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LIOUVILLE-TYPE THEOREMS FOR THE STATIONARY AND NONSTATIONARY NAVIER-STOKES EQUATIONS

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1. Introduction

In this survey article, we review the recent results [6], [7] by the authors on Liouville-type theorems for both the stationary and nonstationary Navier-Stokes equations. Let us first state what Liouville-type theorems for the stationary Navier-Stokes equations are. Let us consider the 3D homogeneous Navier-Stokes equations in the whole space $\mathbb{R}^3$;

$$(1.1)\qquad \begin{cases} -\Delta v+(v\cdot\nabla)v+\nabla p=0 \quad \text{in } \mathbb{R}^3, \\ \operatorname{div} v=0 \quad \text{in } \mathbb{R}^3, \end{cases}$$

where $v=v(x)=(v_1(x),v_2(x),v_3(x))$ and $p=p(x)$ denote the velocity vector and the scalar pressure at the point $x=(x_1,x_2,x_3)\in\mathbb{R}^3$, respectively. We deal with solutions v of (1.1) in the class of the finite Dirichlet integral

$$(1.2)\qquad D(v)\equiv\int_{\mathbb{R}^3}|\nabla v(x)|^2dx<\infty$$

with the homogeneous condition at infinity

$$(1.3)\qquad \lim_{|x|\to\infty}|v(x)|=0 \quad \text{uniformly in } x.$$

Since the pioneer work of Leray [9], it has been an open problem whether $v\equiv 0$ is the only solution of (1.1) under conditions (1.2) and (1.3). This is a Liouville-type statement on the 3D stationary Navier-Stokes equations. A partial answer under some further restrictions is called a Liouville-type theorem for the stationary 3D Navier-Stokes equations. Liouville-type theorems were obtained by several authors up to now. Galdi [4, Theorem X.9.5] showed that if $v\in L^{9/2}(\mathbb{R}^3)$, then it holds that $v\equiv 0$. Chae [2, Theorem 1.2] proved that the condition $\Delta v\in L^{6/5}(\mathbb{R}^3)$ implies that (1.1) has the only trivial solution. He emphasizes that the norm D^2v in $L^{6/5}(\mathbb{R}^3)$ corresponds to that of ∇v in $L^2(\mathbb{R}^3)$ at the level of scaling and that there is no mutual implication relation between their results [4] and [2]. So, it seems to be an interesting question to investigate why the space $L^{9/2}(\mathbb{R}^3)$ necessarily appears from the viewpoint of scaling. On the other hand, recently, Seregin [13] showed that $v\in L^6(\mathbb{R}^3)\cap BMO^{-1}$ leads to the Liouville-type theorem.

It is well-known that if $\{v,p\}$ solves (1.1), so does $\{v_\lambda,p_\lambda\}$ for all $\lambda>0$, where $v_\lambda(x)=\lambda v(\lambda x)$ and $p_\lambda(x)=\lambda^2p(\lambda x)$. A standard method to prove that (1.1) possesses the only trivial solution $v\equiv 0$ is to bound $D(v)$ by means of the quantity of v or ∇v at infinity. For that purpose, Galdi [4] derived such an estimate as

$$\int_{|x|\le R} |\nabla v(x)|^2 dx \le C\|v\|^3_{L^{\frac{9}{2}}(R\le|x|\le 2R)} + CR^{-\frac{1}{3}}\|v\|^2_{L^{\frac{9}{2}}(R\le|x|\le 2R)} + C\|v\|_{L^{\frac{9}{2}}(R\le|x|\le 2R)}\|p\|_{L^{\frac{9}{4}}(R\le|x|\le 2R)} \tag{1.4}$$

for all $R>0$ with a constant C independent of $R>0$, which yields that $v\equiv 0$ is the only solution of (1.1) with (1.2) and (1.3) provided $v\in L^{9/2}(\mathbb{R}^3)$. It is easy to see that both $D(v)$ and each term in the right-hand side have the same scaling with respect to the transformation v_λ for all $\lambda>0$. On the other hand, Chae [2] established an estimate of $p+|v|^2/2$ by a skillful technique which does not need any other bound except for $\|\Delta v\|_{L^{\frac{6}{5}}(\mathbb{R}^3)}$ having the same scaling as $D(v)^{1/2}$.

We shall first establish an a priori estimate of $D(v)$ in terms of the asymptotic behavior of the vortex $\omega = \operatorname{rot} v$ as $|x|\to\infty$. Our estimate is invariant under the change of scaling. As an application, it turns out that if $\omega(x) = o(|x|^{-5/3})$ as $|x|\to\infty$, then $v\equiv 0$ is the only solution of (1.1) with (1.2) and (1.3). In view of the decay rate at the infinity, our result extends Galdi's one and is not included by the previous results such as [2], [3] or [13]. Concerning v itself, introducing the Lorentz space $L^{q,r}(\mathbb{R}^3)$, we extend the result of Galdi [4] to that in the weak-$L^{9/2}$ space, which is based on an a priori estimate of $D(v)$ by means of $\|v\|^3_{L^{\frac{9}{2},\infty}(\mathbb{R}^3)}$.

Our first result on an a priori estimate of $D(v)$ by means of vorticity now reads:

Theorem 1.1. *Let v be a smooth solution of* (1.1) *with* (1.3). *Suppose that $\omega = \operatorname{rot} v$ satisfies*

$$\limsup_{|x|\to\infty} |x|^{5/3}|\omega(x)| < +\infty. \tag{1.5}$$

Then we have that $D(v)<\infty$ as in (1.2) with the estimate

$$D(v) \le C_0 \left(\limsup_{|x|\to\infty} |x|^{5/3}|\omega(x)| \right)^3, \tag{1.6}$$

where $C_0>0$ is an absolute constant independent of v.

Another a priori estimate of $D(v)$ in terms of v itself reads as follows:

Theorem 1.2. *Let v be a smooth solution of* (1.1). *Assume that p is bounded in $\mathbb{R}^3$. Suppose that $v\in L^{9/2,\infty}(\mathbb{R}^3)$, i.e.,*

$$\|v\|_{L^{\frac{9}{2},\infty}} \equiv \sup_{t>0} t\mu\left(\{x\in\mathbb{R}^3; |v(x)|>t\}\right)^{\frac{2}{9}} < \infty, \tag{1.7}$$

where μ is the Lebesgue measure on $\mathbb{R}^3$. Then we have that $D(v)<\infty$ as in (1.2) with the estimate

$$D(v) \le C_0'\|v\|^3_{L^{\frac{9}{2},\infty}}, \tag{1.8}$$

where $C_0'>0$ is an absolute constant independent of v.

As an application of the above theorems, we have the following uniqueness result on (1.1).

Corollary 1.3 (Liouville-type theorem). *Let v be a smooth solution of* (1.1) *in the class* (1.2) *with* (1.3). *Assume that v satisfies either following condition (i) or (ii).*

(i)

$$\limsup_{|x|\to\infty} |x|^{5/3}|\omega(x)| \le (\delta D(v))^{1/3} \tag{1.9}$$

with some constant $\delta < 1/C_0$;

(ii)

$$\|v\|_{L^{\frac{9}{2},\infty}} \le (\delta' D(v))^{1/3} \tag{1.10}$$

with some constant $\delta' < 1/C_0'$.

Then it holds that $v \equiv 0$ *in* $\mathbb{R}^3$.

The article is organized as follows. In Section 2, a proof of Theorem 1.1 is given. In Section 3, Liouvill-type theorems for the nonstationary case are treated. A sketch of their proof is also given.

2. Bound by vorticity; Proof of Theorem 1.1

Based on the Biot-Savart law, we first derive the general estimate of velocity v from that of vorticity $\omega = \operatorname{rot} v$.

Lemma 2.1. *Let* v *be a smooth solenoidal vector field in* $\mathbb{R}^3$ *with (1.3). Suppose that* $\omega = \operatorname{rot} v$ *satisfies*

$$\varepsilon(\alpha) \equiv \limsup_{|x|\to\infty} |x|^{\alpha}|\omega(x)| < \infty \quad \text{for some } 1 < \alpha < 3. \tag{2.1}$$

Then there is $L > 0$ *such that the estimate*

$$|v(x)| \le C_\alpha \varepsilon(\alpha)|x|^{1-\alpha} + \frac{L^3}{6}\|\omega\|_{L^\infty(B_{L/2})}|x|^{-2} \tag{2.2}$$

holds for all $|x| \ge L$ *with a constant* C_α *depending only on* α*, but not on* v *and* L*, where* $B_L \equiv \{x \in \mathbb{R}^3; |x| \le L\}$*. Moreover, it holds that* $\nabla v \in L^q(\mathbb{R}^3)$ *for all* q *with* $3/\alpha < q < \infty$*. When* $\varepsilon(\alpha) = 0$*, we interpret the constants* $\varepsilon(\alpha)$ *and* L *in* (2.2) *as an arbitrary small positive number and a constant depending on* $\varepsilon(\alpha)$*, respectively.*

We next investigate behavior at infinity of the pressure p. For that purpose, we need (1.2).

Lemma 2.2. *Let* v *be a smooth solution to* (1.1) *with* (1.3) *and let* p *be the pressure associated with* v *in* (1.1)*. Assume that* $\omega = \operatorname{rot} v$ *satisfies* (2.1) *for some* α *with* $3/2 \le \alpha < 2$ *and that* L *is the same as in* (2.2)*. Then there exists* $\bar{p} \in \mathbb{R}$ *such that* $p'(x) = p(x) - \bar{p}$ *satisfies the estimate*

$$\begin{aligned} |p'(x)| \quad \le \quad & C_\alpha' \varepsilon(\alpha)^2 |x|^{-2(\alpha-1)} + C_{\alpha,L}\varepsilon(\alpha)(1 + \|\omega\|_{L^\infty(B_{L/2})})^2|x|^{-\alpha} \\ & + \frac{(2L)^3}{3}\|\omega \times v\|_{L^\infty(B_L)}|x|^{-2} \end{aligned} \tag{2.3}$$

for all $|x| \ge 2L$*, where* C_α' *and* $C_{\alpha,L}$ *are constants depending only on* α *and on* α, L*, respectively.*

For the proof of these lemmas, we refer to [6].

Using these lemmas, we are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. In what follows, we shall denote by C various constants which may change from line to line. In particular, we denote by $C = C(*, \cdots, *)$ constants depending only on the quantities appearing in parentheses.

By (1.5) we see that (2.1) holds with $\alpha = 5/3$, and hence it follows from Lemma 2.1 that $D(v) < \infty$ as in (1.2). Let $\psi = \psi(x) \in C_0^\infty(\mathbb{R}^3)$ be a test function satisfying

$$\psi(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| \ge 2 \end{cases}$$

and $0 \le \psi \le 1$. We define a family $\{\psi_R\}$ of cut-off functions with large parameter $R > 0$ by $\psi_R(x) = \psi(x/R)$. Multiplying the equation (1.1) by $\psi_R(x)v(x)$ and then integrating over $\mathbb{R}^3$, we have by integration by parts that

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla v|^2 \psi_R dx &= \int_{\mathbb{R}^3} |v|^2 \Delta\psi_R dx + \frac{1}{2}\int_{\mathbb{R}^3} |v|^2 v \cdot \nabla\psi_R dx \\ &\quad + \int_{\mathbb{R}^3} p' v \cdot \nabla\psi_R dx \\ &=: I_R^{(1)} + I_R^{(2)} + I_R^{(3)}, \end{aligned} \tag{2.4}$$

where $p'(x) = p(x) - \bar{p}$ is as in Lemma 2.2. Let us take R sufficiently large so that $R \ge 4L$, where L is the same constant as in (2.2). Then, taking $\varepsilon_* = \varepsilon(5/3)$ in (2.1), we obtain from Lemmata 2.1 and 2.2 that

$$\begin{aligned} I_R^{(1)} &\le R^{-2}\int_{R\le|x|\le 2R} \left(C\varepsilon_* R^{-\frac{2}{3}} + C(L, \|\omega\|_{L^\infty(B_L)})R^{-2}\right)^2 \|\Delta\psi\|_{L^\infty} dx \\ &\le C\varepsilon_*^2 R^{-1/3} + C(L, \|\omega\|_{L^\infty(B_L)})R^{-4}, \end{aligned}$$

$$\begin{aligned} I_R^{(2)} &\le R^{-1}\int_{R\le|x|\le 2R} \left(C\varepsilon_* R^{-\frac{2}{3}} + C(L, \|\omega\|_{L^\infty(B_L)})R^{-2}\right)^3 \|\nabla\psi\|_{L^\infty} dx \\ &\le C\varepsilon_*^3 + C(L, \|\omega\|_{L^\infty(B_L)})R^{-4}, \end{aligned}$$

and

$$\begin{aligned} & I_R^{(3)} \\ \le\ & R^{-1}\int_{R\le|x|\le 2R} \left(C\varepsilon_*^2 R^{-\frac{4}{3}} + C(L, \|\omega\|_{L^\infty(B_L)})\varepsilon_* R^{-\frac{5}{3}} + C(L, \|\omega\times v\|_{L^\infty(B_L)})R^{-2}\right) \\ & \times \left(C\varepsilon_* R^{-\frac{2}{3}} + C(L, \|\omega\|_{L^\infty(B_L)})R^{-2}\right)\|\nabla\psi\|_{L^\infty} dx \\ \le\ & C\varepsilon_*^3 + C(L, \|\omega\|_{L^\infty(B_L)}, \|v\|_{L^\infty(B_L)})\left(\varepsilon_*^2 R^{-\frac{1}{3}} + \varepsilon_* R^{-\frac{2}{3}} + R^{-2}\right) \end{aligned}$$

for all $R \ge 4L$. Hence, it follows from (2.4) that

$$\int_{\mathbb{R}^3} |\nabla v|^2 \psi_R dx \le C\varepsilon_*^3 + C(\varepsilon_*, L, \|\omega\|_{L^\infty(B_L)}, \|v\|_{L^\infty(B_L)})R^{-\frac{1}{3}} \quad \text{for all } R \ge 4L.$$

Letting $R \to \infty$, we obtain

$$D(v) \le C\varepsilon_*^3,$$

which implies the desired estimate (1.6). In the case when $\varepsilon_* = 0$, similarly to the above one can obtain

$$D(v) \le C\varepsilon^3$$

for an arbitrarily small $\varepsilon > 0$. Hence, in this case we obtain $D(v) = 0$. This completes the proof of Theorem 1.1. We omit the proof of Theorem 1.2 since it is similar to that of Theorem 1.1. We also omit the proof of Corollary 1.3 since it is easy.

3. Liouville-type theorems for the nonstationary case

Next we consider Liouville-type theorems for the Cauchy problem for the Navier-Stokes equations

$$\text{(3.1)}\qquad \begin{cases} v_t - \Delta v + (v\cdot\nabla)v + \nabla p = 0, & (x,t)\in\mathbb{R}^n\times(0,T),\\ \operatorname{div} v = 0, & (x,t)\in\mathbb{R}^n\times(0,T),\\ v(x,0) = v_0(x), & x\in\mathbb{R}^n. \end{cases}$$

Here $v = v(x,t) = (v^1(x,t),\dots,v^n(x,t))$ and $p = p(x,t)$ denote the velocity and the pressure, respectively, while $v_0(x) = (v_0^1(x),\dots,v_0^n(x))$ stands for the given initial velocity. Let the initial data v_0 belong to $L^2_\sigma(\mathbb{R}^n)$, which is the closure of $C^\infty_{0,\sigma}(\mathbb{R}^n)$, compactly supported C^∞-solenoidal vector functions, with respect to the L^2-norm. We recall that a measurable function v on $\mathbb{R}^n\times(0,T)$ is a weak solution of the Leray-Hopf class to (3.1) if $v\in L^\infty(0,T;L^2_\sigma(\mathbb{R}^n))\cap L^2_{loc}([0,T);H^1_\sigma(\mathbb{R}^n))$ and if v satisfies (1.1) in the sense that

$$\int_0^T \{-(v,\partial_\tau\Phi) + (\nabla v,\nabla\Phi) + (v\cdot\nabla v,\Phi)\}\,d\tau = (v_0,\Phi(0))$$

holds for all $\Phi\in H^1_0([0,T);H^1_\sigma(\mathbb{R}^n)\cap L^n(\mathbb{R}^n))$. For every weak solution $v(t)$ of the Leray-Hopf class to (3.1) , it is shown by Prodi [12] and Serrin [14] that, after a redefinition of its value of $v(t)$ on a set of measure zero in the time interval $[0,T]$, $v(\cdot,t)$ is continuous for t in the weak topology of $L^2_\sigma(\mathbb{R}^n)$. See also Masuda [11, Proposition 2].

Serrin [14] proved that if v is a weak solution of the Leray-Hopf class to (3.1) and if $v\in L^s(0,T;L^q(\mathbb{R}^n))$ for $\frac{3}{q}+\frac{2}{s}\le 1$ with some $q>3, s>2$, then the energy identity

$$\text{(3.2)}\qquad \|v(t)\|^2_{L^2} + 2\int_0^t \|\nabla v(\tau)\|^2_{L^2}d\tau = \|v_0\|^2_{L^2}\quad (0\le t<T)$$

is valid. Shinbrot [15] also proved that the same conclusion holds under another assumption for some $s>1, q\ge 4$ such that $\frac{2}{q}+\frac{2}{s}\le 1$. Taniuchi [16] further extended these results to

$$\frac{2}{q}+\frac{2}{s}\le 1,\quad \frac{3}{q}+\frac{1}{s}\le 1\quad (n=3),$$
$$\frac{2}{q}+\frac{2}{s}\le 1,\quad q\ge 4\quad (n\ge 4).$$

We give a new condition which ensures the energy inequality, and as its application, several Liouville-type theorems are established. Let us first introduce our definition of a generalized suitable weak solution.

Definition 3.1 (*Generalized suitable weak solution*). *Let $v_0\in L^2_\sigma(\mathbb{R}^n)$. We say that the pair (v,p) of measurable functions on $\mathbb{R}^n\times(0,T)$ is a generalized suitable weak solution of the Navier-Stokes equations* (3.1) *if*

(i) $v\in L^3_{loc}(\mathbb{R}^n\times[0,T))$, $\nabla v\in L^2_{loc}(\mathbb{R}^n\times[0,T))$ *and* $p\in L^{3/2}_{loc}(\mathbb{R}^n\times[0,T))$;
(ii) *For every compact subset $K\subset\mathbb{R}^n$, $v(\cdot,t)$ is continuous for $t\in[0,T)$ in the weak topology of $L^2(K)$ and is strongly continuous in $L^2(K)$ at $t=0$, that*

is,

$$\int_K v(x,\cdot)\cdot\varphi(x)dx \in C([0,T)) \quad \textit{for all } \varphi \in L^2(K),$$

$$\lim_{t\to 0+}\int_K |v(x,t)-v_0(x)|^2dx = 0;$$

(iii) *The pair* (v,p) *satisfies the Navier-Stokes equations* (3.1) *in the sense of distributions in* $\mathbb{R}^n\times(0,T)$;

(iv) *The pair* (v,p) *fulfills the generalized energy inequality*

$$2\int_0^T\int_{\mathbb{R}^n}|\nabla v|^2\Phi dxdt \le \int_0^T\int_{\mathbb{R}^n}\left[|v|^2(\Phi_t+\Delta\Phi)+(|v|^2+2p)v\cdot\nabla\Phi\right]dxdt \tag{3.3}$$

for any nonnegative test function $\Phi\in C_0^\infty(\mathbb{R}^n\times(0,T))$.

Remark 3.1. (i) *Caffarelli-Kohn-Nirenberg* [1] *first introduced the notion of a suitable weak solution and proved the partial regularity and the the Hausdorff dimension of singularities for such weak solutions. In comparison with the suitable weak solution given by* [1], *we assume neither finite energy* $\sup_{0<t<T}\|v(t)\|_{L^2}^2<\infty$ *nor finite dissipation* $\int_0^T\|\nabla v(\tau)\|_{L^2}^2d\tau<\infty$. *Furthermore, we impose on the pressure* p *only local* $L^{\frac{3}{2}}$*-bound in* $\mathbb{R}^n\times(0,T)$, *while they* [1] *assume such a global bound as* $p\in L^{\frac{5}{4}}(\mathbb{R}^3\times(0,T))$ *for* $n=3$.

(ii) *A similar notion to our generalized suitable weak solution was considered by Lemarié-Rieusset* [8, Chapter 32] *who constructed the local Leray solution based on the uniformly local* L^2*-space.*

Our main result for (3.1) is the following. Here $L^{q,r}$ denotes the Lorenz space with a standard notation.

Theorem 3.2. *Let* $n\ge 2$, $v_0\in L^2_\sigma(\mathbb{R}^n)$ *and let the pair* (v,p) *be a generalized suitable weak solution of* (3.1). *Suppose that there exist* q_1,q_2,r_1,r_2 *satisfying*

$$3\le q_1\le\frac{3n}{n-1},\ 3\le r_1\le\infty \quad \textit{and} \quad (q_1,r_1)\ne\left(\frac{3n}{n-1},\infty\right), \tag{3.4}$$

$$2\le q_2\le\frac{2n}{n-2},\ 2\le r_2\le\infty \quad \textit{and} \quad \begin{cases}(q_2,r_2)\ne\left(\frac{2n}{n-2},\infty\right) & (n\ge 3),\\ q_2\ne\infty & (n=2)\end{cases} \tag{3.5}$$

such that $v\in L^3(0,T;L^{q_1,r_1}(\mathbb{R}^n))\cap L^2(0,T;L^{q_2,r_2}(\mathbb{R}^n))$. *We also assume that the pressure* p *satisfies*

$$\frac{1}{|B_{|x|/2}(x)|}\int_{B_{|x|/2}(x)}p(y,t)dy = o(|x|) \quad \textit{as } |x|\to\infty \tag{3.6}$$

for almost every $t\in(0,T)$. *(*$B_R(x)$ *denotes the ball centered at* $x\in\mathbb{R}^n$ *with radius* $R>0$. *) Then, we have that*

$$v\in L^\infty(0,T;L^2_\sigma(\mathbb{R}^n))\cap L^2(0,T;\dot H^1_\sigma(\mathbb{R}^n))$$

and that

$$\|v(t)\|_{L^2}^2+2\int_0^t\|\nabla v(\tau)\|_{L^2}^2d\tau\le\|v_0\|_{L^2}^2$$

for all $t\in(0,T)$.

Remark 3.2. (i) *Our proof of Theorem 3.2 enables us to show that if the pair* (v, p) *is a smooth solution with such bounds as (3.4) and (3.5) in Theorem 3.2 and if* p *behaves at infinity like (3.6), then we have the energy identity (3.2).*

(ii) *Besides the energy identity (3.2), there is another notion of the strong energy inequality which means that*

$$\|v(t)\|_{L^2}^2 + 2\int_s^t \|\nabla v(\tau)\|_{L^2}^2 d\tau \leq \|v(s)\|_{L^2}^2 \tag{3.7}$$

for almost all $0 \leq s < T$, *including* $s = 0$ *and all* $t > 0$ *such that* $s \leq t \leq T$. *The importance of the strong energy inequality was pointed out by Masuda* [11]. *For every* $v_0 \in L^2_\sigma(\mathbb{R}^n)$, *the existence of the weak solution* v *in the Leray-Hopf class satisfying (3.7) was proved by Leray* [10] *for* $n = 3$ *and by Kato* [5] *for* $n = 4$, *respectively. However, it seems difficult to obtain the corresponding result to the higher dimensional case for* $n \geq 5$. *In addition to the condition (ii) of Definition 3.1, if we assume that*

$$\lim_{t \to s+0} \int_K |v(x,t) - v(x,s)|^2 dx = 0$$

for almost all $0 \leq s < T$, *including* $s = 0$, *then our proof of Theorem 3.2 enables us to see that* v *satisfies the strong energy inequality (3.7).*

(iii) *The condition* (3.6) *is not restrictive. Indeed, if* p *satisfies* $p(x,t) = o(|x|)$ *as* $|x| \to \infty$ *for almost every* $t \in (0,T)$, *then we have* (3.6). *Also, if* p *satisfies* $p \in L^s(0,T; L^{q,r}(\mathbb{R}^n))$ *with some* $s, q, r \in [1, \infty]$, *then* (3.6) *holds.*

An immediate consequence of this theorem is the following Liouville-type theorem.

Corollary 3.3. *Let* $n \geq 2$, *and let* $v_0 \equiv 0$ *in* $\mathbb{R}^n$. *Suppose that the pair* (v,p) *is a generalized suitable weak solution of* (3.1). *If* p *satisfies* (3.6) *and if* $v \in L^3(0,T; L^{q_1,r_1}(\mathbb{R}^n)) \cap L^2(0,T; L^{q_2,r_2}(\mathbb{R}^n))$ *for such* (q_1, r_1) *and* (q_2, r_2) *as in (3.4) and (3.5), respectively, then it holds that* $v(x,t) \equiv 0$ *on* $\mathbb{R}^n \times (0,T)$.

We next deal with the exponents (q_1, r_1) and (q_2, r_2) in the marginal case of (3.4) and (3.5).

Theorem 3.4. *Let* $n \geq 2$, $v_0 \in L^2_\sigma(\mathbb{R}^n)$ *and let the pair* (v,p) *be a generalized suitable weak solution of* (3.1). *Suppose that there exist* q_1, q_2, r_1, r_2 *satisfying*

$$3 \leq q_1 \leq \frac{3n}{n-1}, \; 2 \leq q_2 \leq \frac{2n}{n-2}, \; 3 \leq r_1 \leq \infty, \; 2 \leq r_2 \leq \infty$$

and

$$\text{(Case 1)} \qquad (q_1, r_1) = \left(\frac{3n}{n-1}, \infty\right), \quad \begin{cases} (q_2, r_2) \neq \left(\frac{2n}{n-2}, \infty\right) & (n \geq 3), \\ q_2 \neq \infty & (n = 2), \end{cases}$$

$$\text{(Case 2)} \qquad (q_1, r_1) \neq \left(\frac{3n}{n-1}, \infty\right), \quad \begin{cases} (q_2, r_2) = \left(\frac{2n}{n-2}, \infty\right) & (n \geq 3), \\ q_2 = \infty & (n = 2), \end{cases}$$

$$\text{(Case 3)} \qquad (q_1, r_1) = \left(\frac{3n}{n-1}, \infty\right), \quad \begin{cases} (q_2, r_2) = \left(\frac{2n}{n-2}, \infty\right) & (n \geq 3), \\ q_2 = \infty & (n = 2) \end{cases}$$

such that $v \in L^3(0,T;L^{q_1,r_1}(\mathbb{R}^n)) \cap L^2(0,T;L^{q_2,r_2}(\mathbb{R}^n))$. *We also assume that the pressure* p *satisfies (3.6). Then, we have that*

$$v \in L^\infty(0,T;L^2_\sigma(\mathbb{R}^n)) \cap L^2(0,T;\dot{H}^1_\sigma(\mathbb{R}^n))$$

and that

$$\|v(t)\|_{L^2}^2 + 2\int_0^t \|\nabla v(\tau)\|_{L^2}^2 d\tau \le \|v_0\|_{L^2}^2 + C_0 V_v(t) \tag{3.8}$$

holds for all $t \in (0,T)$ *with some absolute constant* C_0, *where*

$$V_v(t) = \begin{cases} \|v\|^3_{L^3(0,t;L^{q_1,r_1})} & \text{(Case 1)}, \\ \|v\|^2_{L^2(0,t;L^{q_2,r_2})} & \text{(Case 2)}, \\ \|v\|^3_{L^3(0,t;L^{q_1,r_1})} + \|v\|^2_{L^2(0,t;L^{q_2,r_2})} & \text{(Case 3)}. \end{cases}$$

Similarly to Corollary 3.3, we have also the following Liouville-type theorem:

Corollary 3.5. *Let* $n \ge 2$, *and let* $v_0 \equiv 0$ *in* $\mathbb{R}^n$. *Suppose that the pair* (v,p) *is a generalized suitable weak solution of (1.1). We assume that* p *satisfies* (3.6) *and that* $v \in L^3(0,T;L^{q_1,r_1}(\mathbb{R}^n)) \cap L^2(0,T;L^{q_2,r_2}(\mathbb{R}^n))$ *for such* (q_1,r_1) *and* (q_2,r_2) *as in the Cases 1, 2 and 3 in Theorem 3.4. If there exists* $\delta \in (0,1/C_0)$ *such that*

$$V_v(t_0) \le \delta \left(\|v(t_0)\|_{L^2}^2 + 2\int_0^{t_0} \|\nabla v(\tau)\|_{L^2}^2 d\tau \right)$$

for some $t_0 \in (0,T)$, *then it holds that* $v(x,t) \equiv 0$ *on* $\mathbb{R}^n \times [0,t_0]$.

Remark 3.3. (i) *The estimate* (3.8) *is invariant under the scaling transformation* $v_\lambda(x,t) = \lambda v(\lambda x, \lambda^2 t)$ *with* $\lambda > 0$. *Indeed, if* v *satisfies the estimate (3.8) for some* $t \in (0,T)$, *then it holds that*

$$\|v_\lambda(t/\lambda^2)\|_{L^2}^2 + 2\int_0^{t/\lambda^2} \|\nabla v_\lambda(\tau)\|_{L^2}^2 d\tau \le \|v_{0,\lambda}\|_{L^2}^2 + C_0 V_{v_\lambda}(t/\lambda^2)$$

for all $\lambda > 0$.

(ii) *In comparison with the result of Taniuchi* [16], *even for the energy inequality, Theorem 3.2 requires stronger integrability of* v *at the spatial infinity. On the other hand, we do not need to impose on* v *the finite energy and dissipation like*

$$v \in L^\infty(0,T;L^2_\sigma(\mathbb{R}^n)) \cap L^2_{loc}([0,T);H^1_\sigma(\mathbb{R}^n)), \tag{3.9}$$

while [16] *requires such a property as (3.9).*

Let us mention a little bit about the proof of Thorem 3.2 and Theorem 3.4.

First we sketch the proof of Theorem 3.2. Let $\psi = \psi(x) \in C_0^\infty(\mathbb{R}^n)$ be a test function satisfying

$$\psi(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| \ge 2, \end{cases} \quad 0 \le \psi \le 1.$$

We define a family $\{\psi_R\}$ of cut-off functions with large parameter $R > 0$ by $\psi_R(x) = \psi(x/R)$. Using the generalized energy inquality (3.3), we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} |v(t)|^2 \psi_R dx + 2\int_0^t \int_{\mathbb{R}^n} |\nabla v|^2 \psi_R dx d\tau \\
&\le \int_{\mathbb{R}^n} |v_0|^2 \psi_R dx + \int_0^t \int_{\mathbb{R}^n} |v|^2 \Delta \psi_R dx d\tau \\
&\quad + \int_0^t \int_{\mathbb{R}^n} |v|^2 v \cdot \nabla \psi_R dx d\tau + 2\int_0^t \int_{\mathbb{R}^n} p' v \cdot \nabla \psi_R dx d\tau \\
&=: \int_{\mathbb{R}^n} |v_0|^2 \psi_R dx + I_R^{(1)} + I_R^{(2)} + I_R^{(3)}.
\end{aligned}$$

Then we show I_R^1, I_R^2, I_R^3 tends to zero as R tends to infinity under the conditions in Theorem 3.2. This ends a sketch of the proof of Theorem 3.2.

Next we give a skech of the proof of Theorem 3.4. We only treat Case1, because the other cases are quite similar. We first have $\lim_{R\to\infty} I_R^{(1)} = 0$. Concerning $I_R^{(2)}$, we only have $I_R^{(2)} \le C\|v\|^3_{L^3(0,T;L^{\frac{3n}{n-1},\infty})}$, since $-1 + \frac{n(q_1-3)}{q_1} = 0$. Using an estimate of the pressure term by the velocity term, we have $I_R^{(3)} \le C\|v\|^3_{L^3(0,T;L^{\frac{3n}{n-1},\infty})}$.

Thus, letting $R \to \infty$, we conclude that

$$\int_{\mathbb{R}^n} |v(t)|^2 dx + 2\int_0^t \int_{\mathbb{R}^n} |\nabla v|^2 dx d\tau \le \int_{\mathbb{R}^n} |v_0|^2 dx + C_0 \|v\|^3_{L^3(0,T;L^{\frac{3n}{n-1},\infty})}$$

with some absolute constant $C_0 > 0$.

This ends a sketch of the proof of Theorem 3.4.

We omit the proof of Corollary 3.3, Corollary 3.5 since they are easy.

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