数理解析研究所講究録
第2041巻 2017年 47-56

The Split Common Fixed Point Problem for New Classes of Nonlinear Operators in Banach Spaces

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Abstract. The aim of this article is to prove strong convergence theorems by the hybrid method and the shrinking projection method for finding common fixed points of families of new nonlinear mappings in Banach spaces. We first deal with basic properties of new nonlinear mappings. In particular, we prove that the common fixed point sets of new nonlinear mappings are closed and convex. Using these results and the hybrid method introduced by Nakajo and Takahashi [14], we prove a strong convergence theorem which solves the split common fixed point problem in two Banach Spaces. Furthermore, using the shrinking projection method introduced by Takahashi, Takeuchi and Kubota [28], we also prove another strong convergence theorem. Moreover, using these results, we obtain well-known and new strong convergence theorems in Hilbert spaces and Banach spaces.

2010 *Mathematics Subject Classification*: 47H10
Keywords and phrases: Maximal monotone mapping, hybrid method, shrinking projection method, generalized projection, generalized resolvent, split common fixed point problem.

1 Introduction

Recently, Takahashi, Wen and Yao [29] introduced a new class of nonlinear mappings as follows: Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let η and s be real numbers with $\eta \in (-\infty, 1)$ and $s \in [0, \infty)$, respectively. A mapping $T : C \to E$ with $F(T) \neq \emptyset$ is called (η, s)-demigeneralized if, for any $x \in C$ and $q \in F(T)$,

$$2\langle x-q, Jx-JTx\rangle \geq (1-\eta)\phi(x, Tx)+s\phi(Tx, x), \tag{1.1}$$

where $F(T)$ is the set of fixed points of T, J is the duality mapping on E and

$$\phi(x, y)=\|x\|^{2}-2\langle x, Jy\rangle+\|y\|^{2}, \quad \forall x, y\in E.$$

Observe that, in a Hilbert space H, $\phi(x, y)=\|x-y\|^{2}$ for all $x, y\in H$. If $s=0$ in (1.1), then the mapping T is as follows:

$$2\langle x-qJx-JTx\rangle\geq(1-\eta)\phi(x, Tx), \quad \forall x\in C, \quad q\in F(T).$$

Such $(\eta, 0)$-demigeneralized mappings are important.

Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $0 \leq k < 1$. A mapping $U : C \to H$ is called a k-strict pseudo-contraction [5] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + k\|x - Ux - (y - Uy)\|^2$$

for all $x, y \in C$. If U is a k-strict pseudo-contraction and $F(U) \neq \emptyset$, then

$$\|Ux - q\|^2 \leq \|x - q\|^2 + k\|x - Ux\|^2$$

for all $x \in C$ and $q \in F(U)$. From this, we have that

$$\|Ux - x\|^2 + \|x - q\|^2 + 2\langle Ux - x, x - q\rangle \leq \|x - q\|^2 + k\|x - Ux\|^2.$$

Therefore, we get that

$$2\langle x - Ux, x - q\rangle \geq (1 - k)\|x - Ux\|^2 \tag{1.2}$$

for all $x \in C$ and $q \in F(U)$. Thus, from (1.2), a k-strict pseudo-contraction U with $F(U) \neq \emptyset$ is $(k, 0)$-demigeneralized. We also know that there exists such a mapping in a Banach space. Let E be a uniformly convex and smooth Banach space and let B be a maximal monotone operator on E. For each $r > 0$ and $x \in E$, we consider the following equation

$$Jx \in Jx_r + rBx_r.$$

This equation has a unique solution x_r. We define J_r by $x_r = J_r x$. Such a J_r is called the generalized resolvent of B. The set of null points of B is defined by $B^{-1}0 = \{z \in E : 0 \in Bz\}$. We know that $B^{-1}0$ is closed and convex; see [20]. The generalized resolvent has the following property: for any $x \in E$ and $q \in F(J_r) = \{z \in E : 0 \in Bz\}$,

$$2\langle J_r x - q, Jx - JJ_r x\rangle \geq 0.$$

Then we get

$$2\langle J_r x - x + x - q, Jx - JJ_r x\rangle \geq 0$$

and hence

$$\begin{aligned} 2\langle x - q, &Jx - JJ_r x\rangle \\ &\geq 2\langle x - J_r x, Jx - JJ_r x\rangle = \phi(x, J_r x) + \phi(J_r x, x). \end{aligned} \tag{1.3}$$

Thus, from (1.3), the generalized resolvent J_r with $B^{-1}0 \neq \emptyset$ is $(0, 1)$-demigeneralized.

In this article, we first deal with basic properties of new demigeneralized mappings. In particular, we prove that the common fixed point sets of new demigeneralized mappings are closed and convex. Using these results and the hybrid method introduced by Nakajo and Takahashi [14], we prove a strong convergence theorem which solves the split common fixed point problem in Banach Spaces. Furthermore, using the shrinking projection method introduced by Takahashi, Takeuchi and Kubota [28], we also prove another strong convergence theorem. Moreover, using these results, we obtain well-known and new strong convergence theorems in Hilbert spaces and Banach spaces.

2 Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. It is known that a Banach space E is uniformly convex if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. A uniformly convex Banach space is strictly convex and reflexive. We also know that a uniformly convex Banach space has the Kadec-Klee property, i.e., $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists. In this case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. The norm of E is said to be *Fréchet differentiable* if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. The norm of E is said to be *uniformly smooth* if the limit (2.1) is attained uniformly for $x, y \in U$. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [19] and [20]. We know the following result.

Lemma 2.1 ([19]). *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x-y, Jx-Jy \rangle \geq 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x-y, Jx-Jy \rangle = 0$, then $x = y$.*

Let E be a smooth Banach space and let J be the duality mapping on E. Define a function $\phi : E \times E \to \mathbb{R}$ by

$$\phi_E(x,y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \tag{2.2}$$

In the case when E is clear, ϕ_E is simply denoted by ϕ.

Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that

$$\phi(z,x) = \min_{y\in C} \phi(y,x).$$

The mapping $\Pi_C : E \to C$ defined by $z = \Pi_C x$ is called the generalized projection of E onto C. The following are well-known results. For example, see [1, 2, 7].

Lemma 2.2 ([1, 2, 7]). *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = \Pi_C x$;
(2) $\langle z - y, Jx - Jz \rangle \geq 0, \quad \forall y \in C$.

Let E be a Banach space and let A be a mapping of E into 2^{E^*}. The effective domain of A is denoted by $\mathrm{dom}(A)$, that is, $\mathrm{dom}(A) = \{x \in E : Ax \neq \emptyset\}$. A multi-valued mapping A on E is said to be monotone if $\langle x - y, u^* - v^* \rangle \geq 0$ for all $x, y \in \mathrm{dom}(A)$, $u^* \in Ax$, and $v^* \in Ay$. A monotone operator A on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to [4, 16]; see also [20, Theorem 3.5.4].

Theorem 2.3 ([4, 16]). *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let A be a monotone operator of E into 2^{E^*}. Then A is maximal if and only if for any $r > 0$, $R(J + rA) = E^*$, where $R(J + rA)$ is the range of $J + rA$.*

Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let B be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation $Jx \in Jx_r + rBx_r$. This equation has a unique solution x_r. We define J_r by $x_r = J_r x$. Such $J_r, r > 0$ are called the generalized resolvents of B. The set of null points of B is defined by $B^{-1}0 = \{z \in E : 0 \in Bz\}$. We know that $B^{-1}0$ is closed and convex; see [20].

Let E be a smooth and strictly convex Banach space and let J be the duality mapping on E. Let η and s be real numbers with $\eta \in (-\infty, 1)$ and $s \in [0, \infty)$. Then a mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called (η, s)-demigeneralized [29, 12] if, for any $x \in C$ and $q \in F(U)$,

$$2\langle x - q, Jx - JUx \rangle \geq (1 - \eta)\phi(x, Ux) + s\phi(Ux, x),$$

where $F(U)$ is the set of fixed points of U.

Examples.

(1) Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $0 \leq k < 1$. A mapping $U : C \to H$ is called a k-strict pseud-contraction [5] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + k\|x - Ux - (y - Uy)\|^2$$

for all $x, y \in C$. If U is a k-strict pseud-contraction and $F(U) \neq \emptyset$, then U is $(k, 0)$-demigeneralized.

(2) Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called generalized hybrid [8] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Ux - Uy\|^2 + (1 - \alpha)\|x - Uy\|^2 \leq \beta\|Ux - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. Such a mapping U is called $(\alpha,\ \beta)$-generalized hybrid. If U is generalized hybrid and $F(U) \neq \emptyset$, then U is $(0, 0)$-demigeneralized.

(3) Let E be a strictly convex, reflexive and smooth Banach space and let C be a nonempty, closed and convex subset of E. Let Π_C be the generalized projection of E onto C. Then Π_C is $(0,1)$-demigeneralized.

(4) Let E be a uniformly convex and smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Let $\lambda > 0$. Then the generalized resolvent J_λ is $(0,1)$-demigeneralized.

The following lemma is important and crucial in the proofs of our main results.

Lemma 2.4 ([29]). *Let E be a smooth and strictly convex Banach space and let C be a nonempty, closed and convex subset of E. Let η be a real number with $\eta \in (-\infty, 1)$. Let U be an $(\eta, 0)$-demigeneralized mapping of C into E. Then $F(U)$ is closed and convex.*

3 Main Results

In this section, using the hybrid method, we prove a strong convergence theorem for finding a solution of the split common fixed point problem for families of new nonlinear mappings in two Banach spaces. Let E be a Banach space and let C be a nonempty, closed and convex subset of E. Let $\{U_n\}$ be a sequence of mappings of C into E such that $\cap_{n=1}^{\infty} F(U_n) \neq \emptyset$. The sequence $\{U_n\}$ is said to satisfy the condition (I) if for any bounded sequence $\{z_n\}$ of C such that $\lim_{n\to\infty} \|z_n - U_n z_n\| = 0$, every weak cluster point of $\{z_n\}$ belongs to $\cap_{n=1}^{\infty} F(U_n)$.

Theorem 3.1 ([27]). *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let $\{\tau_n\}$ and $\{\eta_n\}$ be sequences of real numbers with $\tau_n, \eta_n \in (-\infty, 1)$ and let $\{t_n\}$ and $\{s_n\}$ be sequences of real numbers with $t_n, s_n \in [0, \infty)$. Let $\{T_n\}$ be a family of (τ_n, t_n)-demigeneralized mappings of E into itself with $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$ satisfying the condition (I) and let $\{U_n\}$ be a family of (η_n, s_n)-demigeneralized mappings of F into itself with $\cap_{n=1}^{\infty} F(U_n) \neq \emptyset$ satisfying the condition (I). Let $A : E \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $\mathcal{F} = \cap_{n=1}^{\infty} F(T_n) \cap A^{-1}(\cap_{n=1}^{\infty} F(U_n)) \neq \emptyset$. Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by*

$$
\begin{cases}
z_n = J_E^{-1}\big(J_E x_n - r_n A^*(J_F A x_n - J_F U_n A x_n)\big), \\
y_n = T_n z_n, \\
C_n = \{z \in E : 2\langle x_n - z, J_E x_n - J_E z_n\rangle \\
\qquad\qquad \geq r_n(1-\eta_n)\phi_F(Ax_n, U_n A x_n) + r_n s_n \phi_F(U_n A x_n, A x_n)\}, \\
D_n = \{z \in E : 2\langle z_n - z, J_E z_n - J_E y_n\rangle \\
\qquad\qquad \geq (1-\tau_n)\phi_E(z_n, y_n) + t_n \phi_E(y_n, z_n)\}, \\
Q_n = \{z \in E : \langle x_n - z, J_E x_1 - J_E x_n\rangle \geq 0\}, \\
x_{n+1} = \Pi_{C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N},
\end{cases}
$$

where $a, b \in \mathbb{R}$, $\{r_n\} \subset (0, \infty)$ and $\{\tau_n\}, \{\eta_n\} \subset (-\infty, 1)$ satisfy the following inequalities

$$0 < a \leq r_n \leq \frac{1}{2\|A\|^2} \quad \textit{and} \quad 0 < b \leq 1 - \tau_n, 1 - \eta_n, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to $z_0 \in \cap_{n=1}^{\infty} F(T_n) \cap A^{-1}(\cap_{n=1}^{\infty} F(U_n))$, where $z_0 = \Pi_{\cap_{n=1}^{\infty} F(T_n) \cap A^{-1}(\cap_{n=1}^{\infty} F(U_n))} x_1$.

Next, using the shrinking projection method [28], we prove a strong convergence theorem for finding a solution of the split common fixed point problem with families of mappings in Banach spaces.

For a sequence $\{C_n\}$ of nonempty, closed and convex subsets of a Banach space E, define $\text{s-Li}_n C_n$ and $\text{w-Ls}_n C_n$ as follows: $x \in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in \text{w-Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies

$$C_0 = \operatorname*{s-Li}_n C_n = \operatorname*{w-Ls}_n C_n, \tag{3.1}$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [13] and we write $C_0 = \text{M-lim}_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [13]. The following lemma was proved by Ibaraki, Kimura and Takahashi [6].

Lemma 3.2 ([6]). *Let E be a smooth Banach space such that E^* has a Fréchet differentiable norm. Let $\{C_n\}$ be a sequence of nonempty, closed and convex subsets of E. If $C_0 =$M-$\lim_{n\to\infty} C_n$ exists and nonempty, then for each $x \in E$, $\{\Pi_{C_n} x\}$ converges strongly to $\Pi_{C_0} x$, where Π_{C_n} and Π_{C_0} are the generalized projections of E onto C_n and C_0, respectively.*

Theorem 3.3 ([27]). *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let $\{\tau_n\}$ and $\{\eta_n\}$ be sequences of real numbers with $\tau_n, \eta_n \in (-\infty, 1)$ and let $\{t_n\}$ and $\{s_n\}$ be sequences of real numbers with $t_n, s_n \in [0, \infty)$. Let $\{T_n\}$ be a family of (τ_n, t_n)-demigeneralized mappings of E into itself with $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$ satisfying the condition (I) and let $\{U_n\}$ be a family of (η_n, s_n)-demigeneralized mappings of F into itself with $\cap_{n=1}^{\infty} F(U_n) \neq \emptyset$ satisfying the condition (I). Let $A : E \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $\mathcal{F} = \cap_{n=1}^{\infty} F(T_n) \cap A^{-1}(\cap_{n=1}^{\infty} F(U_n)) \neq \emptyset$. For $x_1 \in E$ and $C_1 = E$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_E^{-1}\big(J_E x_n - r_n A^*(J_F A x_n - J_F U_n A x_n)\big), \\ y_n = T_n z_n, \\ C_{n+1} = \{z \in C_n : 2\langle x_n - z, J_E x_n - J_E z_n\rangle \\ \qquad\qquad \geq r_n(1-\eta_n)\phi_F(Ax_n, U_n A x_n) + r_n s_n \phi_F(U_n A x_n, A x_n)\} \\ \qquad \cap \{z \in C_n : 2\langle z_n - z, J_E z_n - J_E y_n\rangle \\ \qquad\qquad \geq (1-\tau_n)\phi_E(z_n, y_n) + t_n \phi_E(y_n, z_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b \in \mathbb{R}$ and $\{r_n\} \subset (0, \infty)$ satisfy the following inequalities

$$0 < a \leq r_n \leq \frac{1}{2\|A\|^2} \quad \text{and} \quad 0 < b \leq 1 - \tau_n, 1 - \eta_n, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to $z_0 \in \cap_{n=1}^{\infty} F(T_n) \cap A^{-1}(\cap_{n=1}^{\infty} F(U_n))$, where $z_0 = \Pi_{\cap_{n=1}^{\infty} F(T_n) \cap A^{-1}(\cap_{n=1}^{\infty} F(U_n))} x_1$.

4 Applicationss

In this section, using Theorems 3.1 and 3.3, we get well-known and new strong convergence theorems which are connected with the split common fixed point problem for families of demigeneralized mappings in Banach spaces. We know the following result obtained by Marino and Xu [11].

Lemma 4.1 ([11]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $0 \leq k < 1$ and $U : C \to H$ be a k-strict pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

We also know the following result from Plubtieng and Takahashi [15].

Lemma 4.2 ([15]). *Let H_1 and H_2 be Hilbert spaces and let $\alpha > 0$. Let $G : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I + \lambda G)^{-1}$ be the resolvent of G for $\lambda > 0$. Let $T : H_2 \to H_2$ be an α-inverse strongly monotone mapping and let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $G^{-1}0 \cap A^{-1}(T^{-1}0) \neq \emptyset$. Let $\lambda, r > 0$ and $z \in H_1$. Then the following are equivalent:*

(i) $z = J_\lambda(I - rA^*TA)z$;
(ii) $0 \in A^*TAz + Gz$;
(iii) $z \in G^{-1}0 \cap A^{-1}(T^{-1}0)$.

Using Theorem 3.1 and Lemmas 4.1 and 4.2, we obtain the following theorem.

Theorem 4.3. *Let H_1 and H_2 be Hilbert spaces. Let $k \in [0, 1)$ and $\alpha \in (0, \infty)$. Let $T : H_1 \to H_1$ be an α-inverse strongly monotone mapping with $T^{-1}0 \neq \emptyset$ and let $U : H_2 \to H_2$ be a k-strict pseudo-contraction such that $F(U) \neq \emptyset$. Let G be a maximal monotone operator of H_1 into H_1 and J_λ be the resolvent of G for $\lambda > 0$. Define $T_n = J_{\lambda_n}(I - \lambda_n T)$ for all $n \in \mathbb{N}$ such that $\lambda_n \in (0, \infty)$ and define $U_n = \alpha_n I + (1 - \alpha_n)U$ for all $n \in \mathbb{N}$ such that $0 \leq \alpha_n < 1$ and $\sup_{n\in\mathbb{N}} \alpha_n < 1$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $T^{-1}0 \cap G^{-1}0 \cap A^{-1}F(U) \neq \emptyset$. Let $x_1 \in H_1$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = x_n - r_n A^*(Ax_n - U_n Ax_n), \\ y_n = T_n z_n, \\ C_n = \{z \in H_1 : 2\langle x_n - z, x_n - z_n\rangle \geq r_n(1-k)\|Ax_n - U_n Ax_n\|^2\}, \\ D_n = \{z \in H_1 : 2\langle z_n - z, z_n - y_n\rangle \geq \|z_n - y_n\|^2\}, \\ Q_n = \{z \in H_1 : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\}, \{\lambda_n\} \subset (0, \infty)$ and $a, b \in \mathbb{R}$ satisfy the following inequalities

$$0 < a \leq r_n \leq \frac{1}{2\|A\|^2} \quad \text{and} \quad 0 < b \leq \lambda_n \leq 2\alpha, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in T^{-1}0 \cap G^{-1}0 \cap A^{-1}F(U)$, where $z_0 = P_{T^{-1}0 \cap G^{-1}0 \cap A^{-1}F(U)} x_1$.

Using Theorem 3.1, we get the following result [24].

Theorem 4.4 ([24]). *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let G and B be maximal monotone operators of E into E^* and F into F^*, respectively. Let J_λ and Q_μ be the generalized resolvents of G for $\lambda > 0$ and B for $\mu > 0$, respectively. Let $A : E \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $G^{-1}0 \cap A^{-1}(B^{-1}0) \neq \emptyset$. Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_E^{-1}\big(J_E x_n - r_n A^*(J_F A x_n - J_F Q_{\mu_n} A x_n)\big), \\ y_n = J_{\lambda_n} z_n, \\ C_n = \{z \in E : 2\langle x_n - z, J_E x_n - J_E z_n\rangle \\ \qquad\qquad \geq r_n \phi_F(Ax_n, Q_{\mu_n} A x_n) + r_n \phi_F(Q_{\mu_n} A x_n, A x_n)\}, \\ D_n = \{z \in E : 2\langle z_n - z, J_E z_n - J_E y_n\rangle \\ \qquad\qquad \geq \phi_E(z_n, y_n) + \phi_E(y_n, z_n)\}, \\ Q_n = \{z \in E : \langle x_n - z, J_E x_1 - J_E x_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b \in \mathbb{R}$ and $\{r_n\}$, $\{\lambda_n\}, \{\mu_n\} \subset (0, \infty)$ satisfy the following inequalities

$$0 < a \leq r_n \leq \frac{1}{2\|A\|^2}, \quad \text{and} \quad 0 < b \leq \lambda_n, \mu_n, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $z_0 \in G^{-1}0 \cap A^{-1}(B^{-1}0)$, where $z_0 = \Pi_{G^{-1}0 \cap A^{-1}(B^{-1}0)} x_1$.

Similarly, using Theorem 3.3, we have the following results.

Theorem 4.5. *Let H_1 and H_2 be Hilbert spaces. Let $k \in [0, 1)$ and $\alpha \in (0, \infty)$. Let $T : H_1 \to H_1$ be an α-inverse strongly monotone mapping with $T^{-1}0 \neq \emptyset$ and let $U : H_2 \to H_2$ be a k-strict pseudo-contraction such that $F(U) \neq \emptyset$. Let G be a maximal monotone operator of H_1 into H_1 and J_λ be the resolvent of G for $\lambda > 0$. Define $T_n = J_{\lambda_n}(I - \lambda_n T)$ for all $n \in \mathbb{N}$ such that $\lambda_n \in (0, \infty)$ and define $U_n = \alpha_n I + (1 - \alpha_n)U$ for all $n \in \mathbb{N}$ such that $0 \leq \alpha_n < 1$ and $\sup_{n \in \mathbb{N}} \alpha_n < 1$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $T^{-1}0 \cap G^{-1}0 \cap A^{-1}F(U) \neq \emptyset$. For $x_1 \in H_1$ and $C_1 = H_1$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = x_n - r_n A^*(Ax_n - U_n A x_n), \\ y_n = T_n z_n, \\ C_{n+1} = \{z \in C_n : 2\langle x_n - z, x_n - z_n\rangle \geq r_n(1 - \eta_n)\|Ax_n - U_n A x_n\|^2 \\ \qquad\qquad \text{and} \quad 2\langle z_n - z, z_n - y_n\rangle \geq (1 - \tau_n)\|z_n - y_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\}, \{\lambda_n\} \subset (0, \infty)$ and $a, b \in \mathbb{R}$ satisfy the following inequalities

$$0 < a \leq r_n \leq \frac{1}{2\|A\|^2} \quad \text{and} \quad 0 < b \leq \lambda_n \leq 2\alpha, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in T^{-1}0 \cap G^{-1}0 \cap A^{-1}F(U)$, where $z_0 = P_{T^{-1}0 \cap G^{-1}0 \cap A^{-1}F(U)} x_1$.

Furthermore, using Theorem 3.3, we get the following result [17].

Theorem 4.6 ([17]). *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let G and B be maximal monotone operators of E into E^* and F into F^*, respectively. Let J_λ and Q_μ be the generalized resolvents of G for $\lambda > 0$ and B for $\mu > 0$, respectively. Let $A : E \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $G^{-1}0 \cap A^{-1}(B^{-1}0) \neq \emptyset$. For $x_1 \in E$ and $C_1 = E$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_E^{-1}\big(J_E x_n - r_n A^*(J_F A x_n - J_F Q_{\mu_n} A x_n)\big), \\ y_n = J_{\lambda_n} z_n, \\ C_{n+1} = \{z \in C_n : 2\langle x_n - z, J_E x_n - J_E z_n\rangle \\ \qquad\qquad\qquad \geq r_n \phi_F(Ax_n, Q_{\mu_n} A x_n) + r_n \phi_F(Q_{\mu_n} A x_n, A x_n) \\ \qquad\qquad \text{and} \quad 2\langle z_n - z, J_E z_n - J_E y_n\rangle \\ \qquad\qquad\qquad\qquad \geq \phi_E(z_n, y_n) + \phi_E(y_n, z_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b \in \mathbb{R}$ and $\{r_n\}$, $\{\lambda_n\}, \{\mu_n\} \subset (0, \infty)$ satisfy the following inequalities

$$0 < a \leq r_n \leq \frac{1}{2\|A\|^2} \quad \text{and} \quad 0 < b \leq \lambda_n, \mu_n, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $z_0 \in G^{-1}0 \cap A^{-1}(B^{-1}0)$, where $z_0 = \Pi_{G^{-1}0 \cap A^{-1}(B^{-1}0)} x_1$.

Acknowledgements. The author was partially supported by Grant-in-Aid for Scientific Research No. 15K04906 from Japan Society for the Promotion of Science.

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