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Strong solvability of the Stokes and Navier-Stokes equations in weak L^{n} space

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Abstract

In this résumé we investigate the strong soluvability of the Stokes and the Navier Stokes equations in weak L^{n}-space, where the Stokes semigroup is analytic but not strongly continuous at $t=0$. More precisely, the local in time strong solvability is concerned. To construct a strong solution of the Naiver-Stokes equations in weak L^{n}-space, we clarify the condition on the external forces, which is inherited from the strong solvability of the inhomogeneous Stokes equations.

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1 Introduction

Let $n\geq 3$. We consider the initial value problem of the incompressible Naiver-Stokes equations in the whole space $\mathbb{R}^{n}$.

$$\begin{cases}\partial_{t}u-\Delta u+(u\cdot\nabla)u+\nabla\pi=f & \text{in } \mathbb{R}^{n}\times(0,\infty),\\ \operatorname{div} u=0 & \text{in } \mathbb{R}^{n}\times(0,\infty),\\ u(\cdot,0)=a & \text{in } \mathbb{R}^{n}.\end{cases} \tag{N-S}$$

Here, $u=u(x,t)=\big(u_{1}(x,t),\ldots,u_{n}(x,t)\big)$ and $\pi=\pi(x,t)$ are the unknown velocity and the pressure of the incompressible fluid, respectively, $a=a(x)=(a_{1}(x),\ldots,a_{n}(x))$ and$f=f(x,t)=\big(f_{1}(x,t),\ldots,f_{n}(x,t)\big)$ are the given initial data and the external force, respectively.

The aim and the background are to prove the strong solvability of the time periodic problem of (N-S), instead of the initial value problem of (N-S). Indeed, in [9] we construct a mild solution of (N-S) in $BC(\mathbb{R};L^{n,\infty}(\mathbb{R}^{n}))$ by the real interpolation approach so-called Meyer's method, see Meyer [8]. Let $\mathbb{P}$ be the Fujita-Kato projection and $L_{\sigma}^{n,\infty}(\mathbb{R}^{n})=\mathbb{P}L^{n,\infty}(\mathbb{R}^{n})$.

Theorem 1.1 ([9]). *(i) Let $n\geq 4$. There exists $\varepsilon_{n}>0$ with the following property. Suppose that $f\in BC\big(\mathbb{R};L^{\frac{n}{3},\infty}(\mathbb{R}^{n})\big)$ satisfies $f(t)=f(t+\omega)$ for all $t\in\mathbb{R}$ with some period $\omega>0$. If*

$$\sup_{t\in\mathbb{R}}\|f(t)\|_{\frac{n}{3},\infty}<\varepsilon_{n}$$

then there exists a time periodic solution u of

$$u(t) = \int_{-\infty}^{t} e^{(t-s)\Delta}\mathbb{P}f(s)\,ds - \int_{-\infty}^{t} e^{(t-s)\Delta}\mathbb{P}u\cdot\nabla u(s)\,ds, \qquad t \in \mathbb{R}, \tag{IE}$$

with the same period as f such that $u \in BC\big(\mathbb{R};\, L^{n,\infty}_{\sigma}(\mathbb{R}^n)\big)$ *with* $\nabla u \in BC\big(\mathbb{R};\, L^{\frac{n}{2},\infty}(\mathbb{R}^n)\big)$.

Moreover, for $\frac{n}{3} < p < \infty$, *there exists* $\varepsilon_{n,p} > 0$ *with* $\varepsilon_{n,p} \le \varepsilon_n$ *such that if f additionally belongs to* $BC\big(\mathbb{R};\, L^{p,\infty}(\mathbb{R}^n)\big)$ *and satisfies*

$$\sup_{t\in\mathbb{R}} \|f(t)\|_{\frac{n}{3},\infty} < \varepsilon_{n,p},$$

then the solution u of (IE), obtained above, also satisfies

$$u \in BC\big(\mathbb{R};\, L^{r,\infty}_{\sigma}(\mathbb{R}^n)\big) \quad \text{and} \quad \nabla u \in BC\big(\mathbb{R};\, L^{q,\infty}(\mathbb{R}^n)\big),$$

where the exponents r and q satisfy

$$\begin{cases} n \le r \le \dfrac{np}{n-2p} & \text{if } \ p < \dfrac{n}{2}, \\ n \le r < \infty & \text{if } \ \dfrac{n}{2} \le p, \end{cases} \qquad \begin{cases} \dfrac{n}{2} \le q \le \dfrac{np}{n-p} & \text{if } \ p < n, \\ \dfrac{n}{2} \le q < \infty & \text{if } \ n \le p. \end{cases}$$

(ii) Let $n = 3$. *There exists* $\varepsilon_3 > 0$ *with the following property. Suppose that* $f \in BC\big(\mathbb{R};\, L^1(\mathbb{R}^3)\big)$ *satisfies* $f(t) = f(t+\omega)$ *for* $t \in \mathbb{R}$ *with some period* $\omega > 0$. *If*

$$\sup_{t\in\mathbb{R}} \|f(t)\|_1 < \varepsilon_3,$$

then there exists time periodic function u in $BC\big(\mathbb{R};\, L^{n,\infty}_{\sigma}(\mathbb{R}^3)\big)$ *with the same period* ω *such that*

$$u(t) = \int_{-\infty}^{t} \mathbb{P}e^{(t-s)\Delta}f(s)\,ds - \int_{-\infty}^{t} \nabla\cdot e^{(t-s)\Delta}\mathbb{P}(u\otimes u)(s)\,ds, \qquad t \in \mathbb{R}. \tag{IE*}$$

Moreover, for $1 < p < \infty$ *there exists* $\varepsilon_{3,p} > 0$ *with* $\varepsilon_{3,p} \le \varepsilon_3$ *such that if f additionally belongs to* $BC\big(\mathbb{R};\, L^{p,\infty}(\mathbb{R}^3)\big)$ *and satisfies*

$$\sup_{t\in\mathbb{R}} \|f(t)\|_1 < \varepsilon_{3,p}$$

then the solution u of (IE), obtained above, satisfies (IE) and also satisfies*

$$u \in BC\big(\mathbb{R};\, L^{r,\infty}_{\sigma}(\mathbb{R}^3)\big) \quad \text{and} \quad \nabla u \in BC\big(\mathbb{R};\, L^{q,\infty}(\mathbb{R}^3)\big),$$

where the exponents r and q satisfy

$$\begin{cases} 3 \le r \le \dfrac{3p}{3-2p} & \text{if } \ 1 < p < \dfrac{3}{2}, \\ 3 \le r < \infty & \text{if } \ \dfrac{3}{2} \le p, \end{cases} \qquad \begin{cases} \dfrac{3}{2} < q \le \dfrac{3p}{3-p} & \text{if } \ 1 < p < 3, \\ \dfrac{3}{2} < q < \infty & \text{if } \ 3 \le p. \end{cases}$$

Here, we note that Yamazaki [11] is firstly obtained the time periodic solution in $L^{n,\infty}(\Omega)$ of (N-S) with weak-mild form. In [11], the regularity and strong solvability is discussed in terms of the topology of some *sum space* of the Sobolev spaces with negative differentiability. So we discuss the strong solvability of (N-S) in the topology of $L^{n,\infty}(\mathbb{R}^n)$. Since the Stokes (the heat) semigroup on $L^{n,\infty}(\mathbb{R}^n)$ is not strongly continuous, we may not expect the strong solvability of the Stokes equations for each f. So we introduce the restriction on the external forces, not on initial data, as follows:

$$\lim_{\varepsilon\to 0}\|e^{\varepsilon\Delta}f(t)-f(t)\|_{n,\infty}=0 \quad \text{for each } t, \tag{A}$$

Indeed with the condition (A), we obtain the following theorem.

Theorem 1.2 ([9]). *Let $n\geq 3$. Suppose that $f\in BC\big(\mathbb{R};L^{n,\infty}(\mathbb{R}^n)\big)$ and that $u\in BC\big(\mathbb{R};L^{n,\infty}_\sigma(\mathbb{R}^n)\big)$ is a time periodic solution of (IE) which satisfies $u\in BC\big(\mathbb{R};L^r_\sigma(\mathbb{R}^n)\big)$ with some $r>n$ and $\nabla u\in BC\big(\mathbb{R};L^{q,\infty}(\mathbb{R}^n)\big)$ with some $q\geq \frac{n}{2}$. If $\mathbb{P}f$ is Hölder continuous on $\mathbb{R}$ with values in $L^{n,\infty}(\mathbb{R}^n)$, and if $\mathbb{P}f$ satisfies (A), then the periodic solution u satisfies the following properties,*

(i) $u\in BC\big(\mathbb{R};L^{n,\infty}_\sigma(\mathbb{R}^n)\big)\cap C^1\big(\mathbb{R};L^{n,\infty}(\mathbb{R}^n)\big)$,

(ii) $u(t)\in\{u\in L^{n,\infty}_\sigma(\mathbb{R}^n);\,\partial_j\partial_k u\in L^{n,\infty}(\mathbb{R}^n),\ j,k=1,\dots,n\}$ *for all* $t\in\mathbb{R}$ *and* $\Delta u\in C\big(\mathbb{R};L^{n,\infty}(\mathbb{R}^n)\big)$,

(iii) u *satisfies*

$$\frac{du}{dt}(t)-\Delta u(t)+\mathbb{P}[u\cdot\nabla u](t)=\mathbb{P}f(t) \qquad \text{in } L^{n,\infty}_\sigma(\mathbb{R}^n), \quad t\in\mathbb{R}^n.$$

For the proof of Theorem 1.2, the local in time existence theorem plays an important role. For such a direction, Kozono-Yamazaki [6] construct a local in time strong solution of (N-S) in the sum space $L^{n,\infty}(\Omega)+L^r(\Omega)$, $r>n$. On the other hand, we try to construct a local solution which satisfies the differential equation of (N-S) in the topology of $L^{n,\infty}(\mathbb{R}^n)$.

2 Result

Before stating our results, we introduce the following notations and some function spaces. Let $C^\infty_{0,\sigma}(\mathbb{R}^n)$ denotes the set of all C^∞-solenoidal vectors ϕ with compact support in $\mathbb{R}^n$, i.e., $\operatorname{div}\phi=0$ in $\mathbb{R}^n$. $L^r_\sigma(\mathbb{R}^n)$ is the closure of $C^\infty_{0,\sigma}(\mathbb{R}^n)$ with respect to the L^r-norm $\|\cdot\|_r$, $1<r<\infty$. $(\cdot,\cdot)$ is the duality pairing between $L^r(\mathbb{R}^n)$ and $L^{r'}(\mathbb{R}^n)$, where $1/r+1/r'=1$, $1\leq r\leq\infty$. $L^r(\mathbb{R}^n)$ and $W^{m,r}(\mathbb{R}^n)$ denote the usual (vector-valued) L^r-Lebesgue space and L^r-Sobolev space over $\mathbb{R}^n$, respectively. Moreover, $\mathcal{S}(\mathbb{R}^n)$ denotes the set of all of the Schwartz functions. $\mathcal{S}'(\mathbb{R}^n)$ denotes the set of all tempered distributions. When X is a Banach space, $\|\cdot\|_X$ denotes the norm on X. Moreover, $C(I;X)$, $BC(I;X)$ and $L^r(I;X)$ denote the X-valued continuous and bounded continuous functions over the interval $I\subset\mathbb{R}$ and X-valued L^r functions, respectively.

Moreover, for $1 < p < \infty$ and $1 \le q \le \infty$ let $L^{p,q}(\mathbb{R}^n)$ be the space of all locally integrable functions with (quasi) norm $\|f\|_{p,q} < \infty$, where

$$\|f\|_{p,q} = \begin{cases} \left(\int_0^\infty \left(\lambda \,|\{x \in \mathbb{R}^n\,;\, |f(x)| > \lambda\}|^{\frac{1}{p}}\right)^q \frac{d\lambda}{\lambda}\right)^{\frac{1}{q}}, & 1 \le q < \infty, \\ \sup_{\lambda>0} \lambda \,|\{x \in \mathbb{R}^n\,;\, |f(x)| > \lambda\}|^{\frac{1}{p}}, & q = \infty, \end{cases}$$

where $|E|$ denotes the Lebesgue measure of $E \subset \mathbb{R}^n$. For the case $q = \infty$, $L^{p,\infty}(\mathbb{R}^n)$ is a Banach space with the following norm: with any $1 \le r < p$

$$\|f\|_{L^{p,\infty}} = \sup_{0<|E|<\infty} |E|^{-\frac{1}{r}+\frac{1}{p}} \left(\int_E |f(x)|^r \,dx\right)^{\frac{1}{r}}.$$

Here, we note that $\|\cdot\|_{L^{n,\infty}}$ is equivalent to $\|\cdot\|_{n,\infty}$.

To construct a local solution of (N-S), we introduce the following function spaces.

$$\widetilde{L}^{n,\infty}_\sigma(\mathbb{R}^n) = \overline{L^{n,\infty}_\sigma(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)}^{\|\cdot\|_{n,\infty}} \quad \text{and} \quad L^{n,\infty}_{0,\sigma}(\mathbb{R}^n) = \overline{\{\phi \in C_0^\infty(\mathbb{R}^n); \operatorname{div}\phi = 0\}}^{\|\cdot\|_{n,\infty}}$$

See, Taniuchi [10] and Koba [5].

Theorem 2.1. *Let $a \in \widetilde{L}^{n,\infty}_\sigma(\mathbb{R}^n)$ and $f \in BC([0,\infty); L^{n,\infty}(\mathbb{R}^n))$. Suppose f is Hölder continuous on $[0,\infty)$ with value in $L^{n,\infty}(\mathbb{R}^n)$ and satisfies (A). There are $T > 0$ and a function $u \in BC((0,T); L^{n,\infty}_\sigma(\mathbb{R}^n))$ with $\nabla t^{1/2} u \in BC((0,T); L^{n,\infty}(\mathbb{R}^n))$ which satisfies*

(i) $u \in BC\big((0,T)\,;\, L^{n,\infty}_\sigma(\mathbb{R}^n)\big) \cap C^1\big((0,T)\,;\, L^{n,\infty}_\sigma(\mathbb{R}^n)\big)$,

(ii) $u(t) \in \{u \in L^{n,\infty}_\sigma(\mathbb{R}^n)\,;\, \partial_j\partial_k u \in L^{n,\infty}(\mathbb{R}^n),\ j,k = 1,\dots,n\}$ *for all* $t \in (0,T)$ *and* $\Delta u \in C\big((0,T)\,;\, L^{n,\infty}(\mathbb{R}^n)\big)$,

(iii) u *satisfies*

$$\begin{cases} \dfrac{du}{dt}(t) - \Delta u(t) + \mathbb{P}[u\cdot\nabla u](t) = \mathbb{P}f(t) & \text{in } L^{n,\infty}_\sigma(\mathbb{R}^n), \quad t \in (0,T), \\ u(t) \rightharpoonup a \quad \text{weakly} * \text{ in } L^{n,\infty}_\sigma(\mathbb{R}^n) & \text{as } t \searrow 0. \end{cases}$$

Moreover, if $a \in L^{n,\infty}_\sigma(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$ for some $r > n$, then the existence time $T > 0$ is expressed as

$$T \ge \min\left\{1, \left(\frac{\eta_*}{\|a\|_r + \sup_{0<s<\infty} \|\mathbb{P}f(s)\|_{n,\infty}}\right)^{\frac{2r}{r-n}}\right\},$$

with some absolute constant $\eta_* > 0$.

Remark 2.1. (i) if $a \in L^{n,\infty}_\sigma(\mathbb{R}^n)$ and $f \in L^1(0,\infty; L^{n,\infty}(\mathbb{R}^n)) \cap BC([0,\infty); L^{n,\infty}(\mathbb{R}^n))$ satisfy

$$\|a\|_{n,\infty} + \|\mathbb{P}f\|_{L^1(0,\infty;L^{n,\infty})} + \sup_{s\in\mathbb{R}} s\|\mathbb{P}f(s)\|_{n,\infty} \ll 1,$$

then we can take $T = \infty$.

(ii) Along to Koba [5], if $a \in L^{n,\infty}_{0,\sigma}(\mathbb{R}^n)$ we also see that $\lim_{t\to 0} \|u(t) - a\|_{n,\infty} = 0$. Moreover if $a \in L^{n,\infty}_{0,\sigma}(\mathbb{R}^n)$ is small enough and $f \equiv 0$ then $\lim_{t\to\infty} \|u(t)\|_{n,\infty} = 0$.

3 Key lemma

In this subsection, we reconstruct a theory of abstract evolution equations with the semigroup which is not strongly continuous at $t = 0$. Indeed, the Stokes semigroup is not strongly continuous on $L^{r,\infty}_\sigma(\mathbb{R}^n)$.

For a while, let A be a general closed operator on a Banach space X and $\{e^{tA}\}$ a bounded and analytic on X with the estimates

$$\sup_{0<t<\infty} \|e^{tA}\|_{\mathcal{L}(X)} \le N, \quad \|Ae^{tA}\|_{\mathcal{L}(X)} \le \frac{M}{t}, \quad t > 0, \tag{3.1}$$

where $\mathcal{L}(X)$ is the space of all bounded linear operators on X equipped with the operator norm. Especially, we note that e^{tA} is strongly continuous in X for $t \neq 0$.

Definition 3.1. *Let $\theta \in (0,1]$. We call f is the Hölder continuous on $[0,\infty)$ with value in X with the order θ, if for every $T > 0$ there exists $K_T > 0$ such that*

$$\|f(t) - f(s)\|_X \le K_T |t-s|^\theta, \qquad 0 \le t \le T, 0 \le s \le T.$$

Assumption. *Let $f : [0,\infty) \to X$. We assume for every $t > 0$*

$$\lim_{\varepsilon \searrow 0} \|e^{\varepsilon A} f(t) - f(t)\|_X = 0. \tag{A}$$

Lemma 3.1. *Let $a \in X$ and let $f \in C\big([0,\infty)\,; X\big)$ be the Hölder continous on $[0,\infty)$ with value in X with order $\theta > 0$ and satisfy Assumption. Then*

$$u(t) = e^{tA}a + \int_0^t e^{(t-s)A} f(s)\, ds$$

satisfies

$$\frac{d}{dt}u = Au + f \quad in\ X \quad t > 0.$$

Remark 3.1. We note that we need a restriction only on the external force f not on initial data a. Moreover, Lemma 3.1 does not focus on the verification of the initial condition. If we have some information of the adjoint operator A^* and of the dual space X^*, then we recover the verification of the initial condition with a suitable sense.

4 Outline of proof

The proof of Theorem 2.1 is fulfilled by the standard iteration method developed by Fujita and Kato [1], Kato [4], Giga and Miyakawa [3] and Giga [2]. The difficulty to construct a local in time mild solution comes from the lack of the density of $C^\infty_{0,\sigma}(\mathbb{R}^n)$ in $L^{n,\infty}_\sigma(\mathbb{R}^n)$. For this reason, we restrict initial data within $\widetilde{L}^{n,\infty}_\sigma(\mathbb{R}^n)$. Then once we obtain a local in time solution in a suitable function spaces, Lemma 3.1 guarantees the mild solution is a strong solution, i.e., satisfies the differential equations of (N-S), since it is not difficult to see that the nonlinear term satisfies the assumption (A) by the regularity of the mild solution.

5 Application

As is mentioned in the previous section, our motivation is to prove the strong solvability of the time periodic problem of (N-S), see [9]. For this purpose, to construct a local strong solution and the uniqueness theorem of the mild solution of (N-S) are essential. So we introduce the uniqueness theorem in weak L^n space proved by Kozono and Yamazaki [7].

Theorem 5.1 ([7]). *Let $n < r < \infty$. Then there exists a constant $\kappa = \kappa(n, r) > 0$ with the following property. Let $a \in L^{n,\infty}_\sigma(\mathbb{R}^n) \cap L^r_\sigma(\mathbb{R}^n)$. Suppose v is the mild solution on $[0, T)$ of (N-S) obtained by Theorem 2.1. Suppose w is also a mild solution on $[0, T)$ of (N-S) which satisfies $t^{\frac{1}{2}-\frac{n}{2r}} w \in BC\big((0, T)\,;\, L^r(\mathbb{R}^n)\big)$. If*

$$\limsup_{t\to 0} t^{\frac{1}{2}-\frac{n}{2r}} \|w(t)\|_r \leq \kappa \tag{5.1}$$

then $v \equiv w$ on $(0, T)$.

Then we only give the sketch of proof of Theorem 1.2. Firstly, we construct a mild solution of the time periodic solution of (N-S) with suitable regularity. Then solve the initial value problem of (N-S) where the initial state is the point on the periodic orbit. Finally, by the uniqueness theorem, we may conclude the time periodic mild solution satisfies the differential equation of (N-S).

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