数理解析研究所講究録
第2041巻 2017年 38-46

ATTRACTIVE POINTS, ACUTE POINTS AND APPROXIMATION OF COMMON FIXED POINTS OF FAMILIES OF NONLINEAR MAPPINGS RELATED TO HYBRID MAPPINGS

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ABSTRACT. In this paper, we prove an attractive points theorem and strong convergence theorems of Halpern's type [20] for uniformly asymptotically regular λ-hybrid mappings in a star-shaped subset of a Hilbert space. Using these results, we obtain a fixed point theorem and some strong convergence theorems.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a nonempty subset of H. For a mapping $T : C \to C$, we denote by $F(T)$ the set of *fixed points* of T and by $A(T)$ the set of *attractive points* [28] of T, i.e.,

(i) $F(T) = \{z \in C : Tz = z\}$;
(ii) $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|,\ \forall x \in C\}$.

A mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Kocourek, Takahashi and Yao [22] introduced a broad class of nonlinear mappings called *generalized hybrid* which containing nonexpansive mappings, nonspreading mappings, and hybrid mappings in a Hilbert space. They proved a mean convergence theorem for generalized hybrid mappings which generalizes Baillon's nonlinear ergodic theorem [13]. Aoyama, Iemoto, Kohsaka and Takahashi [4] introduced the class of λ-hybrid mappings in a Hilbert space. This class obtain the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings

2010 *Mathematics Subject Classification.* Primary 47H09, 47H10.
Key words and phrases. Fixed point, attractive point, iteration, nonexpansive mapping, nonexpansive semigroup, strong convergence.

in a Hilbert space. They proved fixed point theorems and mean convergence theorems for such mappings. Motivated by Baillon [13], and Kocourek, Takahashi and Yao [22], Takahashi and Takeuchi [28] introduced the concept of attractive points of a nonlinear mapping in a Hilbert space and they proved a mean convergence theorem of Baillon's type without convexity for generalized hybrid mappings. In 1992, Wittmann [29] proved the following strong convergence theorems of Halpern's type [20] in a Hilbert space;

Theorem 1.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. For any $x_1 = x \in C$, define a sequence $\{x_n\}$ in C by*

$$x_{n+1} = \alpha_n x + (1-\alpha_n)Tx_n,\ \forall n \geq 1$$

where $\{\alpha_n\} \subset [0,1]$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty.$$

Then, $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the metric projection from H onto $F(T)$.

Motivated by Takahashi and Takeuchi [28], Akashi and Takahashi [2] proved a strong convergence theorem of Halpern's type [20] for nonexpansive mappings in a star-shaped subset of a Hilbert space. On the other hand, Domingues Benavides, Acedo and Xu [17] proved strong convergence theorems of Halpern's type [20] for uniformly asymptotically regular one-parameter nonexpansive semigroups. The author [8] studied Halpern's type iterations for nonexpansive semigroups and proved strong convergence theorems for uniformly asymptotically regular nonexpansive semigroups in Hilbert spaces (see also [1, 7, 9, 17, 25, 26]).

In this paper, we prove an attractive points theorem and strong convergence theorems of Halpern's type [20] for uniformly asymptotically regular λ-hybrid mappings in a star-shaped subset of a Hilbert space. Using these results, we obtain a fixed point theorem and some strong convergence theorems.

2. Preliminaries and notations

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the set of all positive integers and the set of all real numbers, respectively. We also denote by

$\mathbb{Z}^+$ and $\mathbb{R}^+$ the set of all nonnegative integers and the set of all nonnegative real numbers, respectively. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We know the following basic equality from [26]. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle \tag{2.1}$$

and

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2.2}$$

Furthermore, we obtain that for all $x, y, w \in H$,

$$\langle (x - y) + (x - w), y - w \rangle = \|x - w\|^2 - \|x - y\|^2. \tag{2.3}$$

In fact, we have that

$$\begin{aligned} &\langle (x - y) + (x - w), y - w \rangle \\ &= \langle (x - y) + (x - w), (y - x) + (x - w) \rangle \\ &= \|x - w\|^2 - \|x - y\|^2 + \langle x - y, x - w \rangle + \langle x - w, y - x \rangle \\ &= \|x - w\|^2 - \|x - y\|^2. \end{aligned}$$

Let C be a closed and convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|$$

for all $y \in C$. The mapping P_C is called the *metric projection* of H onto C. It is characterized by

$$\langle P_C x - y, x - P_C x \rangle \geq 0$$

for all $y \in C$. See [26] for more details. The following result is well-known (see [26]).

Lemma 2.1. *Let C be a nonempty, bounded, closed and convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. Then, $F(T) \neq \emptyset$.*

We write $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in H converges strongly to x. We also write $x_n \rightharpoonup x$ (or w-$\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in H converges weakly to x. In a Hilbert space, it is well known that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$.

A mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. Let $\lambda \in \mathbb{R}$ be given. Following [4], we say that a mapping $T : C \to C$ is λ-hybrid if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + 2(1 - \lambda)\langle x - Tx, y - Ty\rangle$$

for all $x, y \in C$. It is obvious that T is 1-hybrid if and only if T is nonexpansive; T is 0-hybrid if and only if T is nonspreading [23]; T is 1/2-hybrid if and only if T is hybrid [27]); If $\lambda > 1$, then T is λ-hybrid if and only if $T = I$. It is known [3, Proposition 2.2] that if $\lambda < 2$ and $\alpha = (1-\lambda)/(2-\lambda)$, then T is λ-hybrid if and only if it is α-nonexpansive [3], i.e.,

$$\|Tx - Ty\|^2 \le \alpha(\|x - Ty\|^2 + \|Tx - y\|^2 + (1 - 2\alpha)\|x - y\|^2$$

for all $x, y \in C$. In general, nonspreading and hybrid mappings are not continuous mappings. A mapping $T : C \to C$ is called *quasi-nonexpansive* if $F(T)$ is nonempty and $\|w - Tx\| \le \|w - y\|$ for all $w \in F(T)$ and $x \in C$. By Dotson [16, Theorem 1] and Ithoh and Takahashi [21, Corollary 1], we know that $F(T)$ is closed convex whenever T is quasi-nonexpansive. Every λ-hybrid with a fixed point is cleary quasi-nonexpansive. Thus, the set of fixed point of each λ-hybrid mapping is closed convex. The mapping T is said to be firmly nenexpansive if

$$\|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 \le \|x - y\|^2$$

for all $x, y \in C$(see [14, 15, 18, 19]. It is known [4, Lemma 3.1] that if T is firmly nenexpansive, then T is λ-hybrid for each $\lambda \in [0, 1]$.

3. Lemmas

In this section, we give some lemmas which are used in the proofs of our main theorems. We have basic properties of attractive points of nonlinear mappings in a Hilbert space (see [28]).

Lemma 3.1 ([28]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H. Let T be a mappings of C into itself. If $A(T) \neq \emptyset$, then $F(T) \neq \emptyset$.*

Lemma 3.2 ([28]). *Let H be a Hilbert space, let C be a nonempty subset of H. Let T be a mappings of C into H. Then, $A(T)$ is a closed and convex subset of H.*

We also have the following lemma (see also [12, 28]).

Lemma 3.3 ([28]). *Let H be a Hilbert space, let C be a nonempty subset of H. Let T be a mappings of C into H. Let $\{u_n\}$ be a sequence in H such that*

$$\overline{\lim_{n\to\infty}} \langle (u_n - y) + (u_n - Ty), y - Ty \rangle \leq 0$$

for all $y \in C$. If a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ converges weakly to $u \in H$, then $u \in A(T)$.

To prove our main results, we need the following lemma (see [5]; see also [30]).

Lemma 3.4. *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$ and let $\{\gamma_n\}$ be a sequence of real numbers with $\overline{\lim}_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \gamma_n + \beta_n$$

for all $n \in \mathbb{N}$. Then, $\lim_{n\to\infty} s_n = 0$.

4. Main Theorems

In this section, we prove an attractive points theorem and strong convergence to common attractive points of uniformly asymptotically regular λ-hybrid mappings in Hilbert spaces (see also [2, 7, 12, 17, 24, 25, 26, 28]).

Let C be a nonempty subset of H. Then, C is called star-shaped if there exists $z \in C$ such that for any $x \in C$ and any $\gamma \in (0,1)$,

$$\gamma z + (1-\gamma)x \in C.$$

We say that a mapping T of C into itself is asymptotically regular if

$$\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0$$

for all $x \in C$ (see also [26]). We also say that a mapping T of C into itself is uniformly asymptotically regular if for every bounded subset K of C,

$$\lim_{n\to\infty} \sup_{x\in K} \|T^{n+1}x - T^n x\| = 0$$

holds.

Lemma 4.1 ([6]). *Let C be a nonempty subset of a Hilbert space H. Let $\lambda \in \mathbb{R}$ be given. Let T be a λ-hybrid mapping of C into itself. If $A(T) \neq \emptyset$, $\{T^n x\}$ is bouded for each $x \in C$.*

We also get the following attractive point theorems (see also [12, 28]).

Theorem 4.2 ([6]). *Let H be a Hilbert space and let C be a nonempty subset of H. Let λ be a real number. Let T be a uniformly asymptotically regular λ-hybrid mapping of C into itself. Suppose that $\{T^n x\}$ is bounded for some $x \in C$. Then, $A(T) \neq \emptyset$.*

We obtain a strong convergence theorem of Halpern's [20] type for λ-hybrd mappings on a star-shaped subset of H (see [6]).

Theorem 4.3 ([6]). *Let H be a Hilbert space, let C be a star-shaped subset of H with center $z \in C$. Let λ be a real number. Let T be a uniformly asymptotically regular λ-hybrid mapping of C into itself such that $A(T) \neq \emptyset$. Let $\{m_n\}$ be a sequence in $\mathbb{N}$ such that $m_n \to \infty$. Let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n z + (1-\alpha_n) T^{m_n} x_n$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then, $\{x_n\}$ converges strongly to $P_{A(T)}z$, where $P_{A(T)}$ is the metric projection from H onto $A(T)$.

Using Theorem 4.2, we obtain the following fixed point theorem.

Theorem 4.4 ([6]). *Let H be a Hilbert space and let C be a closed and star-shaped subset of H. Let λ be a real number. Let T be a uniformly asymptotically regular λ-hybrid mapping of C into itself. Suppose that $\{T^n x\}$ is bounded for some $x \in C$. Then, $F(T) \neq \emptyset$.*

Using Theorem 4.3, we also get the following strong convergence theorem for λ-hybrid mappings on a star-shaped subset of H (see [20, 29, 30]).

Theorem 4.5 ([6]). *Let H be a Hilbert space, let C be a closed and star-shaped subset of H with center $z \in C$. Let λ be a real number. Let T be a uniformly asymptotically regular λ-hybrid mapping of C into itself such that $F(T) \neq \emptyset$. Let $\{m_n\}$ be a sequence in $\mathbb{N}$ such that $m_n \to \infty$. Let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n z + (1-\alpha_n) T^{m_n} x_n$$

for each $n \in \mathbb{N}$, *where* $\{\alpha_n\} \subset [0,1]$ *satisfies*

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then, $\{x_n\}$ *converges strongly to* u_0, *where* $\|u_0 - z\| = \min\{\|u - z\| : u \in F(T)\}$

We also have the following strong convergence theorem.

Theorem 4.6 ([6]). *Let* H *be a Hilbert space, let* C *be a nonempty subset of* H. *Let* λ *be a real number. Let* T *be a uniformly asymptotically regular* λ*-hybrid mapping of* C *into itself such that* $A(T) \neq \emptyset$. *Let* $\{m_n\}$ *be a sequence in* $\mathbb{N}$ *such that* $m_n \to \infty$. *Let* $\{x_n\}$ *be a sequence in* C *defined by* $x_1 \in C$ *and*

$$x_{n+1} = \alpha_n z + (1-\alpha_n) T^{m_n} x_n$$

for each $n \in \mathbb{N}$, *where* $\{\alpha_n\} \subset [0,1]$ *satisfies*

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty.$$

If $\{x_n\}$ *is in* C, *then* $\{x_n\}$ *converges strongly to* $u_0 \in A(T)$, *where* $u_0 = P_{A(T)}$.

Acknowledgements

The author is supported by Grand-in-Aid for Scientific Research No. 26400196 from Japan Society for the Promotion of Science.

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