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Muckenhoupt-Wheeden conjectures for fractional integral operators

By

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Abstract

The Muckenhoupt-Wheeden conjecture is disproved for fractional integral operator. Both the strong (p,p) and the weak (p,p) type conjecture are proved to be false. The arguments rely upon a property of the characteristic function of an approximating sequence of the Cantor set. An off-diagonal case, $1<p<q<\infty$, is also discussed.

§ 1. Introduction

The purpose of this note is to disprove the joint weighted estimates for fractional integral operators and fractional maximal operators. We mention that the results presented in this paper have been announced in [11] except the last section. We first fix some notation.

By a weight we will always mean a non-negative measurable function on $\mathbb{R}^n$. Given a measurable set E and a weight w, $w(E)=\int_E w(x)\,dx$, $|E|$ denotes the Lebesgue measure of E and $\mathbf{1}_E$ denotes the characteristic function of E. Let $1\le p<\infty$ and w be a weight. We define the weighted Lebesgue space $L^p(w)$ to be a Banach space equipped with the norm

$$\|f\|_{L^p(w)}=\left(\int_{\mathbb{R}^n}|f(x)|^p w(x)\,dx\right)^{1/p}.$$

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We always assume that the support of the weight w contains that of f. Given $1 < p < \infty$, $p' = \frac{p}{p-1}$ will denote the conjugate exponent number of p. Given $0 < \alpha < n$ and a measurable function f, we define the fractional integral operator I_α by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy.$$

We shall consider all cubes in $\mathbb{R}^n$ which have their sides parallel to the coordinate axes. We denote by $\mathcal{Q}$ the family of all such cubes. Given $0 \le \alpha < n$ and a measurable function f, we define the fractional maximal operator M_α by

$$M_\alpha f(x) = \sup_{Q \in \mathcal{Q}} \mathbf{1}_Q(x) |Q|^{\alpha/n} \fint_Q |f(y)| \, dy,$$

where the barred integral $\fint_Q f(y)\,dy$ stands for the usual integral average of f over Q. If $\alpha = 0$ we drop the subscript α. So, $M = M_0$ is the Hardy-Littlewood maximal operator.

Following the excellent survey [2] due to David Cruz-Uribe SFO, we review the Muckenhoupt-Wheeden conjectures for singular integral operators. See [6] for the definition of singular integral operators.

In the late 1970's while studying two weight norm inequalities for the Hilbert transform H, Muckenhoupt and Wheeden made a series of conjectures relating this problem to two weight norm inequalities for the maximal operator M. These conjectures were quickly extended to the general singular integral operators. For a pair of weights (u, σ) and the exponent number $1 < p < \infty$, they conjectured the following:

The strong (p,p) conjecture For any singular integral operator T, the operator $T(\cdot\sigma)$ extends to a bounded linear operator from $L^p(\sigma)$ to $L^p(u)$ if the maximal operator satisfies two inequalities $M(\cdot\sigma) : L^p(\sigma) \to L^p(u)$ and $M(\cdot u) : L^{p'}(u) \to L^{p'}(\sigma)$.

The weak (p,p) conjecture For any singular integral operator T, the operator $T(\cdot\sigma)$ extends to a bounded linear operator from $L^p(\sigma)$ to $L^{p,\infty}(u)$ if the maximal operator only satisfies the dual inequality $M(\cdot u) : L^{p'}(u) \to L^{p'}(\sigma)$.

The weak $(1,1)$ conjecture For any singular integral operator T, the following inequality holds:

$$\sup_{t>0} t\, u(\{x \in \mathbb{R}^n : |Tf(x)| > t\}) \le C \int_{\mathbb{R}^n} |f(x)| Mu(x) \, dx.$$

All three conjectures were recently shown to be false. Reguera and Thiele disproved the weak $(1,1)$ conjecture [9], Reguera and Scurry disproved the strong (p,p) conjecture [8] and Cruz-Uribe, et al. disproved the weak (p,p) conjecture [5].

Their conjectures for singular integral operators extend naturally to fractional integral operators as well though Muckenhoupt and Wheeden did not address them. Such a generalization was first considered by Carro, et al. [1], who disproved the analog of the Muckenhoupt weak $(1,1)$ conjecture:

$$\sup_{t>0} t\, u(\{x \in \mathbb{R}^n : |I_\alpha f(x)| > t\}) \le C \int_{\mathbb{R}^n} |f(x)| M_\alpha u(x) dx. \tag{1.1}$$

In this note, using essentially the same counter-example to (1.1) above, we establish the following theorems, which negatively answer the conjecture posed in [2].

Theorem 1.1. *Let $0 < \alpha < n$ and $1 < p < \infty$. Then, for any integer $N \gg 1$, there exists a weight $w = w_N$ with compact support such that*

$$\frac{1}{N} \|I_\alpha(\cdot w)\|_{L^p(w) \to L^p(w)} \ge C$$

while

$$\|M_\alpha(\cdot w)\|_{L^p(w) \to L^p(w)} \le C \text{ and } \|M_\alpha(\cdot w)\|_{L^{p'}(w) \to L^{p'}(w)} \le C.$$

Here, the positive finite constant C is independent of N.

Theorem 1.2. *Let $0 < \alpha < n$ and $1 < p < \infty$. Then, for any integer $N \gg 1$, there exists a weight $w = w_N$ with compact support such that*

$$\frac{1}{N} \|I_\alpha(\cdot w)\|_{L^p(w) \to L^{p,\infty}(w)} \ge C$$

while

$$\|M_\alpha(\cdot w)\|_{L^{p'}(w) \to L^{p'}(w)} \le C.$$

Here, the positive finite constant C is independent of N.

An off-diagonal version of this conjecture, the case $1 < p < q < \infty$, is true due to Cruz-Uribe, et al. for singular integral operators [3] and Cruz-Uribe and Moen for fractional integral operators [4] (see also the last section).

The letter C will be used for the positive finite constants that may change from one occurrence to another. Constants with subscripts, such as C_1, C_2, do not change in different occurrences. By $A \lesssim B$ ($B \gtrsim A$) we mean that $A \le cB$ with some positive finite constant c independent of appropriate quantities.

§ 2. Proof of Theorems

In what follows we shall prove Theorems 1.1 and 1.2. We need three lemmas to this end.

Lemma 2.1 ([12, Theorem B] also [2, Theorem 5.1]). *Given* $0 \le \alpha < n$, $1 < p \le q < \infty$ *and a pair of weights* (u, σ), *the following are equivalent:*

(1) A pair of weights (u, σ) *satisfies the testing condition*

$$\sup_{Q \in \mathcal{Q}} \sigma(Q)^{-1/p} \left(\int_Q M_\alpha(\mathbf{1}_Q \sigma)(x)^q u(x)\, dx \right)^{1/q} \le C_1;$$

(2) For every $f \in L^p(\sigma)$,

$$\left(\int_{\mathbb{R}^n} M_\alpha(f\sigma)(x)^q u(x)\, dx \right)^{1/q} \le C_2 \left(\int_{\mathbb{R}^n} |f(x)|^p \sigma(x)\, dx \right)^{1/p}.$$

Moreover, the least possible C_1 *and* C_2 *are equivalent.*

Lemma 2.2 ([13, Theorem 1] also [2, Theorem 5.2]). *Given* $0 < \alpha < n$, $1 < p \le q < \infty$ *and a pair of weights* (u, σ), *the following are equivalent:*

(1) The testing condition

$$\sup_{Q \in \mathcal{Q}} \sigma(Q)^{-1/p} \left(\int_Q I_\alpha(\mathbf{1}_Q \sigma)(x)^q u(x)\, dx \right)^{1/q} \le C_1$$

and the dual testing condition

$$\sup_{Q \in \mathcal{Q}} u(Q)^{-1/q'} \left(\int_Q I_\alpha(\mathbf{1}_Q u)(x)^{p'} \sigma(x)\, dx \right)^{1/p'} \le C_1$$

hold;

(2) For all $f \in L^p(\sigma)$,

$$\left(\int_{\mathbb{R}^n} |I_\alpha(f\sigma)(x)|^q u(x)\, dx \right)^{1/q} \le C_2 \left(\int_{\mathbb{R}^n} |f(x)|^p \sigma(x)\, dx \right)^{1/p}.$$

Moreover, the least possible C_1 *and* C_2 *are equivalent.*

Lemma 2.3 ([14, Theorem] also [7, Theorem 1.8]). *Given* $0 < \alpha < n$, $1 < p \le q < \infty$ *and a pair of weights* (u, σ), *the following are equivalent:*

(1) A pair of weights (u, σ) *satisfies the dual testing condition*

$$\sup_{Q \in \mathcal{Q}} u(Q)^{-1/q'} \left(\int_Q I_\alpha(\mathbf{1}_Q u)(x)^{p'} \sigma(x)\, dx \right)^{1/p'} \le C_1;$$

(2) For all $f \in L^p(\sigma)$,

$$\sup_{t>0} t\, u(\{x \in \mathbb{R}^n : |I_\alpha(f\sigma)(x)| > t\})^{1/q} \le C_2 \left(\int_{\mathbb{R}^n} |f(x)|^p \sigma(x)\, dx \right)^{1/p}.$$

Moreover, the least possible C_1 *and* C_2 *are equivalent.*

Thanks to Lemmas 2.1–2.3, Theorems 1.1 and 1.2 can be proved once the following proposition is verified.

Proposition 2.4. *Given* $0 < \alpha < n$, *then for any integer* $N \gg 1$ *there exists a weight* $w = w_N$ *with compact support such that*

$$\sup_{Q \in \mathcal{Q}} \left(\frac{1}{w(Q)} \int_Q I_\alpha(\mathbf{1}_Q w)(x)^p w(x)\, dx \right)^{1/p} \gtrsim N \tag{2.1}$$

while

$$\sup_{Q \in \mathcal{Q}} \left(\frac{1}{w(Q)} \int_Q M_\alpha(\mathbf{1}_Q w)(x)^p w(x)\, dx \right)^{1/p} \lesssim 1 \tag{2.2}$$

holds for all $p > 0$.

Proof. We follow the argument in [10]. Let $0 < \delta < 1$ be the solution to the equation

$$\left(\frac{2}{1-\delta} \right)^{\alpha/n} (1-\delta) = 1. \tag{2.3}$$

Fix a positive large integer $N \gg 1$. Set $\kappa = \frac{2}{1-\delta}$. Let the closed cube

$$E_0 = Q_{0,1} = [0, \kappa^N]^n.$$

Delete from $Q_{0,1}$ all but the 2^n closed corner cubes $Q_{1,j}$, of side κ^{N-1}, to obtain

$$E_1 = \bigcup_{j=1}^{2^n} Q_{1,j}.$$

Continue in this way N steps: at the k stage, $0 < k < N$, replacing each cube of E_{k-1} by the 2^n closed corner cubes $Q_{k,j}$, of side κ^{N-k}, to obtain

$$E_k = \bigcup_{j=1}^{2^{nk}} Q_{k,j}.$$

Thus, E_N contains 2^{nN} closed unit cubes. We have the following:

$$\frac{|E_N \cap Q_{k,j}|}{|Q_{k,j}|} = (1-\delta)^{n(N-k)} \text{ for all } k = 0, 1, \dots, N \text{ and } j = 1, 2, \dots, 2^{nk} \tag{2.4}$$

and, by the use of the equation (2.3),

$$|Q_{k,j}|^{\alpha/n} \left(\frac{|E_N \cap Q_{k,j}|}{|Q_{k,j}|} \right) = 1 \text{ for all } k = 0, 1, \dots, N \text{ and } j = 1, 2, \dots, 2^{nk}. \tag{2.5}$$

We now let $w(x) \equiv \mathbf{1}_{E_N}(x)$ and we shall verify (2.1) and (2.2).

We take the $N+1$ cubes $P_0 = Q_{0,1}$, $P_1 \in \{Q_{1,j}\}$, ..., $P_N \in \{Q_{N,j}\}$ so that

$$P_0 \supset P_1 \supset \cdots \supset P_N.$$

It follows from (2.5) that

$$|P_k|^{\alpha/n} \left(\frac{|E_N \cap P_k|}{|P_k|} \right) = 1 \text{ for all } k = 0, 1, \dots, N. \tag{2.6}$$

Proof of (2.1). In general, for non-negative measurable function f, we have by Fubini's theorem that

$$\begin{aligned}
I_\alpha f(x) &= \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy = \int_0^\infty \int_{|x-y|^{\alpha-n}>s} f(y)\, dy\, ds \\
&\quad \text{by a changing of valuables } s \mapsto t^{\alpha-n} \\
&= (n-\alpha) \int_0^\infty \left(\int_{|x-y|<t} f(y)\, dy \right) t^{\alpha-n-1}\, dt \\
&= C \int_0^\infty |B(x,t)|^{\alpha/n} \left(\fint_{B(x,t)} f(y)\, dy \right) \frac{dt}{t},
\end{aligned}$$

where $B(x,t)$ stands for the ball of radius t around $x \in \mathbb{R}^n$.

This formula and (2.6) enable us that, for all $x \in P_N$, (recalling $w = \mathbf{1}_{P_0} w$)

$$\begin{aligned}
I_\alpha w(x) = I_\alpha[\mathbf{1}_{P_0} w](x) &= C \int_0^\infty |B(x,t)|^{\alpha/n} \left(\fint_{B(x,t)} \mathbf{1}_{P_0}(y) w(y)\, dy \right) \frac{dt}{t} \\
&\gtrsim \sum_{m=1}^N \int_{(\sqrt{n}\kappa)^m}^{(\sqrt{n}\kappa)^{m+1}} |B(x,t)|^{\alpha/n} \left(\fint_{B(x,t)} \mathbf{1}_{P_0}(y) w(y)\, dy \right) \frac{dt}{t} \\
&\gtrsim \sum_{m=1}^N \int_{(\sqrt{n}\kappa)^m}^{(\sqrt{n}\kappa)^{m+1}} |P_{N-m}|^{\alpha/n} \left(\fint_{P_{N-m}} w(y)\, dy \right) \frac{dt}{t} \\
&= \int_{\sqrt{n}\kappa}^{(\sqrt{n}\kappa)^{N+1}} \frac{dt}{t} \\
&\gtrsim N.
\end{aligned}$$

This yields

$$\left(\frac{1}{w(P_0)}\int_{P_0} I_\alpha(\mathbf{1}_{P_0}w)(x)^p w(x)\,dx\right)^{1/p} \gtrsim N,$$

which proves (2.1).

Proof of (2.2). We denote by $Q(x,t)$ the cube of side $2t$ around $x \in \mathbb{R}^n$. It follows that, for all $x \in P_N$,

$$M_\alpha w(x) \lesssim (2\kappa)^\alpha + \sup_{m=1,\dots,N} |Q(x,\kappa^m)|^{\alpha/n} \fint_{Q(x,\kappa^m)} w(y)\,dy,$$

where we have used that, since $w \le 1$, $M_\alpha(\mathbf{1}_{Q(x,\kappa)}w)(x) \le (2\kappa)^\alpha$. By (2.6) we have that

$$\begin{aligned}|Q(x,\kappa^m)|^{\alpha/n} \fint_{Q(x,\kappa^m)} w(y)\,dy &\le 2^n |Q(x,\kappa^m)|^{\alpha/n-1} \int_{P_{N-m}} w(y)\,dy \\ &\lesssim |P_{N-m}|^{\alpha/n} \fint_{P_{N-m}} w(y)\,dy \lesssim 1.\end{aligned}$$

These imply $M_\alpha w(x) \lesssim 1$ for all $x \in P_N$. Invoking the concentration of the density, we further see that $M_\alpha w$ is bounded on $\mathbb{R}^n$. Thus, for any cube $Q \in \mathcal{Q}$,

$$\left(\frac{1}{w(Q)}\int_Q M_\alpha(\mathbf{1}_Q w)(x)^p w(x)\,dx\right)^{1/p} \le \left(\frac{1}{w(Q)}\int_Q M_\alpha w(x)^p w(x)\,dx\right)^{1/p} \lesssim 1,$$

which proves (2.2). The proof of the proposition is now complete. □

Remark. One can construct a positive weight w such that

(2.7) $$\|I_\alpha(\cdot w)\|_{L^p(w)\to L^p(w)} = \infty$$

while

(2.8) $$\|M_\alpha(\cdot w)\|_{L^p(w)\to L^p(w)} \lesssim 1 \text{ and } \|M_\alpha(\cdot w)\|_{L^{p'}(w)\to L^{p'}(w)} \lesssim 1.$$

Indeed, in the proof of Proposition 2.4, for $N = 1, 2, \dots$, we can select the set $E_N \subset [0, \kappa^N]^n$ such that the weight $w_N = \mathbf{1}_{E_N}$ satisfies the following:

$$\begin{cases} c_0 N < I_\alpha(w_N)(x) \text{ for all } x \in E_N, \\ M_\alpha(w_N)(x) < C_0 \text{ for all } x \in \mathbb{R}^n, \end{cases}$$

where c_0 and C_0 are universal constants. Fix a unit vector $\omega \in S_{n-1}$ and let

$$F_N = \kappa^{N^2}\omega + E_N \text{ and } F = \bigcup_N F_N.$$

Then we see that

$$M_\alpha(\mathbf{1}_F)(x) = \sup_N M_\alpha(\mathbf{1}_{F_N})(x) < C_0 \text{ for all } x \in \mathbb{R}^n,$$

since their supports are sufficiently torn apart. Define the weight $w(x) = \mathbf{1}_F(x) + \exp(-|x|^2)$. Then we have that

$$\begin{cases} c_0 N < I_\alpha w(x) \text{ for all } x \in F_N, \\ M_\alpha w(x) < 2C_0 \text{ for all } x \in \mathbb{R}^n. \end{cases}$$

These entail that, for $u > 0$,

$$\begin{aligned} &\sup_{Q\in\mathcal{Q}} \left(\frac{1}{w(Q)} \int_Q I_\alpha(\mathbf{1}_Q w)(x)^u w(x)\,dx \right)^{1/u} \\ &\geq \sup_{Q\in\mathcal{Q}, Q\subset F} \left(\frac{1}{w(Q)} \int_Q I_\alpha(\mathbf{1}_Q w)(x)^u w(x)\,dx \right)^{1/u} \\ &= \infty, \end{aligned}$$

while $M_\alpha(w) \in L^\infty(\mathbb{R}^n)$ and hence

$$\sup_{Q\in\mathcal{Q}} \left(\frac{1}{w(Q)} \int_Q M_\alpha(\mathbf{1}_Q w)(x)^u w(x)\,dx \right)^{1/u} \lesssim 1.$$

Thus, (2.7) and (2.8) follow by Lemmas 2.1 and 2.2.

§ 3. Discussion of an off-diagonal case

In this section, following [2], we prove that an off-diagonal version of Muckenhoupt-Wheeden conjecture is true for fractional integral operators. For this, we need more a lemma.

Lemma 3.1 ([7, Theorem 1.11]). *Given $0 < \alpha < n$, $1 < p < q < \infty$ and a pair of weights (u, σ), the following are equivalent:*

(1) A pair of weights (u, σ) satisfies the dual global testing condition

$$\sup_{x\in\mathbb{R}^n, r>0} u(B(x,r))^{-1/q'} \left(\int_{\mathbb{R}^n} \left(\int_{|x-y|<r} \frac{u(y)}{\max\{r, |y-z|\}^{n-\alpha}}\,dy \right)^{p'} \sigma(z)\,dz \right)^{1/p'} \leq C_1,$$

where $B(x,r)$ is a ball with center x and radius r;

(2) For all $f \in L^p(\sigma)$,

$$\sup_{t>0} t\, u(\{x \in \mathbb{R}^n : |I_\alpha(f\sigma)(x)| > t\})^{1/q} \leq C_2 \left(\int_{\mathbb{R}^n} |f(x)|^p \sigma(x)\,dx \right)^{1/p}.$$

Moreover, the least possible C_1 and C_2 are equivalent.

Proposition 3.2. *Let $0 < \alpha < n$, $1 < p < q < \infty$ and (u, σ) be a pair of weights. The operator $I_\alpha(\cdot\sigma)$ is bounded from $L^p(\sigma)$ to $L^{q,\infty}(u)$ if the maximal operator satisfies the dual inequality $M(\cdot u) : L^{q'}(u) \to L^{p'}(\sigma)$.*

Proof. We merely check the condition posed in Lemma 3.1 (1). Fix a ball $B = B(x, r)$ and $z \in \mathbb{R}^n$. Then we have that

$$\int_{|x-y|<r} \frac{u(y)}{\max\{r, |y-z|\}^{n-\alpha}} \le CM_\alpha(\mathbf{1}_B u)(z).$$

Since the maximal operator satisfies the dual inequality $M(\cdot u) : L^{q'}(u) \to L^{p'}(\sigma)$, we see that

$$\left(\int_{\mathbb{R}^n} \left(\int_{|x-y|<r} \frac{u(y)}{\max\{r, |y-z|\}^{n-\alpha}}\, dy\right)^{p'} \sigma(z)\, dz\right)^{1/p'}$$
$$\le C\left(\int_{\mathbb{R}^n} M_\alpha(\mathbf{1}_B u)(z)^{p'} \sigma(z)\, dz\right)^{1/p'} \le Cu(B)^{1/q'},$$

Which means the condition (1) in Lemma 3.1. □

Proposition 3.3. *Let $0 < \alpha < n$, $1 < p < q < \infty$ and (u, σ) be a pair of weights. The operator $I_\alpha(\cdot\sigma)$ is bounded from $L^p(\sigma)$ to $L^q(u)$ if the maximal operator satisfies two inequalities $M(\cdot\sigma) : L^p(\sigma) \to L^q(u)$ and $M(\cdot u) : L^{q'}(u) \to L^{p'}(\sigma)$.*

Proof. if the maximal operator satisfies $M(\cdot u) : L^{q'}(u) \to L^{p'}(\sigma)$, then, by Proposition 3.2 and Lemma 2.3, a pair of weights (u, σ) satisfies the dual testing condition

$$\sup_{Q\in\mathcal{Q}} u(Q)^{-1/q'} \left(\int_Q I_\alpha(\mathbf{1}_Q u)(x)^{p'} \sigma(x)\, dx\right)^{1/p'} \le C.$$

Similarly, if the maximal operator satisfies $M(\cdot\sigma) : L^p(\sigma) \to L^q(u)$, then a pair of weights (u, σ) satisfies the testing condition

$$\sup_{Q\in\mathcal{Q}} \sigma(Q)^{-1/p} \left(\int_Q I_\alpha(\mathbf{1}_Q \sigma)(x)^q u(x)\, dx\right)^{1/q} \le C.$$

Thus, by Lemma 2.2, the operator $I_\alpha(\cdot\sigma)$ is bounded from $L^p(\sigma)$ to $L^q(u)$. □

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