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Mathematical analysis of Kuramoto-Sakaguchi equation

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1. Introduction

It is remarkable that theoretical investigations of weakly coupled limit cycle oscillators [10] are conducted over several research fields these days. For example, in statistical physics, various network models are being developed, whereas in network science, synchronization on random and complex networks [9] is currently attracting researchers' attention.

As for mathematical arguments, there is study on a partial integro-differential equation called the *Kuramoto-Sakaguchi equation*, which describes the behavior of the probability density of the phase of oscillators as an infinite limit of population [2][3][5][6][8][12].

In this paper, we introduce some of our results concerning the solvability and the existence of the maximal attractor and inertial set concerning the Kuramoto-Sakaguchi equation, which describes the temporal behavior of the phase distribution of weakly coupled oscillators. We also add some detail to the proof of the statements presented in the previous article [8].

This paper is organized as follows. In the next section, we formulate the problem. In Section 3, we overview the existing related results. In Section 4, we introduce function spaces and notations used in the following discussion. In Section 5, the results concerning the existence of the solution are stated. Then, in Section 6, we discuss the vanishing diffusion limit. The existence of the maximal attractor and inertial set are provided in the final section.

2. Formulation

The Kuramoto-Sakaguchi equation is a model equation of the physical theory of coupled oscillators, and describes the temporal evolution of the probability distribution of each oscillator's phase.

By applying the mean field approximation, the temporal evolution of the order parameter $r(t)$ and the phase of the mean field $\psi(t)$ at time t is described as:

$$r(t)\exp(\mathrm{i}\psi(t))=\int_0^{2\pi}\int_{\mathbf{R}}\exp(\mathrm{i}\theta)\varrho(\theta,\omega,t)g(\omega)\,\mathrm{d}\theta\mathrm{d}\omega\quad \mathrm{i}=\sqrt{-1},\ t>0,$$

where $\varrho(\theta,\omega,t)$ is the probability density function of phase θ and natural frequency ω at t, and $g(\omega)$ is the probability distribution function of ω. In addition, it is well known that the time evolution of ϱ is subject to the following evolution equation when the population of oscillators tends to infinity:

$$\frac{\partial\varrho}{\partial t}+\frac{\partial}{\partial\theta}\Big\{\big[\omega+Kr(t)\sin(\psi(t)-\theta)\big]\varrho\Big\}=0\quad \theta\in(0,2\pi),\ t>0.$$

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Combining these yields the following nonlinear partial integro-differential equation:

$$\begin{cases} \dfrac{\partial \varrho}{\partial t} + \omega \dfrac{\partial \varrho}{\partial \theta} + K \dfrac{\partial}{\partial \theta} \left[\varrho(\theta, \omega, t) \displaystyle\int_{\mathbf{R}} g(\omega')\mathrm{d}\omega' \int_0^{2\pi} \sin(\phi - \theta)\varrho(\phi, \omega', t)\mathrm{d}\phi \right] = 0, \\ \qquad\qquad\qquad (\theta, \omega, t) \in (0, 2\pi) \times \mathbf{R} \times (0, \infty), \\ \dfrac{\partial^j \varrho}{\partial \theta^j}\Big|_{\theta=0} = \dfrac{\partial^j \varrho}{\partial \theta^j}\Big|_{\theta=2\pi} \quad (j = 0, 1), \quad (\omega, t) \in \mathbf{R} \times (0, \infty), \\ \varrho\big|_{t=0} = \varrho_0(\theta, \omega), \quad (\theta, \omega) \in (0, 2\pi) \times \mathbf{R}. \end{cases} \tag{2.1}$$

The parabolic regularization of (2.1), which is called the *Kuramoto-Sakaguchi equation* reads:

$$\begin{cases} \dfrac{\partial \varrho}{\partial t} - D\dfrac{\partial^2 \varrho}{\partial \theta^2} + \omega \dfrac{\partial \varrho}{\partial \theta} \\ \qquad + K \dfrac{\partial}{\partial \theta} \left[\varrho(\theta, \omega, t) \displaystyle\int_{\mathbf{R}} g(\omega')\mathrm{d}\omega' \int_0^{2\pi} \sin(\phi - \theta)\varrho(\phi, \omega', t)\mathrm{d}\phi \right] = 0, \\ \qquad\qquad\qquad (\theta, \omega, t) \in (0, 2\pi) \times \mathbf{R} \times (0, \infty), \\ \dfrac{\partial^j \varrho}{\partial \theta^j}\Big|_{\theta=0} = \dfrac{\partial^j \varrho}{\partial \theta^j}\Big|_{\theta=2\pi} \quad (j = 0, 1), \quad (\omega, t) \in \mathbf{R} \times (0, \infty), \\ \varrho\big|_{t=0} = \varrho_0(\theta, \omega), \quad (\theta, \omega) \in (0, 2\pi) \times \mathbf{R}. \end{cases} \tag{2.2}$$

Here, D corresponds to the diffusion coefficient of additive white noise. Hereafter, we mainly deal with (2.2), except for the discussion on the vanishing diffusion limit presented in Section 6.

3. Related works

In this section, we overview the past mathematical arguments concerning (2.1) and (2.2). The classical solvability of (2.2) was first shown by Lavrentiev [11]. Although they assumed that the support of $g(\omega)$ is compact, they later removed the assumption [12]. In it, they also discuss the regularity of the solution with respect to ω. Ha et al. [6] discussed the nonlinear stability of the incoherent state. They showed that the trivial stationary solution $\bar{\varrho} = 1/2\pi$ of (2.2) is stable when the diffusion coefficient D is sufficiently large. Later, they also discussed the nonlinear instability of $\bar{\varrho}$ when D is small [7]. They also discussed the existence of the solution to (2.1) as a vanishing diffusion limit of (2.2).

Concerning the nonlinear stability of *coherence*, Bertini et al. [2] first held the mathematical argument by using the Gelfant's triplet. Later, Giacomin et al. [5] argued the existence of the maximal attractor and inertial manifold. However, their arguments are limited to the case $g = \delta(\omega)$ in (2.2). Chiba [3] discussed the stability of incoherence in (2.1) by generalizing the definition of the spectrum.

4. Function spaces

We introduce the functions spaces and some related notations used throughout this paper. Let $\Omega = (0, 2\pi)$, $\Omega_T = \Omega \times (0, T)$ and $\hat{f}(\theta, \omega, t) \equiv f(\theta - \omega t, \omega, t)$.

By $C^{r+\alpha}(\Omega)$ with a non-negative integer r and $\alpha \in (0, 1)$, we mean the Banach space of functions from $C^r(\overline{\Omega})$, whose rth derivatives satisfy the Hölder condition with exponent α, i.e., the space of functions with the finite norm

$$|u|_{\Omega}^{(r+\alpha)} = \sum_{k=0}^{r} |D^k u|_{\Omega} + [D^r u]_{\Omega}^{(\alpha)},$$

where $D = \partial/\partial x$, and

$$|u|_{\Omega} = \sup_{x\in\Omega} |u(x)|, \quad [u]_{\Omega}^{(\alpha)} = \sup_{x,y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}}.$$

By $C^{r+\alpha,\frac{r+\alpha}{2}}(\Omega_T)$ with $r = 0, 1, 2$, we mean the spaces of functions defined in Ω_T and having the finite norms

$$|u|_{\Omega_T}^{(\alpha,\frac{\alpha}{2})} = |u|_{\Omega_T} + [u]_{\Omega_T}^{(\alpha,\frac{\alpha}{2})} \quad (r = 0),$$

where

$$|u|_{\Omega_T} = \sup_{(x,t)\in\Omega_T} |u(x,t)|, \quad [u]_{\Omega_T}^{(\alpha,\frac{\alpha}{2})} = [u]_{x,\Omega_T}^{(\alpha)} + [u]_{t,\Omega_T}^{(\frac{\alpha}{2})},$$

$$[u]_{x,\Omega_T}^{(\alpha)} = \sup_{x,y,t} \frac{|u(x,t)-u(y,t)|}{|x-y|^{\alpha}}, \quad [u]_{t,\Omega_T}^{(\frac{\alpha}{2})} = \sup_{x,y,t} \frac{|u(x,t)-u(x,\tau)|}{|t-\tau|^{\frac{\alpha}{2}}},$$

and for $r = 1,\ 2$,

$$|u|_{\Omega_T}^{(1+\alpha,\frac{1+\alpha}{2})} = |u|_{\Omega_T} + \left|\frac{\partial u}{\partial x}\right|_{\Omega_T}^{(\alpha,\frac{\alpha}{2})} + [u]_{t,\Omega_T}^{\left(\frac{1+\alpha}{2}\right)},$$

$$|u|_{\Omega_T}^{(2+\alpha,\frac{2+\alpha}{2})} = |u|_{\Omega_T} + \left|\frac{\partial u}{\partial x}\right|_{\Omega_T} + \left|\frac{\partial^2 u}{\partial x^2}\right|_{\Omega_T}^{(\alpha,\frac{\alpha}{2})} + \left|\frac{\partial u}{\partial t}\right|_{\Omega_T}^{(\alpha,\frac{\alpha}{2})} + \left|\frac{\partial u}{\partial x}\right|_{t,\Omega_T}^{(\frac{1+\alpha}{2})},$$

respectively.

Let ϵ be a fixed number satisfying $0 < \epsilon < 1/2$, and define the smooth monotone function $h(\omega)$ as:

$$h(\omega) = \begin{cases} 2 & (|\omega| < 1); \\ 1 + |\omega|^{2+\epsilon} & (|\omega| \geq 1). \end{cases}$$

Then, for $\alpha \in (1/2, 1)$, we define

$$\mathcal{V}_T^{2+\alpha} \equiv \left\{ f(\theta,\omega,t) \middle| h(\omega)\hat{f}(\theta,\omega,t) \in C^{2+\alpha,\frac{2+\alpha}{2}}(\Omega_T),\ \sup_{\omega} \left| h(\omega)\hat{f}(\omega) \right|_{\Omega_T}^{(2+\alpha,\frac{2+\alpha}{2})} < \infty \right\}.$$

For functions independent on t, we define

$$\mathcal{V}^{2+\alpha} \equiv \left\{ f(\theta,\omega) \middle| h(\omega)f(\theta,\omega) \in C^{2+\alpha}(\Omega),\ \sup_{\omega} |h(\omega)f(\omega)|_{\Omega}^{(2+\alpha)} < \infty \right\}.$$

The L_2-norm is denoted by $\|f\| \equiv \|f\|_{L_2(\Omega)}$, and for those depending also on ω, we introduce

$$|||f||| \equiv \sup_{\omega} \|f(\omega)\|.$$

Following the definition by Temam [14], we say that a 2π-periodic function $u(\theta)$ on Ω belongs to the Sobolev space $\mathcal{H}^m$ ($m \in \mathbf{R}$) if the Fourier coefficients $\{a_n\}_{n=-\infty}^{\infty}$ of

$$u(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta},$$

satisfy

$$|\!|\!| u |\!|\!|_m \equiv \sup_{\omega} \|u(\omega)\|_m^2 \equiv \sum_{n=-\infty}^{\infty} (1+|n|^2)^m |a_n(\omega)|^2 < \infty.$$

For periodic functions depending also on ω, we define

$$\overline{\mathcal{H}}^m \equiv \left\{ u(\theta,\omega) = \sum_{n=-\infty}^{\infty} a_n(\omega) e^{in\theta} \,\middle|\, \sup_{\omega\in\mathbf{R}} \|u(\cdot,\omega)\|_m^2 < \infty \right\}.$$

In addition, we define

$$L_1^{(1)} \equiv \left\{ u(\cdot,\omega) \in L_1(\Omega) \,\middle|\, u \geq 0,\ \int_\Omega u(\theta,\omega)\,\mathrm{d}\theta = 1,\ \omega \in \mathbf{R} \right\},$$
$$L_1^{(1)}(T) \equiv \left\{ u(\cdot,\omega,t) \in L_1(\Omega) \,\middle|\, u \geq 0,\ \int_\Omega u(\theta,\omega,t)\,\mathrm{d}\theta = 1,\ t \in (0,T),\ \omega \in \mathbf{R} \right\}.$$

We also use a notation:

$$F[\varrho_1,\varrho_2] \equiv \varrho_1(\theta,t;x,\omega) \int_{\mathbf{R}} G(x-y)\mathrm{d}y \int_{\mathbf{R}} g(\omega')\mathrm{d}\omega' \int_0^{2\pi} \Gamma(\theta-\phi)\varrho_2(\phi,t;y,\omega')\,\mathrm{d}\phi,$$
$$F^{(k)}[\varrho_1,\varrho_2] \equiv$$
$$\varrho_1(\theta,t;x,\omega) \int_{\mathbf{R}} G(x-y)\mathrm{d}y \int_{\mathbf{R}} g(\omega')\mathrm{d}\omega' \int_0^{2\pi} \Gamma^{(k)}(\theta-\phi)\varrho_2(\phi,t;y,\omega')\,\mathrm{d}\phi$$
$$(k=1,2,\ldots).$$

Hereafter, c's represent constants in the estimate of some quantities. When we denote $c(t)$ with suffixes, it depends on t. For simplicity, we hereafter use notations $f^{(j,k)} \equiv \left(\frac{\partial}{\partial\theta}\right)^j \left(\frac{\partial}{\partial t}\right)^k f$ ($j,k = 0,1,2,\ldots$) for a function $f = f(\theta,t)$ in general.

5. Existence of solution to (2.2)

The following theorem is one of our main results.

Theorem 5.1. *Let ϵ, α and $h(\omega)$ be those defined in the previous section, and assume: Let us assume $0 < \varepsilon < 1/2$, $1/2 < \alpha < 1$ and the following issues:*

(i) $g(\omega) \in C^\infty(\mathbf{R})$, $g(\omega) \geq 0\ \forall\omega \in \mathbf{R}$ *and* $\int_{\mathbf{R}} g(\omega)\,\mathrm{d}\omega = 1$;

(ii) $\varrho_0 \in \mathcal{V}^{2+\alpha} \bigcap L_1^{(1)}$.

Then, there exists a certain $T_* > 0$ and a unique solution $\varrho \in \mathcal{V}_{T_*}^{2+\alpha}$ to (2.2).

Next, we state the global-in-time existence of the solution to (2.2).

Theorem 5.2. *In addition to the assumptions in* Theorem 5.1, *if we assume* $\varrho_0 \in \overline{\mathcal{H}}^4$, *there exists a unique solution* $\varrho \in \mathcal{V}_T^{2+\alpha}$ *to* (2.2) *for arbitrary* $T > 0$. *In addition, it satisfies*

$$\varrho(\theta,\omega,t) \in \mathcal{K}^4(T) \equiv L_\infty(0,T;\overline{\mathcal{H}}^4) \bigcap C(0,T;\overline{\mathcal{H}}^2) \bigcap C^1(0,T;\overline{\mathcal{H}}^0).$$

Actually, ϱ stated above has additional regularity with respect to t (see, for instance, Lemma II.3.2 in [14]).

Corollary 5.1. *Under the assumptions in* Theorem 5.2, *the solution* $\varrho(\theta,\omega,t)$ *to* (2.2) *stated in* Theorem 5.2 *satisfies*

$$\varrho(\theta,\omega,t) \in \widetilde{\mathcal{K}}^4(T) \equiv C(0,T;\overline{\mathcal{H}}^4) \bigcap L_1^{(1)}(T).$$

Proof. The statement follows from the fact

$$\frac{\partial \varrho}{\partial t} \in L_\infty(0,T;(\overline{\mathcal{H}}^4)'),$$

and Lemma II.3.2 in [14], where $(\overline{\mathcal{H}}^4)'$ is the dual of $\overline{\mathcal{H}}^2$. □

For the proof of Theorem 5.2, we first show the following lemma.

Lemma 5.1. *Let* $T > 0$ *be an arbitrary number. If there exists a solution to* (2.2) *on* $(0,T)$, *estimates of the form*

$$|\!|\!| \varrho^{(k,0)}(t) |\!|\!| \le c_{5(k)} \quad (k = 1,2,\ldots,4) \tag{5.1}$$

hold with certain constants $c_{5(k)}$ *independent of* t.

Proof. For the sake of simplicity, we introduce the notation $\tilde{\varrho} \equiv \varrho - \bar{\varrho}$ and derive the estimate of its norm which leads to the desired estimates. From (2.2), it is obvious that $\tilde{\varrho}$ satisfies

$$\begin{cases} \dfrac{\partial \tilde{\varrho}}{\partial t} + \omega \dfrac{\partial \tilde{\varrho}}{\partial \theta} + \dfrac{\partial}{\partial \theta}\left(F[\tilde{\varrho}+\bar{\varrho},\tilde{\varrho}+\bar{\varrho}]\right) - D\dfrac{\partial^2 \tilde{\varrho}}{\partial \theta^2} = 0 \\ \qquad\qquad \theta \in \Omega,\ t \in (0,T),\ \omega \in \mathbf{R}, \\ \dfrac{\partial^i \tilde{\varrho}}{\partial \theta^i}\bigg|_{\theta=0} = \dfrac{\partial^i \tilde{\varrho}}{\partial \theta^i}\bigg|_{\theta=2\pi} \quad (i=0,1),\ t \in (0,T),\ \omega \in \mathbf{R}, \\ \tilde{\varrho}\big|_{t=0} = \tilde{\varrho}_0 \equiv \varrho_0 - \bar{\varrho} \quad \theta \in \Omega,\ \omega \in \mathbf{R}. \end{cases} \tag{5.2}$$

Multiply $(5.2)_1$ by $\tilde{\varrho}$. Then, making use of Lemma 5.1 and the periodicity of $F[\tilde{\varrho} + \bar{\varrho}, \tilde{\varrho} + \bar{\varrho}]$ with respect to θ yield

$$\begin{aligned} \int_\Omega \tilde{\varrho}(\theta,\omega,t)\frac{\partial}{\partial \theta}\Big(F[\tilde{\varrho}+\bar{\varrho},\tilde{\varrho}+\bar{\varrho}]\Big)\,\mathrm{d}\theta &= \int_\Omega \varrho(\theta,t,\omega)\frac{\partial}{\partial \theta}\Big(F[\tilde{\varrho}+\bar{\varrho},\tilde{\varrho}+\bar{\varrho}]\Big)\,\mathrm{d}\theta \\ &= -\frac{1}{2}\int_\Omega F^{(1)}[\varrho^2,\varrho]\,\mathrm{d}\theta \\ &\le \frac{c_{55}}{2}\|\varrho(\cdot,t,\omega)\|^2. \end{aligned}$$

On the other hand, in the same line with the arguments by Lavrentiev [11], we have

$$
\begin{aligned}
\|\varrho(\cdot,\omega,t)\|^2 &\le \int_\Omega \varrho(\theta,\omega,t)\left(\frac{1}{2\pi}+\sqrt{2\pi}\|\varrho^{(1,0)}(\cdot,\omega,t)\|\right)\,\mathrm{d}\theta\\
&=\frac{1}{2\pi}+\sqrt{2\pi}\|\varrho^{(1,0)}(\cdot,\omega,t)\|\\
&\le \frac{1}{2\pi}+C_{\varepsilon'}+\varepsilon'\|\tilde{\varrho}^{(1,0)}(\cdot,\omega,t)\|^2,
\end{aligned}
$$

where ε is a certain positive constant, and C_ε is a constant dependent on ε (hereafter we use these notations in the same meaning). We applied the Young's inequality in the last inequality.

Thus, after taking the supremum with respect to ω, we have the estimate of the form

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\!|\tilde{\varrho}(t)\|\!|^2 + D\|\!|\tilde{\varrho}^{(1,0)}(t)\|\!|^2 \le c_{56}+\varepsilon'\|\!|\tilde{\varrho}^{(1,0)}(t)\|\!|^2. \tag{5.3}
$$

Therefore, if we take ε' so small that $\varepsilon' < D$ holds, then by virtue of the classical Gronwall's inequality, we have the estimate of the form (see, for instance, p. 85 of [14])

$$
\begin{aligned}
\|\!|\tilde{\varrho}(t)\|\!|^2 &\le \|\!|\tilde{\varrho}_0\|\!|^2\exp\Big(-2(D-\varepsilon')t\Big)+\frac{c_{57}}{D-\varepsilon'}\left(1-\exp\Big(-2(D-\varepsilon')t\Big)\right)\\
&\le c_{58}\quad \forall t\in(0,T).
\end{aligned} \tag{5.4}
$$

Next, we show the estimate of $\tilde{\varrho}^{(1,0)}$, which satisfies

$$
\frac{\partial\tilde{\varrho}^{(1,0)}}{\partial t}+\omega\frac{\partial\tilde{\varrho}^{(1,0)}}{\partial\theta}-D\frac{\partial^2\tilde{\varrho}^{(1,0)}}{\partial\theta^2}+\frac{\partial}{\partial\theta}\Big(F^{(1)}[\tilde{\varrho}+\bar{\varrho},\tilde{\varrho}]+F[\tilde{\varrho}^{(1,0)},\tilde{\varrho}]\Big)=0.
$$

Then, due to the estimates

$$
\begin{aligned}
&\int_\Omega \tilde{\varrho}^{(1,0)}(\theta,t,\omega)\frac{\partial}{\partial\theta}\Big(F^{(1)}[\tilde{\varrho}+\bar{\varrho},\tilde{\varrho}]\Big)\,\mathrm{d}\theta\\
&\quad=\int_\Omega F^{(1)}[\big(\tilde{\varrho}^{(1,0)}(\theta,t,\omega)\big)^2,\tilde{\varrho}]\,\mathrm{d}\theta+\frac{1}{2}\int_\Omega F^{(2)}\Big[\frac{\partial}{\partial\theta}\big(\tilde{\varrho}(\theta,t,\omega)\big)^2,\tilde{\varrho}\Big]\,\mathrm{d}\theta\\
&\qquad+\frac{1}{2\pi}\int_\Omega F^{(2)}[\tilde{\varrho}^{(1,0)},\tilde{\varrho}]\,\mathrm{d}\theta\\
&\quad\le c_{59}\|\tilde{\varrho}^{(1,0)}(\cdot,\omega,t)\|^2+c_{510}\|\tilde{\varrho}(\cdot,\omega,t)\|^2+c_{511}\|\tilde{\varrho}^{(1,0)}(\cdot,\omega,t)\|\,\|\!|\tilde{\varrho}(t)\|\!|,
\end{aligned}
$$

$$
\begin{aligned}
\int_\Omega \tilde{\varrho}^{(1,0)}(\theta,\omega,t)\frac{\partial}{\partial\theta}\Big(F[\tilde{\varrho}^{(1,0)},\tilde{\varrho}]\Big)\,\mathrm{d}\theta &= -\frac{1}{2}\int_\Omega F\Big[\frac{\partial}{\partial\theta}\big(\tilde{\varrho}^{(1,0)}(\theta,\omega,t)\big)^2,\tilde{\varrho}\Big]\,\mathrm{d}\theta\\
&\le c_{512}\|\tilde{\varrho}^{(1,0)}(\cdot,\omega,t)\|^2,
\end{aligned}
$$

and the Young's inequality, and taking the supremum with respect to x and ω, we have the estimate of the form

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\!|\tilde{\varrho}^{(1,0)}(t)\|\!|^2+D\|\!|\tilde{\varrho}^{(2,0)}(t)\|\!|^2\le \chi_1^{(0,0)}\|\!|\tilde{\varrho}(t)\|\!|^2+\chi_1^{(1,0)}\|\!|\tilde{\varrho}^{(1,0)}(t)\|\!|^2. \tag{5.5}
$$

with constants $\chi_1^{(i,0)}$ $(i = 0, 1)$. Now we divide the second term in the left-hand side of (5.3) into two terms by using a small constant $\varepsilon > 0$, and apply the Poincaré's inequality

$$\|\tilde{\varrho}(\cdot, \omega, t)\| \leq 2\pi \|\tilde{\varrho}^{(1,0)}(\cdot, \omega, t)\|$$

to the first term:

$$(D - \varepsilon)\|\tilde{\varrho}^{(1,0)}(t)\|^2 + \varepsilon\|\tilde{\varrho}^{(1,0)}(t)\|^2 \geq \frac{D-\varepsilon}{4\pi^2}\|\tilde{\varrho}(t)\|^2 + \varepsilon\|\tilde{\varrho}^{(1,0)}(t)\|^2.$$

Then, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\varrho}(t)\|^2 + \frac{D-\varepsilon}{4\pi^2}\|\tilde{\varrho}(t)\|^2 + \varepsilon\|\tilde{\varrho}^{(1,0)}(t)\|^2 \leq c_{513} + \varepsilon'\|\tilde{\varrho}^{(1,0)}(t)\|^2.$$

Summing up this and (5.5) multiplied by a positive constant $m^{(1,0)}$, which will be specified later, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\Big(\|\tilde{\varrho}(t)\|^2 + m^{(1,0)}\|\tilde{\varrho}^{(1,0)}(t)\|^2\Big) + \Big\{\frac{(D-\varepsilon)}{4\pi^2} - m^{(1,0)}\chi_1^{(0,0)}\Big\}\|\tilde{\varrho}(t)\|^2 \\ &+ \Big\{\varepsilon - \varepsilon' - m^{(1,0)}\chi_1^{(1,0)}\Big\}\|\tilde{\varrho}^{(1,0)}(t)\|^2 + m^{(1,0)}D\|\tilde{\varrho}^{(2,0)}(t)\|^2 \\ &\leq c_{514}.\end{aligned}$$

Therefore, we take ε, ε' and $m^{(1,0)}$ in the following manner:

(i) Take ε and ε' so that $\varepsilon' < \varepsilon < D$ holds;

(ii) Then, take $m^{(1,0)} > 0$ so small that

$$\begin{cases} \dfrac{D-\varepsilon}{4\pi^2} - \chi_1^{(0,0)}m^{(1,0)} > 0, \\ \varepsilon - \varepsilon' - \chi_1^{(1,0)}m^{(1,0)} > 0 \end{cases}$$

hold.

Then, in the same line with the deduction of (5.4), we have

$$\|\tilde{\varrho}^{(1,0)}(t)\|^2 \leq c_{515} \quad \forall t > 0. \tag{5.6}$$

Similarly, for $k = 2, 3, 4$ we have the estimates of the form

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\varrho}^{(k,0)}(t)\|^2 + D\|\tilde{\varrho}^{(k+1,0)}(t)\|^2 \leq \sum_{j=0}^{k}\chi_k^{(j,0)}\|\tilde{\varrho}^{(j,0)}(t)\|^2 \qquad (k = 2, 3, 4). \tag{5.7}$$

□

Thanks to Lemma 5.1, $\varrho\big|_{t=T_*}$ satisfies the assumptions in Theorem 5.1. Therefore, we can extend it onto the time interval $(T_*, 2T_*)$, and it again satisfies the estimate (5.1). Iterating this procedure finitely many times, we obtain the solution of (2.2) on the desired time interval.

6. Vanishing diffusion limit

In this section, we show the existence of the solution when the diffusion D tends to zero. For the sake of simplicity, we denote the solution of (2.2) with $D > 0$ by $\varrho_{(D)}$, and we use $\varrho_{(0)}$ to stand for the solution of (2.1).

As we have stated, Ha and Xiao [6] held a similar discussion for the original Kuramoto-Sakaguchi equation (2.2). However, they estimated the norm of $\varrho_{(D)}$ by using the polynomial of D, which resulted in the convergence in $L_\infty(\Omega)$ with respect to θ. In the discussion below, we apply the compactness argument for deriving convergence of a higher order than their result. The framework of this discussion was provided in our previous paper [8], but the details of the proof are presented here since they were omitted in it.

Theorem 6.1. *Let $T > 0$ be an arbitrary number. Under the same assumptions as in* Theorem 5.2, *there exists a solution $\varrho_{(0)}$ of* (2.1) *in $\mathcal{K}^4(T)$.*

Before the proof of Theorem 6.1, we first prepare some lemmas below.

Lemma 6.1. *Let $T > 0$, and ϱ_0 satisfies the same assumptions as in* Theorem 5.2. *Then, the sequence $\{\varrho_{(D)}^{(k,l)}\}_{D>0}$ is bounded in $\mathcal{K}^4(T)$.*

Proof. What we have to verify are

$$\sup_{t\in(0,T)} |||\varrho_{(D)}^{(k,l)}(t)|||_0 \le c_{k,l}(T) \quad (k + 2l \le 4), \tag{6.1}$$

but these are verified by the arguments similar to those in Lemma 5.1, so we omit them. □

By virtue of Lemma 6.1, we see that the sequence $\{\varrho_{(D)}\}_{D>0}$ includes a subsequence, denoted as $\{\varrho_{(D)}\}$ again, which is convergent in the weak-star sense as D tends to zero:

$$\varrho_{(D)} \to \exists\hat{\varrho} \quad \text{in } L_\infty(0,T;\overline{\mathcal{H}}^4) \quad \text{weakly star;} \tag{6.2}$$

$$\frac{\partial \varrho_{(D)}}{\partial t} \to \exists\hat{\varrho}' \quad \text{in } L_\infty(0,T;\overline{\mathcal{H}}^2) \quad \text{weakly star.} \tag{6.3}$$

Then, in the relationship

$$\varrho_{(D)} = \varrho_0 + \int_0^t \frac{\partial \varrho_{(D)}}{\partial t}(\tau)\,\mathrm{d}\tau \quad \text{in } L_\infty(0,T;\overline{\mathcal{H}}^2),$$

if we make D tend to zero, we have

$$\hat{\varrho} = \varrho_0 + \int_0^t \hat{\varrho}'(\tau)\,\mathrm{d}\tau \quad \text{in } L_\infty(0,T;\overline{\mathcal{H}}^2),$$

which means $\hat{\varrho}' = \frac{\partial\hat{\varrho}}{\partial t}$.

The next lemma clarifies the space to which this sequence converges.

Lemma 6.2. *The sequence $\{\varrho_{(D)}\}_{D>0}$ forms a Cauchy sequence in $\mathcal{K}^4(T)$.*

Proof. By subtracting (2.2) with D replaced by D' from the original one, $\tilde{\tilde{\varrho}} \equiv \varrho_{(D)} - \varrho_{(D')}$ satisfies

$$\begin{aligned}\frac{\partial\tilde{\tilde{\varrho}}}{\partial t} + \omega\frac{\partial\tilde{\tilde{\varrho}}}{\partial\theta} - D\frac{\partial^2\tilde{\tilde{\varrho}}}{\partial\theta^2} - (D-D')\frac{\partial^2\varrho_{(D')}}{\partial\theta^2}\\ + K\frac{\partial}{\partial\theta}\left[\tilde{\tilde{\varrho}}(\theta,\omega,t)\int_{\mathbf{R}} g(\omega')\mathrm{d}\omega'\int_\Omega \sin(\phi-\theta)\varrho_{(D)}(\phi,\omega',t)\mathrm{d}\phi\right]\\ + K\frac{\partial}{\partial\theta}\left[\varrho_{(D')}(\theta,\omega,t)\int_{\mathbf{R}} g(\omega')\mathrm{d}\omega'\int_\Omega \sin(\phi-\theta)\tilde{\tilde{\varrho}}(\phi,\omega',t)\mathrm{d}\phi\right] = 0 \qquad (6.4)\end{aligned}$$

Multiplying (6.4) by $\tilde{\tilde{\varrho}}$, integrating by parts over Ω, and taking the supremum with respect to ω yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|||\tilde{\tilde{\varrho}}(t)|||^2 \le c_{61}\left(|||\tilde{\tilde{\varrho}}(t)|||^2 + |D-D'|^2\right).$$

Here, we used the estimates as an example:

$$\begin{aligned}K\int_\Omega \tilde{\tilde{\varrho}}(\theta,\omega,t)\frac{\partial}{\partial\theta}\left[\tilde{\tilde{\varrho}}(\theta,\omega,t)\int_{\mathbf{R}} g(\omega')\mathrm{d}\omega'\int_\Omega \sin(\phi-\theta)\varrho_{(D)}(\phi,\omega',t)\ \mathrm{d}\phi\right]\mathrm{d}\theta\\ = -\frac{K}{2}\int_\Omega \frac{\partial}{\partial\theta}(\tilde{\tilde{\varrho}}(\theta,\omega,t))^2\left(\int_{\mathbf{R}} g(\omega')\mathrm{d}\omega'\int_\Omega \sin(\phi-\theta)\varrho_{(D)}(\phi,\omega',t)\ \mathrm{d}\phi\right)\mathrm{d}\theta\\ \le \frac{K}{2}\|\tilde{\tilde{\varrho}}(t)\|^2,\end{aligned}$$

$$\begin{aligned}\int_\Omega \tilde{\tilde{\varrho}}(\theta,\omega,t)\frac{\partial}{\partial\theta}\left[\varrho_{(D')}(\theta,\omega,t)\int_{\mathbf{R}} g(\omega')\mathrm{d}\omega'\int_\Omega \sin(\phi-\theta)\tilde{\tilde{\varrho}}(\phi,\omega',t)\ \mathrm{d}\phi\right]\mathrm{d}\theta\\ \le |||\varrho_{(D')}(t)|||\,|||\tilde{\tilde{\varrho}}^{(1,0)}(t)|||\left|\left|\left|\left(\int_{\mathbf{R}} g(\omega')\mathrm{d}\omega'\int_\Omega \sin(\phi-\theta)\tilde{\tilde{\varrho}}(\phi,\omega',t)\right)\right|\right|\right|.\end{aligned}$$

Thus, by virtue of the Gronwall's inequality, we have

$$|||\tilde{\tilde{\varrho}}(t)|||^2 \le c_{62}|D-D'|^2 t e^{c_{63}t},$$

which implies that $\{\varrho_{(D)}\}_{D>0}$ makes a Cauchy sequence in $L_\infty(0,T;\overline{\mathcal{H}}^0)$.

Similar arguments hold for $\varrho_{(D)}^{(k,l)}$ for $(k,l) \ne (0,0)$, and by summing them up multiplied by appropriate constants, we arrive at the desired result. □

By Lemma 6.2, we see that $\hat{\varrho}$ belongs to $\mathcal{V}^4(T)$. Now, we show that $\hat{\varrho}$ certainly satisfies (2.1). To do this, we take an arbitrary function $h(\theta,t) \in C^1(0,T;C_0^\infty(\Omega))$ satisfying $h(\theta,t)|_{t=T} = 0$, $h(\theta,t)|_{t=0} \ne 0$, and consider

$$\int_0^T \mathrm{d}t\int_\Omega\left\{\frac{\partial\varrho_{(D)}}{\partial t} + \omega\frac{\partial\varrho_{(D)}}{\partial\theta} - D\frac{\partial^2\varrho_{(D)}}{\partial\theta^2} + \frac{\partial}{\partial\theta}\Big(F[\varrho_{(D)},\varrho_{(D)}]\Big)\right\}h(\theta,t)\ \mathrm{d}\theta = 0$$

$$\forall\omega\in\mathbf{R}. \qquad (6.5)$$

In virtue of (6.2)–(6.3), if we make D tend to zero,

$$\int_0^T \mathrm{d}t \int_\Omega \left\{ \frac{\partial \varrho_{(D)}}{\partial t} + \omega \frac{\partial \varrho_{(D)}}{\partial \theta} - D \frac{\partial^2 \varrho_{(D)}}{\partial \theta^2} \right\} h(\theta, t) \mathrm{d}\theta$$

$$\to \int_0^T \mathrm{d}t \int_\Omega \left\{ \frac{\partial \hat{\varrho}}{\partial t} + \omega \frac{\partial \hat{\varrho}}{\partial \theta} \right\} h(\theta, t) \mathrm{d}\theta \quad \forall \omega \in \mathbf{R}.$$

Thanks to the Rellich's theorem [13], we have

$$\varrho_{(D)} \to \hat{\varrho} \quad \text{in } L_2(0, T; \overline{\mathcal{H}}^0)$$

strongly as $D \to 0$; therefore,

$$\int_0^T \mathrm{d}t \int_\Omega \frac{\partial}{\partial \theta} \Big(F[\varrho_{(D)}, \varrho_{(D)}] \Big) h(\theta, t)\, \mathrm{d}\theta \to \int_0^T \mathrm{d}t \int_\Omega \frac{\partial}{\partial \theta} \Big(F[\hat{\varrho}, \hat{\varrho}] \Big) h(\theta, t)\, \mathrm{d}\theta$$

holds. Thus, we have

$$\int_0^T \mathrm{d}t \int_\Omega \left\{ \frac{\partial \hat{\varrho}}{\partial t} + \omega \frac{\partial \hat{\varrho}}{\partial \theta} + \frac{\partial}{\partial \theta} \Big(F[\hat{\varrho}, \hat{\varrho}] \Big) \right\} h(\theta, t) \mathrm{d}\theta = 0, \tag{6.6}$$

which means that $\hat{\varrho}$ certainly satisfies $(2.1)_1$. Next, integrate (6.5) and (6.6) by part with respect to t, and the assumptions on $h(\theta, t)$ yield

$$\begin{aligned} &- \varrho_0(\theta, \omega) h(\theta, 0) - \int_0^T \mathrm{d}t \int_\Omega \varrho_{(D)}(\theta, \omega, t) \frac{\partial h}{\partial t}(\theta, t)\, \mathrm{d}\theta \\ &+ \int_0^T \mathrm{d}t \int_\Omega \left\{ \omega \frac{\partial \varrho_{(D)}}{\partial \theta} - D \frac{\partial^2 \varrho_{(D)}}{\partial \theta^2} + \frac{\partial}{\partial \theta} \Big(F[\varrho_{(D)}, \varrho_{(D)}] \Big) \right\} h(\theta, t)\, \mathrm{d}\theta = 0, \end{aligned} \tag{6.7}$$

$$\begin{aligned} &- \hat{\varrho}(\theta, \omega, 0) h(\theta, 0) - \int_0^T \mathrm{d}t \int_\Omega \hat{\varrho}(\theta, \omega, t) \frac{\partial h}{\partial t}(\theta, t)\, \mathrm{d}\theta \\ &+ \int_0^T \mathrm{d}t \int_\Omega \left\{ \omega \frac{\partial \hat{\varrho}}{\partial \theta} + \frac{\partial}{\partial \theta} \Big(F[\hat{\varrho}, \hat{\varrho}] \Big) \right\} h(\theta, t)\, \mathrm{d}\theta = 0, \end{aligned} \tag{6.8}$$

respectively. Comparing (6.7) and (6.8) implies $\hat{\varrho}|_{t=0} = \varrho_{(0)}$, so the initial condition $(2.1)_3$ is satisfied. The periodicity of $\hat{\varrho}$ obviously holds due to the function space to which $\hat{\varrho}$ belongs. Thus, $\hat{\varrho} = \varrho_{(0)}$.

This completes the proof of Theorem 6.1. As in the case of Theorem 5.2, we have the following statement.

Corollary 6.1. *Under the assumptions in* Theorem 5.2, *the solution* $\varrho(\theta, \omega, t)$ *to* (2.1) *stated in* Theorem 6.1 *satisfies*

$$\varrho(\theta, \omega, t) \in \widetilde{\mathcal{K}}^4(T).$$

7. Existence of maximal attractor and inertial set

In this section, we discuss the existence of the maximal attractor and inertial set. Hereafter, let H be a separable Hilbert space equipped with a norm $\|\cdot\|_H$, and define a semigroup $\{S(t)\}_{t \geq 0}$ as a family of operators:

$$S(t) : u_0 \in H \longmapsto u(t) \in H,$$

where $u(t)$ is subject to a certain dynamical system with initial data u_0 in general.

First, we define the attractor of a semigroup [14].

Definition 7.1. *An attractor is a set $\mathcal{A} \subset H$ that enjoys the following properties:*

(i) *$\mathcal{A}$ is an invariant set, that is, $S(t)\mathcal{A} = \mathcal{A}\ \forall t \geq 0$ holds;*

(ii) *$\mathcal{A}$ possesses an open neighborhood $\mathcal{U}$ such that, for every $u_0 \in \mathcal{U}$,*

$$\mathrm{dist}_H(S(t)u_0, \mathcal{A}) \equiv \inf_{y \in \mathcal{A}} \|S(t)u_0 - y\|_H \to 0 \text{ as } t \to 0.$$

Next, we define the *maximal attractor* [14].

Definition 7.2. *We say that $\mathcal{A} \subset H$ is a maximal attractor for the semigroup $\{S(t)\}_{t\geq 0}$ if $\mathcal{A}$ is a compact attractor that attracts the bounded sets of H.*

We discuss the existence of the maximal attractor in our problem. Below, let $\overline{S}(t)$ be a semigroup associated with problem (2.2) and defined on $\overline{\mathcal{H}}^4$:

$$\overline{S}(t) : \varrho_0 \in \overline{\mathcal{H}}^4 \longmapsto \varrho(t) \in \overline{\mathcal{H}}^4,$$

where $\varrho(t)$ is a solution to (2.2) with initial data ϱ_0. Theorem 5.2 implies that $\overline{S}(t)$ is a continuous mapping from $\overline{\mathcal{H}}^4$ to itself for each $t > 0$.

Theorem 7.1. *Under the assumptions in* Theorem 1, *the semigroup $\overline{S}(t)$ possesses a compact maximal attractor in $\overline{\mathcal{H}}^4$ that is connected.*

The proof of Theorem 7.1 is achieved by the direct application of the *a-priori* estimate that we already obtained and Theorem I.1.1 in [14].

Next, we introduce the definition of *inertial set.* It is well known that the orbits of dissipative systems are sometimes absorbed in a finite dimensional set rapidly [14]. Hereafter, let B be a compact subset of H.

Definition 7.3. *Let B be invariant under a continuous semigroup $S(t)$, that is, $S(t)B = B\ \forall t \geq 0$ holds. Let $\mathcal{A}$ be the maximal attractor for $\{S(t)\}_{t\geq 0}$ on B. Then, set $\mathcal{M}$ is called an "inertial set" for $(\{S(t)\}_{t\geq 0}, B)$ if it has finite fractal dimension $d_f(\mathcal{M})$ and moreover satisfies*

(i) *$\mathcal{A} \subset \mathcal{M} \subset B, \quad S(t)\mathcal{M} \subset \mathcal{M}$ for every $t \geq 0$;*

(ii) *for every $u_0 \in B$, $\mathrm{dist}_H(S(t)u_0, \mathcal{M}) \leq c_{71}e^{-c_{72}t}$ with positive constants c_{7j} $(j = 1, 2)$ independent of u_0.*

Next, we show the existence of the inertial set for the solution of (2.2) [1][4].

Theorem 7.2. *Under the assumptions in* Theorem 5.1, *the semigroup $\overline{S}(t)$ possesses an inertial set $\mathcal{M}$ for $(\{\overline{S}(t)\}_{t\geq 0}, \overline{\mathcal{H}}^2)$ satisfying*

$$\mathrm{dist}_{\overline{\mathcal{H}}^0}(\overline{S}(t)u_0, \mathcal{M}) \leq c_{73}e^{-\frac{c_{74}t}{t_*}} \quad \forall u_0 \in \overline{\mathcal{H}}^2$$

with positive constants c_{7j} $(j = 3, 4)$ independent of u_0 by taking t_* and N_0 sufficiently large.

Theorem 7.2 is proved with the aid of the result by Eden et al. [4], which claims that the *squeezing property* of a semigroup implies the existence of an inertial set.

Definition 7.4. *A continuous semigroup* $\{S(t)\}_{t\geq 0}$ *is said to satisfy the squeezing property on a compact subset* $B \subset H$ *if there exists* $t_* > 0$ *such that* $S_* = S(t_*)$ *satisfies the following.*
There exists an orthogonal projection P_{N_0} *of rank* N_0 *such that, if for every* u *and* v *in* B

$$\|P_{N_0}(S_*u - S_*v)\|_H \leq \|(I - P_{N_0})(S_*u - S_*v)\|_H$$

holds, then

$$\|S_*u - S_*v\|_H \leq \frac{1}{8}\|u - v\|_H.$$

The following theorem is due to Eden et al. [4].

Theorem 7.3. *If* $\{S(t)_{t\geq 0}\}$ *satisfies the squeezing property on* B *and if* $S_* = S(t_*)$ *is Lipschitz continuous on* B *with Lipschitz constant* L*, then there exists an inertial set* $\mathcal{M}$ *for* $(\{S(t)\}_{t\geq 0}, B)$ *such that*

$$d_f(\mathcal{M}) \leq N_0 \max\{1, \ln(16L+1)/\ln 2\},$$
$$\mathrm{dist}_H(S(t)u_0, \mathcal{M}) \leq c_{75}\exp\Big(-c_{76}t/t_*\Big)\ \forall u_0 \in B$$

with positive constants N_0 *and* c_{7j} $(j = 5, 6)$.

Thanks to Theorem 7.3 it is sufficient to verify the squeezing property of $\{\overline{S}(t)\}_{t\geq 0}$ to prove Theorem 7.2. In the following, we state the proof of Theorem 7.2. First, let us define two solutions ϱ_j $(j = 1, 2)$ of (2.2), whose initial data are ϱ_{j0} $(j = 1, 2)$, respectively:

$$\frac{\partial \varrho_j}{\partial t} + \omega\frac{\partial \varrho_j}{\partial \theta} - D\frac{\partial^2 \varrho_j}{\partial \theta^2} + K\frac{\partial}{\partial \theta}\left[\varrho_j(\theta,\omega,t)\int_{\mathbf{R}} g(\omega')\mathrm{d}\omega'\int_{\Omega}\varrho_j(\phi,\omega',t)\sin(\phi-\theta)\,\mathrm{d}\phi\right] = 0 \quad (i = 1, 2), \tag{7.1}$$

from which we derive the problem for $\breve{\varrho} \equiv \varrho_1 - \varrho_2$:

$$\begin{cases} \dfrac{\partial \breve{\varrho}}{\partial t} + \omega\dfrac{\partial \breve{\varrho}}{\partial \theta} - D\dfrac{\partial^2 \breve{\varrho}}{\partial \theta^2} + K\Big[R[\varrho_1] - R[\varrho_2]\Big] = 0 \quad (\theta,\omega,t) \in \Omega\times\mathbf{R}\times(0,T), \\ \dfrac{\partial^j \breve{\varrho}}{\partial \theta^j}\Big|_{\theta=0} = \dfrac{\partial^j \breve{\varrho}}{\partial \theta^j}\Big|_{\theta=2\pi} \quad (j = 0, 1),\ (\omega, t) \in \mathbf{R}\times(0,T), \\ \breve{\varrho}\big|_{t=0} = \breve{\varrho}_0 \equiv \varrho_{10} - \varrho_{20} \quad (\theta,\omega) \in \Omega\times\mathbf{R}, \end{cases} \tag{7.2}$$

where

$$R[\varrho] \equiv \frac{\partial}{\partial \theta}\left[\varrho(\theta,\omega,t)\int_{\mathbf{R}} g(\omega')\mathrm{d}\omega'\int_{\Omega}\varrho(\phi,\omega',t)\sin(\phi-\theta)\,\mathrm{d}\phi\right]$$
$$\equiv \frac{\partial}{\partial \theta}\mathcal{T}[\varrho,\varrho].$$

We also define the eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ of the operator $\partial^2/\partial\theta^2$ under the periodic boundary condition in the order of magnitude. $\{V_i\}_{i=1}^{\infty}$ is the corresponding eigenvector functions, $N \in \mathbf{N}$ is specified later, $H_N \equiv \mathrm{span}\{V_1, V_2, \ldots, V_N\}$, $P_N : H \to H_N$, the orthogonal projection onto H_N, $Q_N \equiv I - P_N$, $\breve{\varrho}_N \equiv Q_N[\breve{\varrho}]$, and $\breve{\varrho}_{N0} \equiv Q_N[\breve{\varrho}_0]$. Then, after operating Q_N to (7.2), we obtain

$$\frac{\partial \breve{\varrho}_N}{\partial t} + \omega \frac{\partial \breve{\varrho}_N}{\partial \theta} - D\frac{\partial^2 \breve{\varrho}_N}{\partial \theta^2} + KQ_N\Big[R[\varrho_1] - R[\varrho_2]\Big] = 0, \tag{7.3}$$

Then, multiplying $\breve{\varrho}_N$ to (7.3) and integrating over Ω lead to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\breve{\varrho}_N(t)\|^2 + D\|\breve{\varrho}_N^{(1,0)}(t)\|^2 + \int_\Omega \breve{\varrho}_N(\theta,\omega,t)Q_N\Big[R[\varrho_1] - R[\varrho_2]\Big]\,\mathrm{d}\theta = 0.$$

We note that the following estimate holds:

$$\begin{aligned}
&\int_\Omega \breve{\varrho}_N(\theta,\omega,t)Q_N\Big[R[\varrho_1] - R[\varrho_2]\Big]\,\mathrm{d}\theta \\
&\qquad = \int_\Omega \breve{\varrho}_N(\theta,\omega,t)\frac{\partial}{\partial\theta}\Big[Q_N[\mathcal{T}[\varrho_1,\varrho_1] - \mathcal{T}[\varrho_2,\varrho_2]]\Big]\,\mathrm{d}\theta \\
&\qquad \le K\|\breve{\varrho}_N^{(1,0)}(t)\|\Big\|Q_N\big[\mathcal{T}[\varrho_1,\varrho_1] - \mathcal{T}[\varrho_2,\varrho_2]\big]\Big\| \\
&\qquad \le K\lambda_{N+1}^{-\frac{1}{2}}\|\breve{\varrho}_N^{(1,0)}(t)\|\Big\|\mathcal{T}[\varrho_1,\varrho_1] - \mathcal{T}[\varrho_2,\varrho_2]\Big\|_1 \\
&\qquad \le c_{77}K\lambda_{N+1}^{-\frac{1}{2}}\|\breve{\varrho}_N^{(1,0)}(t)\|\|\breve{\varrho}(t)\|_1 \\
&\qquad \le \frac{D}{2}\|\breve{\varrho}_N^{(1,0)}(t)\|^2 + \frac{c_{77}^2K^2}{2D\lambda_{N+1}}\|\breve{\varrho}(t)\|_1^2.
\end{aligned}$$

From this, together with the Poincaré's inequality $\|\breve{\varrho}_N(t)\|^2 \le \lambda_{N+1}^{-1}\|\breve{\varrho}_N^{(1,0)}(t)\|^2$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\breve{\varrho}_N(t)\|^2 + \frac{D\lambda_{N+1}}{2}\|\breve{\varrho}_N(t)\|^2 \le \frac{c_{77}^2K^2}{2D\lambda_{N+1}}\|\breve{\varrho}(t)\|_1^2.$$

Thus, the Gronwall's inequality yields

$$\|\breve{\varrho}_N(t)\|^2 \le e^{-D\lambda_{N+1}t}\|\breve{\varrho}_{N0}\|^2 + \frac{c_{77}^2K^2}{D\lambda_{N+1}}\int_0^t \|\breve{\varrho}(\tau)\|_1^2\,\mathrm{d}\tau. \tag{7.4}$$

On the other hand, by multiplying $\breve{\varrho}$ to (7.2) and integrating over Ω, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\breve{\varrho}(t)\|^2 + D\|\breve{\varrho}(t)\|^2 \\
&\quad + K\int_\Omega \breve{\varrho}(\theta,\omega,t)\frac{\partial}{\partial\theta}\left[\breve{\varrho}(\theta,\omega,t)\int_{\mathbf{R}} g(\omega')\mathrm{d}\omega'\int_\Omega \varrho_1(\phi,\omega',t)\sin(\phi-\theta)\,\mathrm{d}\phi\right]\mathrm{d}\theta \\
&\quad + K\int_\Omega \breve{\varrho}(\theta,\omega,t)\frac{\partial}{\partial\theta}\left[\varrho_2(\theta,\omega,t)\int_{\mathbf{R}} g(\omega')\mathrm{d}\omega'\int_\Omega \breve{\varrho}(\phi,\omega',t)\sin(\phi-\theta)\,\mathrm{d}\phi\right]\mathrm{d}\theta = 0.
\end{aligned} \tag{7.5}$$

We note that the following estimates hold:

$$\begin{aligned}
&\int_\Omega \breve{\varrho}(\theta,\omega,t)\frac{\partial}{\partial\theta}\Big(\breve{\varrho}(\theta,\omega,t)\int_{\mathbf{R}} g(\omega')\mathrm{d}\omega'\int_\Omega \varrho_1(\phi,\omega',t)\sin(\phi-\theta)\,\mathrm{d}\phi\Big)\,\mathrm{d}\theta \\
&\qquad = \frac{1}{2}\int_\Omega \big(\breve{\varrho}(\theta,\omega,t)\big)^2\Big(\int_{\mathbf{R}} g(\omega')\mathrm{d}\omega'\int_\Omega \varrho_1(\phi,\omega',t)\cos(\phi-\theta)\,\mathrm{d}\phi\Big)\,\mathrm{d}\theta \\
&\qquad \le c_{78}\|\breve{\varrho}(t)\|^2,
\end{aligned}$$

$$\int_\Omega \breve{\varrho}(\theta,\omega,t)\frac{\partial}{\partial\theta}\Big(\varrho_2(\theta,\omega,t)\int_{\mathbf{R}} g(\omega')\mathrm{d}\omega'\int_\Omega \breve{\varrho}(\phi,\omega',t)\sin(\phi-\theta)\ \mathrm{d}\phi\Big)\ \mathrm{d}\theta$$
$$= \int_\Omega \breve{\varrho}(\theta,\omega,t)\frac{\partial\varrho_2}{\partial\theta}(\theta,\omega,t)\Big(\int_{\mathbf{R}} g(\omega')\mathrm{d}\omega'\int_\Omega \breve{\varrho}(\phi,\omega',t)\sin(\phi-\theta)\ \mathrm{d}\phi\Big)\ \mathrm{d}\theta$$
$$+ \int_\Omega \breve{\varrho}(\theta,\omega,t)\varrho_2(\theta,\omega,t)\Big(\int_{\mathbf{R}} g(\omega')\mathrm{d}\omega'\int_\Omega \breve{\varrho}(\phi,\omega',t)\cos(\phi-\theta)\ \mathrm{d}\phi\Big)\ \mathrm{d}\theta$$
$$\equiv J_1 + J_2.$$

It is easy to see that

$$J_1 \le \sup_{\theta,\omega}\|\varrho_2^{(1,0)}(t)\|\|\breve{\varrho}(t)\|\left\|\int_{\mathbf{R}} g(\omega')\mathrm{d}\omega'\int_\Omega \breve{\varrho}(\phi,\omega',t)\sin(\phi-\theta)\ \mathrm{d}\phi\right\|$$
$$\le c_{79}\|\breve{\varrho}(t)\|^2.$$

A similar estimate holds for J_2. Thus, (7.5) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\breve{\varrho}(t)\|^2 + D\|\breve{\varrho}^{(1,0)}(t)\|^2 \le c_{710}\|\breve{\varrho}(t)\|^2,$$

which, together with the Gronwall's inequality again, leads to

$$\int_0^t \|\breve{\varrho}^{(1,0)}(\tau)\|^2\ \mathrm{d}\tau \le \frac{1}{2}e^{2c_{711}t}\|\breve{\varrho}_0\|^2.$$

Applying this to (7.4) and the fact $\|\breve{\varrho}_{N0}\| \le \|\breve{\varrho}\|$ yield

$$\|\breve{\varrho}_N(t)\|^2 \le \left(e^{-D\lambda_{N+1}t} + \frac{c_{77}^2K^2}{D\lambda_{N+1}}e^{2c_{711}t}\right)\|\breve{\varrho}_0\|^2.$$

Now we show that there exists t_* and N_0 such that if

$$\|P_{N_0}[\breve{\varrho}](t_*)\| \le \|Q_{N_0}[\breve{\varrho}](t_*)\| \tag{7.6}$$

holds, then

$$\|\breve{\varrho}(t_*)\| \le \frac{1}{8}\|\breve{\varrho}(0)\|$$

holds. To show that, we first take t_* large enough so that

$$e^{-D\lambda_{N+1}t_*} \le \frac{1}{256}.$$

This is achieved by taking, for instance, $t_* \ge \frac{8}{D\lambda_1}\log 2$. Then, for this t_*, we take N_0 so that

$$\frac{c_{77}^2K^2e^{2c_{711}t_*}}{D\lambda_{N_0+1}} \le \frac{1}{256}$$

holds. Then, we have

$$\|\breve{\varrho}_{N_0}(t_*)\|^2 \le \frac{1}{128}\|\breve{\varrho}(0)\|^2. \tag{7.7}$$

On the other hand, under the assumption (7.6), we have

$$\|\breve{\varrho}(t_*)\|^2 = \|P_{N_0}[\breve{\varrho}](t_*)\|^2 + \|Q_{N_0}[\breve{\varrho}](t_*)\|^2 \le 2\|Q_{N_0}[\breve{\varrho}](t_*)\|^2 = 2\|\breve{\varrho}_{N_0}(t_*)\|^2. \tag{7.8}$$

Therefore, by combining (7.7) and (7.8), we obtain

$$\|\breve{\varrho}(t_*)\|^2 \le \frac{1}{64}\|\breve{\varrho}(0)\|^2,$$

which supports the desired squeezing property. Higher derivative terms are estiamted in a similar manner. By virtue of Theorem 7.3, this completes the proof of Theorem 7.2.

8. Conclusion

In this paper, we provided local and global-in-time solvability of the Kuramoto-Sakaguchi equation. We also showed the existence of the solution to the vanishing diffusion limit problem. The existence of the maximal attractor and inertial set were also discussed. Our future work will concern the existence of the inertial manifold and the stability analysis of the coherent state. We will also tackle bifurcation analysis in the future.

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