数理解析研究所講究録
第2038巻 2017年 1-8

On Chorin's method for the Oberbeck-Boussinesq equations

Yoshiyuki Kagei

Faculty of Mathematics, Kyushu University, Japan

1 Introduction

This article gives a summary of [2]. We consider the artificial compressible system:

$$\varepsilon^2 \partial_t p + \operatorname{div} \boldsymbol{v} = 0, \tag{1.1}$$

$$\mathrm{Pr}^{-1}(\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}) - \Delta \boldsymbol{v} + \nabla p - \mathrm{Ra}\theta \boldsymbol{e}_3 = \boldsymbol{0}, \tag{1.2}$$

$$\partial_t \theta + \boldsymbol{v} \cdot \nabla \theta - \Delta \theta - \mathrm{Ra}\boldsymbol{v} \cdot \boldsymbol{e}_3 = 0. \tag{1.3}$$

Here $\boldsymbol{v} = {}^\top(v^1(x,t), v^2(x,t), v^3(x,t))$, $p = p(x,t)$ and $\theta = \theta(x,t)$ denote the unknown velocity field, pressure and temperature deviation from the heat conductive state, respectively, at time $t > 0$ and position $x \in \mathbb{R}^3$; $\boldsymbol{e}_3 = {}^\top(0,0,1) \in \mathbb{R}^3$; $\mathrm{Pr} > 0$ and $\mathrm{Ra} > 0$ are non-dimensional parameters, called Prandtl and Rayleigh numbers, respectively; and $\varepsilon > 0$ is a small parameter, called artificial Mach number. Here and in what follows, the superscript ${}^\top \cdot$ stands for the transposition. The system (1.1)–(1.3) is considered in the infinite layer $\Omega = \{x = (x', x_3); x' = (x_1, x_2) \in \mathbb{R}^2, 0 < x_3 < 1\}$.

By putting $\varepsilon = 0$ in (1.1) we obtain an incompressible system, called the Oberbeck-Boussinesq equation, which is a system of equations describing convection phenomena of viscous fluid in Ω heated from below (heated at $x_3 = 0$) under the gravitational force. As for the Oberbeck-Boussinesq equation (1.1)-(1.3)$|_{\varepsilon=0}$, it is well known that under the boundary condition: $\boldsymbol{v}|_{x_3=0,1} = \boldsymbol{0}$, $\theta|_{x_3=0,1} = 0$, there exists a critical number $\mathrm{Ra}_c > 0$ such that when $\mathrm{Ra} < \mathrm{Ra}_c$, the heat conductive state $\boldsymbol{v} = \boldsymbol{0}$, $\theta = 0$ is stable, while, when $\mathrm{Ra} > \mathrm{Ra}_c$, the heat conductive state is unstable and convective cellular stationary solutions bifurcate from the heat conductive state.

A. Chorin ([1]) proposed the artificial compressible system such as (1.1)–(1.3) with $\varepsilon > 0$ to find stationary solutions of equations for viscous incompressible fluid numerically. In the context of the Oberbeck-Boussinesq equation (1.1)–(1.3) with $\varepsilon = 0$, the idea is stated as follows. Obviously, the sets of stationary solutions for the systems with $\varepsilon = 0$ and $\varepsilon > 0$ are the same ones. If solutions of the artificial compressible system (1.1)–(1.3)$|_{\varepsilon>0}$ converge to a function $u_s = {}^\top(p_s, \boldsymbol{v}_s, \theta_s)$ as $t \to \infty$, then the limit u_s is a stationary solution of (1.1)–(1.3)$|_{\varepsilon>0}$ which is thus a stationary solution of (1.1)–(1.3)$|_{\varepsilon=0}$. By using this method, Chorin numerically obtained stationary cellular convection solutions of (1.1)–(1.3)$|_{\varepsilon=0}$.

Since the limit u_s in Chorin's method described above is a large time limit of solutions of (1.1)–(1.3)$|_{\varepsilon>0}$, u_s is stable as a solution of (1.1)–(1.3)$|_{\varepsilon>0}$. It is of interest to consider whether u_s is stable as a solution of (1.1)–(1.3)$|_{\varepsilon=0}$, in other words, whether u_s represents an observable stationary flow in the real world, and, conversely, what kind of stationary flows can be computed by Chorin's method. These questions are to be formulated as stability problem for stationary solutions of the systems (1.1)–(1.3)$|_{\varepsilon=0}$ and (1.1)–(1.3)$|_{\varepsilon>0}$. Since the system with $\varepsilon = 0$ is obtained from the one with $\varepsilon > 0$ as the limit $\varepsilon \to 0$, one could expect that solutions of (1.1)–(1.3)$|_{\varepsilon=0}$ would be approximated by solutions of (1.1)–(1.3)$|_{\varepsilon>0}$ when $\varepsilon \ll 1$. However, this limiting process is a singular limit, and hence, it is not straightforward to conclude that stability properties of u_s as a solution of (1.1)–(1.3)$|_{\varepsilon=0}$ are the same as those as a solution of (1.1)–(1.3)$|_{\varepsilon>0}$ even when $0 < \varepsilon \ll 1$.

The purpose of this article is to study the stability relations of stationary solutions between the systems with $\varepsilon = 0$ and $\varepsilon > 0$ when ε is sufficiently small. We thus consider the spectra of the linearized operators around a stationary solution of (1.1)–(1.3)$|_{\varepsilon=0}$ and (1.1)–(1.3)$|_{\varepsilon>0}$ for $\varepsilon \ll 1$.

2 Main Results

Let $u_s = {}^\top(p_s, \boldsymbol{v}_s, \theta_s)$ be a stationary solution of (1.1)–(1.3) satisfying

$$\int_{\Omega_{per}} p_s(x)\, dx = 0$$

under the boundary condition:

$$\boldsymbol{v}|_{x_3=0,1} = \boldsymbol{0}, \quad \theta|_{x_3=0,1} = 0,$$

and the periodicity condition:

$$p,\ \boldsymbol{v} \text{ and } \theta \text{ are } \mathcal{Q}\text{-periodic in } (x_1, x_2).$$

Here $\mathcal{Q} = [-\pi/\alpha_1, \pi/\alpha_1) \times [-\pi/\alpha_2, \pi/\alpha_2)$ with positive constants α_j, $j = 1, 2$; and $\Omega_{per} = \mathcal{Q} \times (0, 1)$ is the basic period domain.

We consider the linearized problem around $u_s = {}^\top(p_s, \boldsymbol{v}_s, \theta_s)$:

$$\varepsilon^2 \partial_t p + \operatorname{div} \boldsymbol{w} = 0, \tag{2.1}$$

$$\mathrm{Pr}^{-1} \partial_t \boldsymbol{w} - \Delta \boldsymbol{w} + \mathrm{Pr}^{-1}(\boldsymbol{v}_s \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{v}_s) + \nabla p - \mathrm{Ra}\theta \boldsymbol{e}_3 = \boldsymbol{0}, \tag{2.2}$$

$$\partial_t \theta - \Delta \theta + \boldsymbol{v}_s \cdot \nabla \theta + \boldsymbol{w} \cdot \nabla \theta_s - \mathrm{Ra}\boldsymbol{w} \cdot \boldsymbol{e}_3 = 0 \tag{2.3}$$

under the boundary condition

$$\boldsymbol{w}|_{x_3=0,1} = \boldsymbol{0}, \quad \theta|_{x_3=0,1} = 0, \tag{2.4}$$

and the periodicity condition

$$p,\ \boldsymbol{w} \text{ and } \theta \text{ are } \mathcal{Q}\text{-periodic in } (x_1, x_2). \tag{2.5}$$

By applying the Helmholtz projection $\mathbb{P}$ to the system (2.1)-(2.3)$|_{\varepsilon=0}$, we have the linearized operator around $\boldsymbol{U}_s = {}^\top(\boldsymbol{v}_s, \theta_s)$ associated with problem (2.1)-(2.3)$|_{\varepsilon=0}$ under (2.4) and (2.5). We define the operator $L : L^2_{per,\sigma} \times L^2_{per} \to L^2_{per,\sigma} \times L^2_{per}$ by

$$L = \begin{pmatrix} -\mathrm{Pr}\mathbb{P}\Delta + \mathbb{P}(\boldsymbol{v}_s \cdot \nabla + {}^\top(\nabla \boldsymbol{v}_s)) & -\mathrm{PrRa}\mathbb{P}\boldsymbol{e}_3 \\ {}^\top(\nabla\theta_s) - \mathrm{Ra}{}^\top\boldsymbol{e}_3 & -\Delta + \boldsymbol{v}_s \cdot \nabla \end{pmatrix}$$

with domain $D(L) = [(H^2_{per} \cap H^1_{0,per})^3 \cap L^2_{per,\sigma}] \times [H^2_{per} \cap H^1_{0,per}]$. Here L^2_{per}, H^k_{per}, $\cdots$, denote L^2, H^k, $\cdots$ spaces over Ω_{per} with periodicity condition in x'; $H^1_{0,per}$ denotes the set of all functions in H^1_{per} that vanish on $\{x_3 = 0, 1\}$; and $L^2_{per,\sigma}$ denotes the set of all vector fields $\boldsymbol{w}$ in $(L^2_{per})^3$ that satisfy $\operatorname{div} \boldsymbol{w} = 0$ in Ω_{per}, $w^3 = 0$ on $\{x_3 = 0, 1\}$ and $w^j|_{x_j = -\frac{\pi}{\alpha_j}} = w^j|_{x_j = \frac{\pi}{\alpha_j}}$, $j = 1, 2$.

We also introduce the linearized operator around $u_s = {}^\top(p_s, \boldsymbol{w}_s, \theta_s)$ associated with (2.1)–(2.3)$|_{\varepsilon>0}$ under (2.4) and (2.5). We define the operator $L_\varepsilon : H^1_{per,*} \times (L^2_{per})^3 \times L^2_{per} \to H^1_{per,*} \times (L^2_{per})^3 \times L^2_{per}$ by

$$L_\varepsilon = \begin{pmatrix} 0 & \frac{1}{\varepsilon^2}\mathrm{div} & 0 \\ \mathrm{Pr}\nabla & -\mathrm{Pr}\Delta + \boldsymbol{v}_s \cdot \nabla + {}^\top(\nabla \boldsymbol{v}_s) & -\mathrm{PrRa}\boldsymbol{e}_3 \\ 0 & {}^\top(\nabla\theta_s) - \mathrm{Ra}{}^\top\boldsymbol{e}_3 & -\Delta + \boldsymbol{v}_s \cdot \nabla \end{pmatrix}$$

with domain $D(L_\varepsilon) = H^1_{per,*} \times [H^2_{per} \cap H^1_{0,per}]^3 \times [H^2_{per} \cap H^1_{0,per}]$. Here $H^1_{per,*} = H^1_{per} \cap L^2_{per,*}$, $L^2_{per,*} = \{\phi \in L^2_{per}; \int_{\Omega_{per}} \phi \, dx = 0\}$.

We state our main results. See [2] for more general forms. We begin with

Theorem 2.1. ([2]) *If there exists a positive number b_0 such that $\rho(-L_{\varepsilon_n}) \supset \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \geq -b_0\}$ for some sequence $\varepsilon_n \to 0$ as $n \to \infty$, then there exists a constant $b_1 > 0$ such that $\rho(-L) \supset \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \geq -b_1\}$.*

Theorem 2.1 shows that if u_s is obtained by Chorin's method with $0 < \varepsilon \ll 1$, then it is stable as a solution of the Oberbeck-Boussinesq system. In particular, if u_s is unstable as a solution with $\varepsilon = 0$, then so is u_s as a solution with $0 < \varepsilon \ll 1$, and hence, unstable stationary solutions of the Oberbeck-Boussinesq system cannot be obtained by Chorin's method with $0 < \varepsilon \ll 1$. We next give a sufficient condition for u_s to be computed by Chorin's method with $0 < \varepsilon \ll 1$.

We denote by $\|\cdot\|_p$ the L^p norm over Ω_{per}. We also denote by (f, g) the L^2 inner product of f and g over Ω_{per}.

Theorem 2.2. ([2]) *Suppose that $\rho(-L) \supset \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \geq -b_0\}$ for some constant $b_0 > 0$. Then there exist constants $\varepsilon_0 > 0$, $\delta_0 > 0$ and $b_1 > 0$ such that if*

$$\inf_{\boldsymbol{w} \in (H^1_{per,0})^3, \boldsymbol{w} \neq \boldsymbol{0}} \frac{\operatorname{Re}(\boldsymbol{w} \cdot \nabla \boldsymbol{v}_s, \boldsymbol{w})}{\|\nabla \boldsymbol{w}\|_2^2} \geq -\delta_0, \tag{2.6}$$

then $\rho(-L_\varepsilon) \supset \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \geq -b_1\}$ for all $0 < \varepsilon \leq \varepsilon_0$.

In Theorem 2.2 we require smallness condition only for the velocity field $\boldsymbol{v}_s$ but not for the temperature θ_s.

Since the velocity fields of cellular stationary convective patterns bifurcating from the heat conductive state are small when $\mathrm{Ra} \sim \mathrm{Ra}_c$, Theorem 2.2 is applicable.

Remark 2.3. *Due to the translation invariance in x_1 and x_2 variables, 0 is an eigenvalue of $-L_\varepsilon$ whenever $\partial_{x_1} u_s \neq 0$ or $\partial_{x_2} u_s \neq 0$. In this case the theorems above also hold with reasonable modifications. See* [2].

3 Outline of proof of Theorem 2.2

In this section, following [2], we give an outline of the proof of Theorem 2.2. We assume that

$$\rho(-L) \supset \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \geq -b_0\}.$$

Since $-L$ is a sectorial operator with compact resolvent, we have the following resolvent estimate for $-L$ by the standard energy method.

Proposition 3.1. *There exist a constant $a_0 > 0$ such that*

$$\Sigma := \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \geq -a_0|\mathrm{Im}\,\lambda|^2 - b_0\} \subset \rho(-L)$$

and the estimates

$$\|(\lambda + L)^{-1}\boldsymbol{F}\|_2 \leq \frac{C}{|\lambda| + 1}\|\boldsymbol{F}\|_2,$$

$$\|\partial_x^2(\lambda + L)^{-1}\boldsymbol{F}\|_2 \leq C\|\boldsymbol{F}\|_2$$

hold uniformly for $\lambda \in \Sigma$.

We set $Y = H^1_{per,*} \times (L^2_{per})^3 \times L^2_{per}$. We consider the resolvent problem for $-L_\varepsilon$:

$$\lambda u + L_\varepsilon u = F, \tag{3.1}$$

where $u = {}^\top(p, \boldsymbol{w}, \theta) \in D(L_\varepsilon)$ and $F = {}^\top(f, \boldsymbol{g}, h) \in Y$. Problem (3.1) is written as

$$\varepsilon^2 \lambda p + \mathrm{div}\, \boldsymbol{w} = \varepsilon^2 f, \tag{3.2}$$

$$\mathrm{Pr}^{-1}\lambda \boldsymbol{w} - \Delta \boldsymbol{w} + \mathrm{Pr}^{-1}(\boldsymbol{v}_s \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{v}_s) + \nabla p - \mathrm{Ra}\theta \boldsymbol{e}_3 = \mathrm{Pr}^{-1}\boldsymbol{g}, \tag{3.3}$$

$$\lambda\theta - \Delta\theta + \boldsymbol{v}_s \cdot \nabla\theta + \boldsymbol{w} \cdot \nabla\theta_s - \mathrm{Ra}\boldsymbol{w} \cdot \boldsymbol{e}_3 = h, \tag{3.4}$$

and $u = {}^\top(p, \boldsymbol{w}, \theta)$ satisfies the boundary conditions (2.4) and (2.5).

Proposition 3.2. *There exist constants $a_1 > 0$ and $b_2 > 0$ such that $\{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \geq -a_1\varepsilon^2|\mathrm{Im}\,\lambda|^2 + b_2\} \subset \rho(-L_\varepsilon)$ for all $0 < \varepsilon \leq 1$.*

This proposition can be proved by the Matsumura-Nishida energy method ([3]). See [2] for the detail.

We next show that the spectrum of $-L_\varepsilon$ in a disc with radius $O(\varepsilon^{-1})$ can be viewed as a perturbation of the one of $-L$. We introduce the operator $\mathscr{L}_{\varepsilon,\lambda}$: $H^1_{per,*} \times (L^2_{per})^3 \times L^2_{per} \to H^1_{per,*} \times (L^2_{per})^3 \times L^2_{per}$ defined by

$$D(\mathscr{L}_{\varepsilon,\lambda}) = H^1_{per,*} \times [H^2_{per} \cap H^1_{0,per}]^3 \times [H^2_{per} \cap H^1_{0,per}],$$

$$\mathscr{L}_{\varepsilon,\lambda} = \begin{pmatrix} 0 & \frac{1}{\varepsilon^2}\mathrm{div} & 0 \\ \mathrm{Pr}\nabla & \lambda - \mathrm{Pr}\Delta + \boldsymbol{v}_s\cdot\nabla + {}^\top(\nabla \boldsymbol{v}_s) & -\mathrm{PrRa}\boldsymbol{e}_3 \\ 0 & {}^\top(\nabla\theta_s) - \mathrm{Ra}{}^\top\boldsymbol{e}_3 & \lambda - \Delta + \boldsymbol{v}_s\cdot\nabla \end{pmatrix}.$$

Note that

$$\mathscr{L}_{\varepsilon,0} = L_\varepsilon.$$

We prepare the following estimates for $\mathscr{L}_{\varepsilon,\lambda}^{-1}$.

Proposition 3.3. *Let $\varepsilon > 0$. If $\lambda \in \Sigma$, then $\mathscr{L}_{\varepsilon,\lambda}$ has a bounded inverse $\mathscr{L}_{\varepsilon,\lambda}^{-1}$ and ${}^\top(p, \boldsymbol{v}, \theta) = \mathscr{L}_{\varepsilon,\lambda}^{-1}F$ for $F = {}^\top(f, \boldsymbol{g}, h) \in Y$ satisfies*

$$\|\boldsymbol{U}\|_2 \le C\left\{\varepsilon^2\|f\|_{H^1} + \frac{1}{|\lambda|+1}\|\boldsymbol{F}\|_2\right\},$$

$$\|\partial_x^2\boldsymbol{U}\|_2 + \|\partial_x p\|_2 \le C\left\{\varepsilon^2(|\lambda|+1)\|f\|_{H^1} + \|\boldsymbol{F}\|_2\right\},$$

where $\boldsymbol{U} = {}^\top(\boldsymbol{w}, \theta)$ and $\boldsymbol{F} = {}^\top(\boldsymbol{g}, h)$.

See [2] for a proof of Proposition 3.3.

Proposition 3.4. *There exist positive numbers ε_1 and a_2 such that*

$$\Sigma \cap \{\lambda \in \mathbb{C}; |\lambda| \le a_2\varepsilon^{-1}\} \subset \rho(-L_\varepsilon)$$

for all $0 < \varepsilon \le \varepsilon_1$.

Proof. We follow the argument in [2]. We write the resolvent problem

$$(\lambda + L_\varepsilon)u = F$$

as

$$\mathscr{L}_{\varepsilon,\lambda}u + \lambda J u = F, \tag{3.5}$$

where $F = {}^\top(f, \boldsymbol{g}, h) \in Y$. If $\lambda \in \Sigma$, then it follows from Proposition 3.3 that (3.5) is written as

$$\mathscr{L}_{\varepsilon,\lambda}(I + \lambda\mathscr{L}_{\varepsilon,\lambda}^{-1}J)u = F,$$

and, furthermore, we have

$$\|\mathscr{L}_{\varepsilon,\lambda}^{-1}JF\|_{H^1\times H^2\times H^2} \le \varepsilon^2 C_1(|\lambda|+1)\|f\|_{H^1}$$

for all $F = {}^\top(f, \boldsymbol{g}, h) \in Y$. It then follows that there exists $\varepsilon_1 > 0$ such that if $\lambda \in \Sigma$ and $|\lambda| \le 1/(4\sqrt{C_1}\varepsilon)$, then $\mathscr{L}_{\varepsilon,\lambda}^{-1}JF \in D(\mathscr{L}_{\varepsilon,\lambda}) = D(L_\varepsilon)$ and $\|\lambda\mathscr{L}_{\varepsilon,\lambda}^{-1}JF\|_{H^1\times H^2\times H^2} \le \frac{1}{2}\|F\|_{H^1\times L^2\times L^2}$ for $0 < \varepsilon \le \varepsilon_1$. Therefore, $(I + \lambda\mathscr{L}_{\varepsilon,\lambda}^{-1}J)$ is boundedly invertible both on Y and $D(L_\varepsilon)$ with estimates

$$\|(I + \lambda\mathscr{L}_{\varepsilon,\lambda}^{-1}J)^{-1}F\|_{H^1\times L^2\times L^2} \le 2\|F\|_{H^1\times L^2\times L^2}$$

for $F \in Y$ and

$$\|(I + \lambda\mathscr{L}_{\varepsilon,\lambda}^{-1}J)^{-1}F\|_{H^1\times H^2\times H^2} \le 2\|F\|_{H^1\times H^2\times H^2}$$

for $F \in D(L_\varepsilon)$. We thus find that $\lambda + L_\varepsilon = \mathscr{L}_{\varepsilon,\lambda} + \lambda J$ has a bounded inverse $(\lambda + L_\varepsilon)^{-1} = (\mathscr{L}_{\varepsilon,\lambda} + \varepsilon^2 \lambda J)^{-1}$ on Y which satisfies

$$(\lambda + L_\varepsilon)^{-1} = \mathscr{L}_{\varepsilon,\lambda}^{-1} - \lambda \mathscr{L}_{\varepsilon,\lambda}^{-1} J \sum_{N=0}^{\infty} (-\lambda)^N (\mathscr{L}_{\varepsilon,\lambda}^{-1} J)^N \mathscr{L}_{\varepsilon,\lambda}^{-1}$$

and

$$\begin{aligned} \|(\lambda + L_\varepsilon)^{-1} F\|_{H^1 \times L^2 \times L^2} &\leq 2C_1 \left\{ \varepsilon^2 (|\lambda| + 1) \|f\|_{H^1} + \|\boldsymbol{F}\|_2 \right\} \\ &\leq 2C_1 \left\{ \varepsilon \|f\|_{H^1} + \|\boldsymbol{F}\|_2 \right\} \end{aligned}$$

with $\boldsymbol{F} = {}^\top(\boldsymbol{g}, \boldsymbol{h})$. This completes the proof. □

Theorem 2.2 follows from Propositions 3.2 and 3.4 if $\sqrt{b_2/a_1} < a_2$ for $0 < \varepsilon \ll 1$. In the case $\sqrt{b_2/a_1} \geq a_2$, there is some range of λ near the imaginary axis with $|\mathrm{Im}\,\lambda| = O(\varepsilon^{-1})$ to be proved that it belongs to $\rho(-L_\varepsilon)$.

To prove Theorem 2.2 when $\sqrt{b_2/a_1} \geq a_2$, we prepare estimates for the θ-component. We recall that the Poincaré inequality

$$\|\nabla \theta\|_2 \geq \beta \|\theta\|_2$$

holds for $\theta \in H^1_{0,per}$ with some positive constant β.

Proposition 3.5. *Let* ${}^\top(p, \boldsymbol{w}, \theta)$ *be a solution of* (3.2)–(3.4) *under boundary conditions* (2.4) *and* (2.5). *Then if* $\mathrm{Re}\,\lambda \geq -\frac{\beta^2}{2}$, *the following estimates hold:*

$$\|\theta\|_2 \leq \frac{1}{|\mathrm{Im}\,\lambda|} \left(1 + \frac{2\|\boldsymbol{v}_s\|_\infty}{\beta}\right) \{(\|\nabla \theta_s\|_\infty + \mathrm{Ra})\|\boldsymbol{w}\|_2 + \|h\|_2\},$$

$$\|\nabla \theta\|_2 \leq \frac{2}{\beta} \{(\|\nabla \theta_s\|_\infty + \mathrm{Ra})\|\boldsymbol{w}\|_2 + \|h\|_2\}.$$

This proposition can be proved by the standard energy method. The idea is that $-\Delta$ with zero-Dirichlet bondary condition is sectorial (self-adjoint) and so $(\lambda - \Delta)^{-1} \to 0$ as $|\mathrm{Im}\,\lambda| \to 0$.

We are now ready to complete the proof of Theorem 2.2.

Proposition 3.6. *For given* $\mu_* > 0$ *and* $\eta_* > 0$ *there exist constants* $\varepsilon_1 > 0$ *and* $c_2 > 0$ *such that if*

$$\inf_{\boldsymbol{w} \in (H^1_{0,per})^3, \boldsymbol{w} \neq 0} \frac{\mathrm{Re}\,(\boldsymbol{w} \cdot \nabla \boldsymbol{v}_s, \boldsymbol{w})}{\|\nabla \boldsymbol{w}\|_2^2} \geq -\frac{\mathrm{Pr}}{32},$$

then

$$\left\{\lambda = \mu + i\frac{\eta}{\varepsilon};\ -c_2 \leq \mu \leq \mu_*,\ |\eta| \geq \eta_*\right\} \subset \rho(-L_\varepsilon)$$

for all $0 < \varepsilon \leq \varepsilon_1$. *Here* ε_1 *and* c_2 *are positive constants depending only on* Pr, Ra, $\|\boldsymbol{v}_s\|_{C^1}$, $\|\nabla \theta_s\|_\infty$, β, μ_* *and* η_*

Proof. We give an outline. The details can be found in [2]. We see from (3.2) that

$$p = -\frac{1}{\varepsilon^2\lambda}\operatorname{div}\boldsymbol{w} + \frac{1}{\lambda}f. \tag{3.6}$$

Substituting (3.6) into (3.3), we have

$$\frac{\varepsilon^2\lambda^2}{\mathrm{Pr}}\boldsymbol{w} - \varepsilon^2\lambda\Delta\boldsymbol{w} - \nabla\operatorname{div}\boldsymbol{w} + \frac{\varepsilon^2\lambda}{\mathrm{Pr}}(\boldsymbol{v}_s\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{v}_s) - \varepsilon^2\lambda\mathrm{Ra}\theta\boldsymbol{e}_3 = \varepsilon^2\boldsymbol{G}_\lambda, \tag{3.7}$$

where $\boldsymbol{G}_\lambda = \frac{\lambda}{\mathrm{Pr}}\boldsymbol{g} - \nabla f$.

Let $\lambda = \mu + i\frac{\eta}{\varepsilon}$ with $|\eta| \geq \eta_*(>0)$. Without loss of generality we may assume $\eta \geq \eta_*$. Taking the inner product of (3.7) with $\boldsymbol{w}$, we have

$$\begin{aligned}
&\frac{\varepsilon^2\lambda^2}{\mathrm{Pr}}\|\boldsymbol{w}\|_2^2 + \varepsilon^2\lambda\|\nabla\boldsymbol{w}\|_2^2 + \|\operatorname{div}\boldsymbol{w}\|_2^2 \\
&= -\varepsilon^2\lambda\left(\mathrm{Pr}^{-1}(\boldsymbol{v}_s\cdot\nabla\boldsymbol{w},\boldsymbol{w}) + \mathrm{Pr}^{-1}(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s,\boldsymbol{w}) - \mathrm{Ra}(\theta, w^3)\right) + \varepsilon^2(\boldsymbol{G}_\lambda,\boldsymbol{w}).
\end{aligned} \tag{3.8}$$

The real and imaginary parts of (3.8) yield

$$\begin{aligned}
&\frac{1}{\mathrm{Pr}}(\varepsilon^2\mu^2 - \eta^2)\|\boldsymbol{w}\|_2^2 + \varepsilon^2\mu\|\nabla\boldsymbol{w}\|_2^2 + \|\operatorname{div}\boldsymbol{w}\|_2^2 \\
&= -\varepsilon^2\mu\mathrm{Re}\left(\mathrm{Pr}^{-1}(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s,\boldsymbol{w}) - \mathrm{Ra}(\theta, w^3)\right) \\
&\quad +\varepsilon\eta\mathrm{Im}\left(\mathrm{Pr}^{-1}(\boldsymbol{v}_s\cdot\nabla\boldsymbol{w},\boldsymbol{w}) + \mathrm{Pr}^{-1}(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s,\boldsymbol{w}) - \mathrm{Ra}(\theta, w^3)\right) \\
&\quad +\varepsilon^2\mathrm{Re}\,(\boldsymbol{G}_\lambda,\boldsymbol{w})
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
&\frac{2\varepsilon\mu\eta}{\mathrm{Pr}}\|\boldsymbol{w}\|_2^2 + \varepsilon\eta\|\nabla\boldsymbol{w}\|_2^2 \\
&= -\varepsilon^2\mu\mathrm{Im}\left(\mathrm{Pr}^{-1}(\boldsymbol{v}_s\cdot\nabla\boldsymbol{w},\boldsymbol{w}) + \mathrm{Pr}^{-1}(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s,\boldsymbol{w}) - \mathrm{Ra}(\theta, w^3)\right) \\
&\quad -\varepsilon\eta\mathrm{Re}\left(\mathrm{Pr}^{-1}(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s,\boldsymbol{w}) - \mathrm{Ra}(\theta, w^3)\right) \\
&\quad +\varepsilon^2\mathrm{Im}\,(\boldsymbol{G}_\lambda,\boldsymbol{w}).
\end{aligned} \tag{3.10}$$

By Proposition 3.5, we see from (3.9) and (3.10) that

$$\begin{aligned}
&\frac{1}{\mathrm{Pr}}(\eta^2 - \varepsilon^2\mu^2)\|\boldsymbol{w}\|_2^2 \\
&\leq \left(\varepsilon^2\mu + \frac{\eta}{\eta_*} + \varepsilon\eta\right)\|\nabla\boldsymbol{w}\|_2^2 + \left((\varepsilon^2|\mu| + \varepsilon\eta)\mathrm{Pr}^{-1}\|\nabla\boldsymbol{v}_s\|_\infty + \varepsilon\eta\mathrm{Pr}^{-2}\|\boldsymbol{v}_s\|_\infty^2\right)\|\boldsymbol{w}\|_2^2 \\
&\quad +(\varepsilon^2|\mu| + \varepsilon\eta)\frac{\mathrm{Ra}\varepsilon}{\eta}\left(1 + \frac{2\|\boldsymbol{v}_s\|_\infty}{\beta}\right)\left\{(\|\nabla\theta_s\|_\infty + \mathrm{Ra})\|\boldsymbol{w}\|_2^2 + \|h\|_2\|\boldsymbol{w}\|_2\right\} \\
&\quad +\varepsilon^2\|\boldsymbol{G}_\lambda\|_2\|\boldsymbol{w}\|_2
\end{aligned} \tag{3.11}$$

and

$$
\begin{aligned}
&\frac{2\mu\eta}{\Pr}\|\boldsymbol{w}\|_2^2 + \frac{3\eta}{4}\|\nabla \boldsymbol{w}\|_2^2 \\
&\le\ -\eta \Pr^{-1} \mathrm{Re}\,(\boldsymbol{w}\cdot\nabla \boldsymbol{v}_s, \boldsymbol{w}) + \left(\frac{\varepsilon^2|\mu|^2\Pr^{-2}\|\boldsymbol{v}_s\|_\infty^2}{\eta} + \varepsilon|\mu|\Pr^{-1}\|\nabla \boldsymbol{v}_s\|_\infty\right)\|\boldsymbol{w}\|_2^2 \\
&\quad +(\varepsilon|\mu|+\eta)\frac{\mathrm{Ra}\varepsilon}{\eta}\left(1+\frac{2\|\boldsymbol{v}_s\|_\infty}{\beta}\right)\{(\|\nabla\theta_s\|_\infty + \mathrm{Ra})\|\boldsymbol{w}\|_2^2 + \|h\|_2\|\boldsymbol{w}\|_2\} \\
&\quad +\varepsilon\|\boldsymbol{G}_\lambda\|_2\|\boldsymbol{w}\|_2.
\end{aligned}
\tag{3.12}
$$

It then follows from (3.11) and (3.12) that there exists a positive constant c_2 such that if $0 < \varepsilon \ll 1$, then

$$
\frac{\beta^2}{16}\eta\|\boldsymbol{w}\|_2^2 + \frac{\eta}{32}\|\nabla \boldsymbol{w}\|_2^2 \le C_{\varepsilon,\lambda}(\|F\|_{H^1\times(L^2)^3\times L^2})\|\boldsymbol{w}\|_2 \tag{3.13}
$$

for $-c_2 \le \mu \le \mu_*$ and $\eta \ge \eta_*$, provided that $\inf_{\boldsymbol{w}\in(H^1_{0,per})^3, \boldsymbol{w}\neq \mathbf{0}} \frac{\mathrm{Re}\,(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s, \boldsymbol{w})}{\|\nabla \boldsymbol{w}\|_2^2} \ge -\frac{\Pr}{32}$. This completes the proof. □

Theorem 2.2 now follows by taking $\eta_* = \frac{a_2}{2}$, $\mu_* = 2b_2$ and $\varepsilon > 0$ sufficiently small.

References

[1] A. Chorin, A numerical method for solving incompressible viscous flow problems, J. Comput. Phys., **2** (1967), pp. 12–26.

[2] Y. Kagei and T. Nishida, On Chorin's method for stationary solutions of the Oberbeck-Boussinesq equation, to appear in J. Math. Fluid Mech.

[3] A. Matsumura and T. Nishida, Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids. Comm. Math. Phys., **89** (1983) , pp. 445–464.