a_h^d(\cdot;\cdot)$ hold.

**Lemma 1.** The following hold.

(1) (Consistency) Let $u$ be the exact solution of (2.1) and $\widehat{u}$ denote the trace of $u$. Then

$$a_h^d(u, \widehat{u}; v_h, \widehat{v}_h) = (f, v_h)_\Omega \quad \forall v_h \in W_h, \widehat{v}_h \in M_h.$$

(2) (Boundedness) There exists a constant $C$ such that

$$|a_h^d(w_h, \widehat{w}_h; v_h, \widehat{v}_h)| \leq C\, |||(w_h, \widehat{w}_h)|||_d\, |||(v_h, \widehat{v}_h)|||_d \quad \forall w_h, v_h \in W_h, \widehat{w}_h, \widehat{v}_h \in M_h.$$

(3) (Coercivity) There exists a constant $C$ such that

$$a_h^d(v_h, \widehat{v}_h; v_h, \widehat{v}_h) \geq C\, |||(v_h, \widehat{v}_h)|||_d^2 \quad \forall v_h \in W_h, \widehat{v}_h \in M_h.$$

*Proof.* The full proof was firstly given in the present author's Master's thesis [30], which, however, is written in Japanese. So, we refer to [32]. $\square$

*Remark.* To obtain an error estimate, we need not only the consistency and boundedness for $W_h$ and $M_h$ but also those for $H^2(\Omega)$. However, we omitted it because those are not directly used in this article and we have to introduce the auxiliary norm and notations.

From the above lemma, the following optimal error estimates can be deduced.

**Theorem 2.** Assume that $\{\mathcal{T}_h\}_h$ satisfies the chunkiness condition and the quasi-uniformity and that the approximation property holds. Let $u$ be the solution of (2.1) and $\widehat{u}$ denote the trace of $u$. If $u \in H^{k+1}(\Omega)$, then

$$|||(u - u_h, \widehat{u} - \widehat{u}_h)|||_d \leq Ch^k |u|_{k+1,\Omega}. \tag{2.6}$$

By Aubin-Nitsche's trick, we have

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^{k+1} |u|_{k+1,\Omega}.$$

2.5. **Nonsymmetric schemes.** The third term in (2.5) is called a consistent term because $\langle \boldsymbol{n}\cdot\nabla v, \widehat{u}-u\rangle_{\partial\mathcal{T}_h}=0$ holds for the exact solution $u$ and its trace $\widehat{u}$. We note that the nonsymmetric version of the scheme [30, 35] can also be deduced, for a real number $s$,

$$\begin{aligned} a_h^d(u_h,\widehat{u}_h;v_h,\widehat{v}_h) = (\nabla u_h,\nabla v_h)_{\mathcal{T}_h} &+ \langle \boldsymbol{n}\cdot\nabla u_h, \widehat{v}_h - v_h\rangle_{\partial\mathcal{T}_h} \\ &+ s\langle \boldsymbol{n}\cdot\nabla v_h, \widehat{u}_h - u_h\rangle_{\partial\mathcal{T}_h} + S_h(u_h,\widehat{u}_h;v_h,\widehat{v}_h). \end{aligned}$$

The IP-H (symmetric) scheme (2.2) is included as $s=1$. We call the scheme with $s\neq 1$ the nonsymmetric scheme. Although an optimal $H^1$-error estimate for the nonsymmetric scheme was proved as well as the symmetric scheme, an optimal $L^2$-error estimate could not be proved because of the lack of the adjoint consistency. Note that the order of convergence in the $L^2$ norm is greater than or equal to that of the energy norm, which follows easily from the fact that the energy norm is stronger than the $L^2$ norm. In [30, 35], it was shown by numerical experiments that the $L^2$-orders of convergence are actually suboptimal when the degrees of polynomials are even. For odd polynomials, the optimal convergence in the $L^2$ norm was observed in some cases. The $L^2$ suboptimality of the nonsymmetric DG method were investigated in [15, 12, 4], whereas there are few studies for the HDG method. For the DG method, in [15], the suboptimal convergence was in fact demonstrated in the one and two dimensions in special cases where the mesh and exact solution are carefully designed. It might be the case for the HDG method.

2.6. **A lifting operator.** In [31], a lifting operator was introduced and the HDG method using it was also proposed. The local lifting operator $\boldsymbol{R}_h^{\partial K}: L^2(\partial K)\to W_h(K)^n$ is defined by, for $\widehat{\mu}\in L^2(\partial K)$,

$$(\boldsymbol{R}_h^{\partial K}(\widehat{\mu}),\boldsymbol{w})_K = \langle \widehat{\mu}, \boldsymbol{w}\cdot\boldsymbol{n}\rangle_{\partial K} \quad \forall \boldsymbol{w}\in W_h(K)^n.$$

For $\mu\in H^1(K)$, we define $\boldsymbol{R}_h^{\partial K}(\mu)=\boldsymbol{R}_h^{\partial K}(\mu|_{\partial K})$.

The (global) lifting operator $\boldsymbol{R}_h:\prod_{K\in\mathcal{T}_h}L^2(\partial K)\to W_h^n$ is defined by $\boldsymbol{R}_h(\mu)|_K = \boldsymbol{R}_h(\mu|_{\partial K})$ for all $K\in\mathcal{T}_h$. Note that the lifting operator satisfies

$$(\boldsymbol{R}_h(\mu),\boldsymbol{w})_{\mathcal{T}_h} = \langle \mu, \boldsymbol{w}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h}\langle \mu|_{\partial K}, \boldsymbol{w}\cdot\boldsymbol{n}\rangle_{\partial K} \quad \forall \boldsymbol{w}\in W_h^n$$

for $\mu\in\prod_{K\in\mathcal{T}_h}L^2(\partial K)$.

The IP-H method using the lifting operator, which is going to be called the IPL-H method in this article, read as: find $u_h\in W_h$ and $\widehat{u}_h\in M_h$ such that

(2.7)
$$a_h^d(u_h,\widehat{u}_h;v_h,\widehat{v}_h) + (\boldsymbol{R}_h(\widehat{u}_h-u_h),\boldsymbol{R}_h(\widehat{v}_h-v_h))_{\mathcal{T}_h} = (f,v_h)_\Omega \quad \forall v_h\in W_h, \widehat{v}_h\in M_h.$$

Thanks to the additional stabilization term $(\boldsymbol{R}_h(\widehat{u}_h - u_h), \boldsymbol{R}_h(\widehat{v}_h - v_h))_{\mathcal{T}_h}$, the scheme is coercive for all $\tau > 0$. We note that the scheme (2.7) can be rewritten as

$$(\nabla u_h + \boldsymbol{R}_h(\widehat{u}_h - u_h), \nabla v_h + \boldsymbol{R}_h(\widehat{v}_h - v_h))_{\mathcal{T}_h} + S_h(u_h, \widehat{u}_h; v_h, \widehat{v}_h) = (f, v_h)_\Omega.$$

2.7. **Equivalence between the IPL-H and LDG-H methods.** We are going to show that the IPL-H method is essentially equivalent to the LDG-H method. In the LDG-H method (cf. [10]), the mixed formulation of (2.1) is considered:

$$\boldsymbol{q} + \nabla u = 0, \qquad \nabla \cdot \boldsymbol{q} = f.$$

Let $\boldsymbol{V}_h$ be the finite element space for approximating $\boldsymbol{q}$. The LDG-H method reads as: find $\boldsymbol{q}_h \in \boldsymbol{V}_h$, $u_h \in W_h$ and $\widehat{u}_h \in M_h$ such that

$$(\boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} - (u_h, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} + \langle \widehat{u}_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial\mathcal{T}_h} = 0, \qquad \forall \boldsymbol{v} \in \boldsymbol{V}_h, \tag{2.8a}$$

$$-(\boldsymbol{q}_h, \nabla w)_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, w \rangle_{\partial\mathcal{T}_h} = (f, w)_\Omega, \qquad \forall w \in W_h, \tag{2.8b}$$

$$\langle \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, \widehat{w} \rangle_{\partial\mathcal{T}_h} = 0, \qquad \forall \widehat{w} \in M_h, \tag{2.8c}$$

where the numerical flux $\widehat{\boldsymbol{q}}_h$ is defined by $\widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h + \tau(u_h - \widehat{u}_h)\boldsymbol{n}$.

**Proposition 3.** The IPL-H method is equivalent to the LDG-H method with $\boldsymbol{V}_h = W_h^n$.

*Proof.* To begin with, integrating (2.8a) by parts, we have

$$(\boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} + (\nabla u_h, \boldsymbol{v})_{\partial\mathcal{T}_h} + \langle \widehat{u}_h - u_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial\mathcal{T}_h} = 0. \tag{2.9}$$

Substituting $\boldsymbol{v} = \nabla w$ into (2.9) yields

$$(\boldsymbol{q}_h, \nabla w)_{\mathcal{T}_h} + (\nabla u_h, \nabla w)_{\partial\mathcal{T}_h} + \langle \widehat{u}_h - u_h, \boldsymbol{n} \cdot \nabla w \rangle_{\partial\mathcal{T}_h} = 0.$$

From this and (2.8b), it follows that

$$(\nabla u_h, \nabla w)_{\mathcal{T}_h} + \langle \widehat{u}_h - u_h, \boldsymbol{n} \cdot \nabla w \rangle_{\partial\mathcal{T}_h} + \langle \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, w \rangle_{\partial\mathcal{T}_h} = (f, w)_\Omega.$$

By (2.8c) and the definition of $\widehat{\boldsymbol{q}}_h$, we have

$$\begin{aligned}(\nabla u_h, \nabla w)_{\mathcal{T}_h} + \langle \widehat{u}_h - u_h, \boldsymbol{n} \cdot \nabla w \rangle_{\partial\mathcal{T}_h} + \langle (\boldsymbol{q}_h + \tau(u_h - \widehat{u}_h)) \cdot \boldsymbol{n}, w - \widehat{w} \rangle_{\partial\mathcal{T}_h} \\ = (f, w)_\Omega \quad \forall w \in W_h, \widehat{w} \in M_h.\end{aligned} \tag{2.10}$$

Substituting $\boldsymbol{v} = \boldsymbol{R}_h(w - \widehat{w})$ into (2.9), we deduce

$$\begin{aligned}\langle \boldsymbol{q}_h \cdot \boldsymbol{n}, w - \widehat{w} \rangle_{\partial\mathcal{T}_h} &= (\boldsymbol{q}_h, \boldsymbol{R}_h(w - \widehat{w}))_{\mathcal{T}_h} \\ &= -(\nabla u_h, \boldsymbol{R}_h(w - \widehat{w}))_{\mathcal{T}_h} - \langle \widehat{u}_h - u_h, \boldsymbol{R}_h(w - \widehat{w}) \cdot \boldsymbol{n} \rangle_{\partial\mathcal{T}_h} \\ &= -\langle \boldsymbol{n} \cdot \nabla u_h, w - \widehat{w} \rangle_{\partial\mathcal{T}_h} + (\boldsymbol{R}_h(\widehat{u}_h - u_h), \boldsymbol{R}_h(\widehat{w} - w))_{\mathcal{T}_h}.\end{aligned}$$

From this and (2.10), we obtain the scheme (2.7). □

## 3. HDG METHOD FOR CONVECTION-DIFFUSION PROBLEMS

We consider the stationary convection-diffusion equation:

$$\begin{aligned} -\epsilon \Delta u + \boldsymbol{b} \cdot \nabla u + cu &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\epsilon > 0$ is the diffusion coefficient, $\boldsymbol{b} \in W^{1,\infty}(\Omega)^n$ is divergence-free, $c$ is a positive constant. We are concerned with the convection-dominated case, i.e. $\epsilon \ll \|\boldsymbol{b}\|_\infty$, because the standard finite element method gets unstable in such a case as is well known.

### 3.1. The upwind scheme.

The convection-diffusion problem is decomposed into diffusive and convective parts. In the HDG method, they are discretized separately. The HDG method proposed in [32] is as follows: find $u_h \in W_h$ and $\widehat{u}_h \in M_h$ such that

$$\epsilon a_h^d(u_h, \widehat{u}_h; v_h, \widehat{v}_h) + a_h^c(u_h, \widehat{u}_h; v_h, \widehat{v}_h) = (f, v_h)_\Omega \quad \forall v_h \in W_h, \widehat{v}_h \in M_h,$$

where

$$\begin{aligned} a_h^c(u_h, \widehat{u}_h; v_h, \widehat{v}_h) := (\boldsymbol{b} \cdot \nabla u_h, v_h)_{\mathcal{T}_h} + \langle \widehat{u}_h - u_h, (\boldsymbol{b} \cdot \boldsymbol{n})\widehat{v}_h \rangle_{\partial\mathcal{T}_h^+} \\ + \langle \widehat{u}_h - u_h, (\boldsymbol{b} \cdot \boldsymbol{n}) v_h \rangle_{\partial\mathcal{T}_h^-} + (cu_h, v_h)_\Omega. \end{aligned} \tag{3.1}$$

The inner products are defined by

$$\langle w, v \rangle_{\partial\mathcal{T}_h^\pm} := \sum_{K \in \mathcal{T}_h} \int_{\partial K^\pm} wv ds,$$

where

$$\begin{aligned} \partial K^- &:= \{x \in \partial K : \boldsymbol{b}(x) \cdot \boldsymbol{n}(x) < 0\}, \\ \partial K^+ &:= \{x \in \partial K : \boldsymbol{b}(x) \cdot \boldsymbol{n}(x) \geq 0\}, \end{aligned}$$

see also Figure 1. We note that, before the upwind scheme (3.1) was proposed in [32], the HDG discretization for the convective part was already proposed in [10, 13]. The scheme of [10, 13] was as follows:

$$\begin{aligned} \widetilde{a}_h^c(u_h, \widehat{u}_h; v_h, \widehat{v}_h) := -(u_h, \boldsymbol{b} \cdot \nabla v_h)_{\mathcal{T}_h} + \langle (\boldsymbol{b} \cdot \boldsymbol{n}) u_h, v_h - \widehat{v}_h \rangle_{\partial\mathcal{T}_h^+} \\ + \langle (\boldsymbol{b} \cdot \boldsymbol{n}) \widehat{u}_h, v_h - \widehat{v}_h \rangle_{\partial\mathcal{T}_h^-} + (cu_h, v_h)_\Omega. \end{aligned}$$

We can show that both the schemes are equivalent to each other.

**Proposition 4.** It holds that $a_h^c(u_h, \widehat{u}_h; v_h, \widehat{v}_h) = \widetilde{a}_h^c(u_h, \widehat{u}_h; v_h, \widehat{v}_h)$ for all $u_h, v_h \in W_h$ and $\widehat{u}_h, \widehat{v}_h \in M_h$.

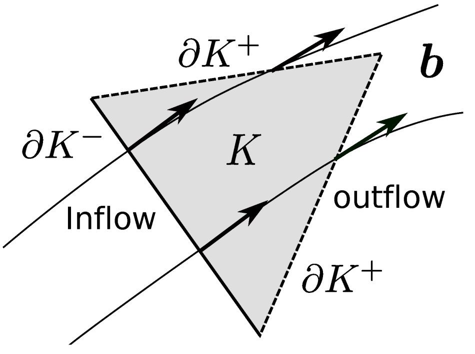


FIGURE 1. Inflow and outflow element boundaries.

*Proof.* We assume that $c = 0$ only for simplicity. Integrating by parts and recalling that $\boldsymbol{b}$ is divergence-free, we have

$$(\boldsymbol{b}\cdot\nabla u_h, v_h) = -(u_h, \boldsymbol{b}\cdot\nabla v_h) + \langle(\boldsymbol{b}\cdot\boldsymbol{n})u_h, v_h\rangle_{\partial\mathcal{T}_h^+} + \langle(\boldsymbol{b}\cdot\boldsymbol{n})u_h, v_h\rangle_{\partial\mathcal{T}_h^-}.$$

The bilinear form $a_h^c$ can be rewritten as

$$\begin{aligned} a_h^c(u_h, \widehat{u}_h; v_h, \widehat{v}_h) = -(u_h, \boldsymbol{b}\cdot\nabla v_h) &+ \underbrace{\langle(\boldsymbol{b}\cdot\boldsymbol{n})u_h, v_h\rangle_{\partial\mathcal{T}_h^+}}_{=:I_1} + \underbrace{\langle(\boldsymbol{b}\cdot\boldsymbol{n})u_h, v_h\rangle_{\partial\mathcal{T}_h^-}}_{=:I_2} \\ &+ \underbrace{\langle\widehat{u}_h - u_h, (\boldsymbol{b}\cdot\boldsymbol{n})\widehat{v}_h\rangle_{\partial\mathcal{T}_h^+}}_{=:I_3} + \underbrace{\langle\widehat{u}_h - u_h, (\boldsymbol{b}\cdot\boldsymbol{n})v_h\rangle_{\partial\mathcal{T}_h^-}}_{=:I_4}, \end{aligned}$$

and we have

$$\begin{aligned} I_1 + I_3 &= \langle u_h, (\boldsymbol{b}\cdot\boldsymbol{n})(v_h - \widehat{v}_h)\rangle_{\partial\mathcal{T}_h^+} + \langle\widehat{u}_h, (\boldsymbol{b}\cdot\boldsymbol{n})\widehat{v}_h\rangle_{\partial\mathcal{T}_h^+} =: I_5 + I_6, \\ I_2 + I_4 &= \langle\widehat{u}_h, (\boldsymbol{b}\cdot\boldsymbol{n})v_h\rangle_{\partial\mathcal{T}_h^-} =: I_7. \end{aligned}$$

Using the transmission condition

$$\langle(\boldsymbol{b}\cdot\boldsymbol{n})\widehat{u}_h, \widehat{v}_h\rangle_{\partial\mathcal{T}_h} = \langle(\boldsymbol{b}\cdot\boldsymbol{n})\widehat{u}_h, \widehat{v}_h\rangle_{\partial\mathcal{T}_h^+} + \langle(\boldsymbol{b}\cdot\boldsymbol{n})\widehat{u}_h, \widehat{v}_h\rangle_{\partial\mathcal{T}_h^-} = 0,$$

we get

$$I_6 + I_7 = \langle(\boldsymbol{b}\cdot\boldsymbol{n})\widehat{u}_h, v_h - \widehat{v}_h\rangle_{\partial\mathcal{T}_h^-}.$$

Thus, it follows that

$$I_1 + I_2 + I_3 + I_4 = \langle(\boldsymbol{b}\cdot\boldsymbol{n})u_h, v_h - \widehat{v}_h\rangle_{\partial\mathcal{T}_h^+} + \langle(\boldsymbol{b}\cdot\boldsymbol{n})\widehat{u}_h, v_h - \widehat{v}_h\rangle_{\partial\mathcal{T}_h^-},$$

which completes the proof. $\square$

3.2. **Error analysis.** We quote theoretical results proved in [42]; the coercivity, inf-sup condition of $\widetilde{a}_h^c(\cdot,\cdot)(= a_h^c(\cdot,\cdot)$ as proved in the previous section) and state the error estimates. The norm corresponding to the convective term is defined by

$$\|\!|(v_h,\widehat{v}_h)|\!\|_c^2 := c\|u_h\|_{L^2(\Omega)}^2 + \|h_K^{1/2}\boldsymbol{b}\cdot\nabla v_h\|_{\mathcal{T}_h}^2 + \||\boldsymbol{b}\cdot\boldsymbol{n}|^{1/2}(\widehat{v}_h - v_h)\|_{\partial\mathcal{T}_h}^2.$$

**Lemma 5.** The following hold.

(1) (Coercivity) [42, Lemma 4.2] For all $v_h \in W_h$ and $\widehat{v}_h \in M_h$, we have the equality

$$\widetilde{a}_h^c(v_h,\widehat{v}_h;v_h,\widehat{v}_h) = c\|v_h\|_{L^2(\Omega)}^2 + \frac{1}{2}\||\boldsymbol{b}\cdot\boldsymbol{n}|^{1/2}(\widehat{v}_h - v_h)\|_{\partial\mathcal{T}_h}^2.$$

(2) (Inf-sup stability) [42, Lemma 4.5] There exists a constant $C$ such that

$$C\,\|\!|(v_h,\widehat{v}_h)|\!\|_c \le \sup_{w_h\in W_h,\widehat{w}_h\in M_h} \frac{\widetilde{a}_h^c(v_h,\widehat{v}_h;w_h,\widehat{w}_h)}{\|\!|(w_h,\widehat{w}_h)|\!\|_c} \quad \forall v_h\in W_h, \widehat{v}_h \in M_h.$$

From this and Lemma 1, the following error estimate can be obtained, which is an improved result shown in [32, Theorem 3].

**Theorem 6.** If $u \in H^{k+1}(\Omega)$, then we have

$$\epsilon^{1/2}\,\|\!|(u-u_h,u-\widehat{u}_h)|\!\|_d + \|\!|(u-u_h,u-\widehat{u}_h)|\!\|_c \le C(\epsilon^{1/2}+h^{1/2})h^k\|u\|_{k+1}.$$

In particular, if $\epsilon$ is smaller than $h$ and $\{\mathcal{T}_h\}_h$ is quasi-uniform, then the errors in the streamline and $L^2$ norms are bounded as

$$\|\boldsymbol{b}\cdot\nabla(u-u_h)\|_{\mathcal{T}_h} \le Ch^k\|u\|_{k+1,\Omega},$$
$$\|u-u_h\|_{L^2(\Omega)} \le Ch^{k+1/2}\|u\|_{k+1,\Omega}.$$

*Remark.* The above error estimates are no longer useful when $\epsilon$ is very small since $\|u\|_{k+1,\Omega}$ may depend on negative powers of $\epsilon$. In [32], to show the robustness of the HDG method, it was shown that the HDG solution is close to the solution of the reduced problem.

3.3. **Numerical results.** To validate the stability of the HDG method in the convection-dominated case, we provide numerical results and compare the HDG method with the standard finite element method and the streamline upwind Petrov-Galerkin (SUPG) method. The test problem is as follows.

$$-\epsilon\Delta u + (1,0)^T\cdot\nabla u = 1 \quad \text{in } \Omega := (0,1)^2,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

When $\epsilon$ is very small, in this problem, boundary layers appear along the characteristic boundary (near $y=0$ and $y=1$) and the outflow boundary (near $x=1$), which causes the extreme numerical instability of the standard finite element method.

We fixed the mesh size $h \approx 1/10$ and the stabilization parameter $\tau_0 = 8$ and, an unstructured triangular mesh was used. All computations were carried out with `FreeFem++` [16]. In Figure 2, the numerical solutions are displayed for $\epsilon = 10^{-k}(k = 1, 3, 5, 7)$. The standard finite element solutions break down due to numerical instability when $\epsilon \leq 10^{-3}$. The SUPG method seems to be robust even for very small $\epsilon$, however, overshoot appears near the outflow boundary. We observe that the HDG method is robust and free from the overshoot phenomena unlike the SUPG method. We can also see that the HDG solutions get closer to the solution of the reduced problem, $u(x, y) = x$, as $\epsilon$ tends to zero.

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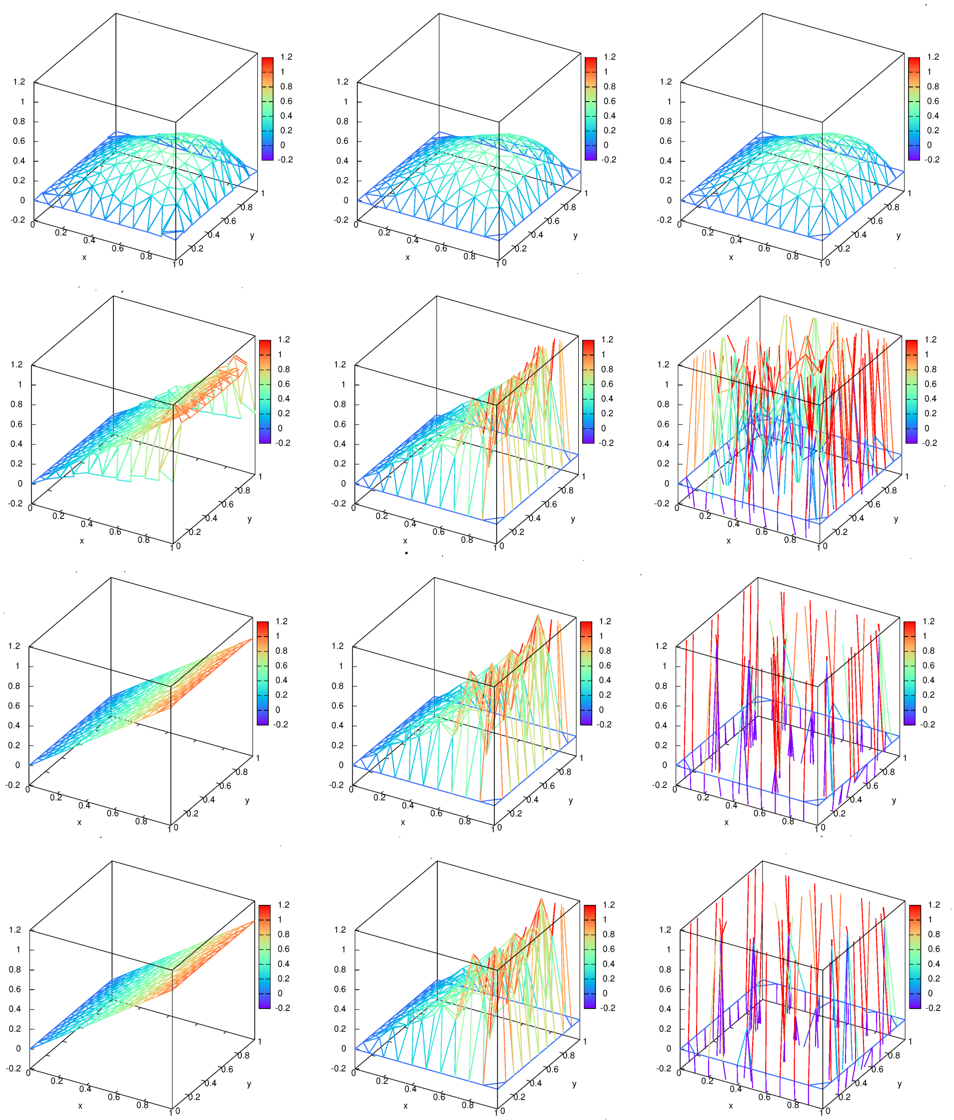

FIGURE 2. Solutions of the HDG (left), SUPG (center) and standard finite element (right) methods, where $\epsilon = 10^{-k}$ ($k = 1, 3, 5, 7$) from top to bottom.