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Growth properties for generalized Riesz potentials in central Herz-Morrey spaces

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Abstract

Riesz decomposition theorem says that a superharmonic function on the punctured unit ball $\boldsymbol{B}_0$ is represented as the sum of a generalized potential and a harmonic function outside the origin. Our first aim in this note is to study growth properties near the origin for generalized Riesz potentials of functions in central Herz-Morrey spaces on $\boldsymbol{B}_0$.

We know another Riesz decomposition theorem which says that a superharmonic function on the unit ball $\boldsymbol{B}$ is represented as the sum of another generalized potential and a harmonic function on $\boldsymbol{B}$. Our second aim in this note is to obtain growth properties near the boundary $\partial\boldsymbol{B}$ for generalized Riesz potentials of functions in central Herz-Morrey spaces on $\boldsymbol{B}$.

A continuous function u on an open set Ω is called monotone in the sense of Lebesgue [18] if for every relatively compact open set $G \subset \Omega$,

$$\max_{\overline{G}} u = \max_{\partial G} u \qquad \text{and} \qquad \min_{\overline{G}} u = \min_{\partial G} u.$$

Harmonic functions on Ω are monotone in Ω. More generally, solutions of elliptic partial differential equations of second order and weak solutions for variational problems may be monotone (see [15]). Our final aim in this note is concerned with growth properties for monotone Sobolev functions in central Herz-Morrey spaces.

Contents

Part I
Isolated Singularities

1 Generalized Riesz potentials

Let us consider the Riesz kernel $I_\alpha(x) = |x|^{\alpha-n}$ of order α and a generalized kernel

$$I_{\alpha,\ell}(x,y) = I_\alpha(x-y) - \sum_{|\lambda|\le\ell} \frac{(-y)^\lambda}{\lambda!} D^\lambda I_\alpha(x)$$

for an integer ℓ; when $\ell \le -1$, $I_{\alpha,\ell}(x,y) = I_\alpha(x-y)$. Note here that $I_{\alpha,\ell}(x,y)$ is Taylor's remainder of $|x+z|^{\alpha-n}$ around $z=0$.

We define the generalized Riesz potential of order α for a locally integrable function f on the puncture unit ball $\boldsymbol{B}_0$ by

$$I_{\alpha,\ell} f(x) = \int_{\boldsymbol{B}_0} I_{\alpha,\ell}(x,y) f(y)\, dy.$$

Here we prepare the estimates for generalized Riesz kernels.

LEMMA **1.1** (cf. [12, Lemma 3.2]). *Let $\ell \ge 0$.*

(1) $|I_{\alpha,\ell}(x,y)| \le C|x-y|^{\alpha-n}$ *when* $|x|/2 < |y| < 2|x|$.

(2) $|I_{\alpha,\ell}(x,y)| \le C|y|^{\ell+1}|x|^{\alpha-n-\ell-1}$ *when* $|y| < |x|/2$.

(3) $|I_{\alpha,\ell}(x,y)| \le C|y|^{\ell}|x|^{\alpha-n-\ell}$ *when* $2|x| < |y|$.

REMARK **1.2.** If u is a superharmonic function on $\boldsymbol{B}$, then u is represented as the sum of a potential and a harmonic function (see e.g. [2], [3], [14], [23]).

Let u be a superharmonic function on $\boldsymbol{B}_0$. In view of Theorems 1.3 and 3.4 in [12], if $r^a S(|u|, r)$ is bounded in $(0,1)$ for some $a > n-2$, then

$$\sup_{0<r<1/2} r^{a+2-n}\mu(A(0,r)) < \infty \quad (A(0,r) = B(0,2r) \setminus B(0,r))$$

and u is represented as

$$u(x) = I_{2,\ell}\mu(x) + \text{ a harmonic function}$$

near the origin (except at the origin), where $(2-n+a)-1 < \ell \le 2-n+a$ and $\mu = c(-\Delta)u$ is the Riesz measure; see also [8], [10], [11].

2 Central Herz-Morrey spaces

For $1 \le p < \infty$ and a real number ν, we consider the family $M^{p,q,\nu}(\boldsymbol{B})$ of all measurable functions f on $\boldsymbol{B}$ satisfying

$$\|f\|_{M^{p,q,\nu}(\boldsymbol{B})} = \left(\int_0^1 \left(r^{\nu} \|f\|_{L^p(A(0,r))} \right)^q dr/r \right)^{1/q} < \infty$$

when $0 < q < \infty$ and

$$\|f\|_{M^{p,\infty,\nu}(\boldsymbol{B})} = \sup_{0<r<1} r^{\nu} \|f\|_{L^p(A(0,r))} < \infty;$$

set $f = 0$ outside $\boldsymbol{B}$ as before; see e.g. [4], [5], [16].

If $0 < q_1 < q_2 < \infty$, then

$$M^{p,q_1,\nu}(\boldsymbol{B}) \subset M^{p,q_2,\nu}(\boldsymbol{B}) \subset M^{p,\infty,\nu}(\boldsymbol{B})$$

Our space is somewhat a family of functions with finite weighted mixed norm

$$\|f\|_{p,q,\omega} = \left(\int \left(\int |f(x,y)|^p \omega(x,y) dy \right)^{q/p} dx \right)^{1/q} < \infty.$$

3 Sobolev's inequality

Let $p^\sharp$ be the Sobolev exponent of $p > 1$:

$$1/p^\sharp = 1/p - \alpha/n > 0.$$

Lemma **3.1** (Sobolev's inequality (cf. [1], [23])). *There is a constant $C > 0$ such that*

$$\|I_\alpha f\|_{L^{p^\sharp}(\mathbb{R}^n)} \le C \|f\|_{L^p(\mathbb{R}^n)}.$$

Sobolev's inequality is extended to generalize Riesz potentials of functions in central Herz-Morrey spaces.

Theorem **3.2.** *Assume that $\alpha - n/p < \nu < n - n/p$. Then*

$$\|I_\alpha f\|_{M^{p^\sharp,q,\nu}(\boldsymbol{B})} \le C \|f\|_{M^{p,q,\nu}(\boldsymbol{B})}.$$

Theorem **3.3** (Sobolev's inequality for generalized Riesz potentials). *Assume that $\ell \ge 0$ and $n - n/p + \ell < \nu < n - n/p + \ell + 1$. Then*

$$\|I_{\alpha,\ell} f\|_{M^{p^\sharp,q,\nu}(\boldsymbol{B})} \le C \|f\|_{M^{p,q,\nu}(\boldsymbol{B})}.$$

To prove Sobolev's inequality, for a real number β and $0 < r < 1$, let us consider the Hardy type operators

$$H_\beta^- f(r) = r^{-\beta} \int_{B(0,r)} |y|^{\beta-n} f(y)\, dy$$

and

$$H_\beta^+ f(r) = r^{-\beta} \int_{\boldsymbol{B} \setminus B(0,r)} |y|^{\beta-n} f(y)\, dy$$

for measurable functions f on $\boldsymbol{B}$.

LEMMA **3.4.** *Let $\beta - \nu - n/p > \varepsilon > 0$. Then*

$$H_\beta^- f(r) \le C r^{-\varepsilon - n/p - \nu} \left(\int_0^r \left(t^{\varepsilon+\nu} \|f\|_{L^p(A(0,t))} \right)^q \frac{dt}{t} \right)^{1/q}$$

for all $0 < r < 1$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

LEMMA **3.5.** *Let $0 < \varepsilon < -\beta + \nu + n/p$. Then*

$$H_\beta^+ f(r) \le C r^{\varepsilon - n/p - \nu} \left(\int_{r/2}^1 \left(t^{-\varepsilon+\nu} \|f\|_{L^p(A(0,t))} \right)^q \frac{dt}{t} \right)^{1/q}$$

for all $0 < r < 1$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof of Theorem 3.3. Let $\|f\|_{M^{p,q,\nu}(\boldsymbol{B})} \le 1$ and $f \ge 0$. For $x \in \boldsymbol{B}$, set

$$\begin{aligned} I_{\alpha,\ell} f(x) &= \int_{B(0,|x|/2)} I_{\alpha,\ell}(x,y) f(y)\, dy \\ &\quad + \int_{B(0,2|x|)\setminus B(0,|x|/2)} I_{\alpha,\ell}(x,y) f(y)\, dy \\ &\quad + \int_{B(0,1)\setminus B(0,2|x|)} I_{\alpha,\ell}(x,y) f(y)\, dy \\ &= u_1(x) + u_2(x) + u_3(x). \end{aligned}$$

Let $0 < r < 1$. By Lemma 1.1 we have

$$|u_2(x)| \le C \int_{A(0,r/2)\cup A(0,r)} |x-y|^{\alpha-n} |f(y)| dy$$

for $x \in A(0,r)$, so that Lemma 3.4 gives

$$\|u_2\|_{L^{p^\sharp}(A(0,r))} \le C \|f\|_{L^{p^\sharp}(A(0,r/2)\cup A(0,r))}.$$

Hence,

$$\int_0^1 \left(r^\nu \|u_2\|_{L^{p(\cdot)}(A(0,r))} \right)^q \frac{dr}{r} \le C \int_0^1 \left(t^\nu \|f\|_{L^p(A(0,r/2)\cup A(0,r))} \right)^q \frac{dt}{t}.$$

By Lemma 1.1 we see that

$$\begin{aligned} |u_1(x)| &\le C|x|^{\alpha-n-\ell-1} \int_{B(0,|x|/2)} |y|^{\ell+1} f(y)\, dy \\ &\le C r^\alpha H^-_{n+\ell+1} f(r) \end{aligned}$$

for $x \in A(0,r)$. Hence, using Lemma 3.4, we find

$$\|u_1\|_{L^{p(\cdot)}(A(0,r))} \le C r^{-\varepsilon-\nu} \left(\int_0^r \left(t^{\varepsilon+\nu} \|f\|_{L^p(A(0,t))} \right)^q \frac{dt}{t} \right)^{1/q}$$

for $0 < \varepsilon < n + \ell + 1 - n/p - \nu$. Consequently,

$$\begin{aligned} \int_0^1 \left(r^\nu \|u_1\|_{L^p(A(0,r))}\right)^q \frac{dr}{r} &\le C \int_0^1 \left(r^{-\varepsilon} \int_0^r \left(t^{\varepsilon+\nu} \|f\|_{L^p(A(0,t))}\right)^q \frac{dt}{t}\right) \frac{dr}{r} \\ &\le C \int_0^1 \left(t^{\varepsilon+\nu} \|f\|_{L^p(A(0,t))}\right)^q \left(\int_t^1 r^{-\varepsilon q} \frac{dr}{r}\right) \frac{dt}{t} \\ &\le C \int_0^1 \left(t^\nu \|f\|_{L^p(A(0,t))}\right)^q \frac{dt}{t} \end{aligned}$$

Similarly, by Lemma 1.1 we see that

$$\begin{aligned} |u_3(x)| &\le C|x|^{\alpha-n-\ell} \int_{\boldsymbol{B}\setminus B(0,2|x|)} |y|^\ell f(y)\, dy \\ &\le C r^\alpha H^+_{n+\ell} f(r) \end{aligned}$$

for $x \in A(0,r)$. Hence, using Lemma 3.5, we find

$$\|u_3\|_{L^p(A(0,r))} \le C r^{\varepsilon-\nu} \left(\int_r^1 \left(t^{-\varepsilon+\nu} \|f\|_{L^p(A(0,t))}\right)^q \frac{dt}{t}\right)^{1/q}$$

for $0 < \varepsilon < -(n + \ell - n/p - \nu)$. Thus,

$$\begin{aligned} \int_0^1 \left(r^\nu \|u_3\|_{L^{p(\cdot)}(A(0,r))}\right)^q \frac{dr}{r} &\le C \int_0^1 \left(r^{\varepsilon} \int_r^1 \left(t^{-\varepsilon+\nu} \|f\|_{L^p(A(0,t))}\right)^q \frac{dt}{t}\right) \frac{dr}{r} \\ &\le C \int_0^1 \left(t^{-\varepsilon+\nu} \|f\|_{L^p(A(0,t))}\right)^q \left(\int_0^t r^{\varepsilon q} \frac{dr}{r}\right) \frac{dt}{t} \\ &\le C \int_0^1 \left(t^\nu \|f\|_{L^p(A(0,t))}\right)^q \frac{dt}{t} \end{aligned}$$

□

4 Growth near the origin of spherical means

The L^q ($1 \le q < \infty$) means over the spherical surface $S(0,r)$ for a function u is defined by

$$\begin{aligned} S_q(u,r) &= \left(\frac{1}{|S(0,r)|} \int_{S(0,r)} |u(x)|^q \, dS(x)\right)^{1/q} \\ &= \left(\frac{1}{\omega_{n-1}} \int_{S(0,1)} |u(r\sigma)|^q \, dS(\sigma)\right)^{1/q}, \end{aligned}$$

where $S(0,r) = \partial B(0,r)$ and $|S(0,r)| = \omega_{n-1} r^{n-1}$ with ω_{n-1} denoting the area of the unit sphere.

Our aim is to find $d > 0$ such that

$$\liminf_{r\to 0+} r^d S_q(I_{\alpha,\ell} f, r) = 0$$

for a function f on $\boldsymbol{B}$ satisfying Herz-Morrey type conditions.

Our result is a continuation of Gardiner's result ([13, 1988]) :

REMARK 4.1. For a Green potential $G\mu$ on $\boldsymbol{B}$,

(1) if $(n-1)/(n-2) \le q < (n-1)/(n-3)$, then

$$\liminf_{r\to 1} (1-r)^{n-1-(n-1)/q} S_q(G\mu, r) = 0;$$

(2) if $1 \le q < (n-1)/(n-2)$, then

$$\lim_{r\to 1} (1-r)^{n-1-(n-1)/q} S_q(G\mu, r) = 0.$$

THEOREM 4.2. *Suppose* $n - n/p + \ell < \nu < n - n/p + \ell + 1$. *If* $(n - \alpha p - 1)/(p(n-1)) < 1/q \le 1/p$, *then*

$$\liminf_{r\to 0+} r^{(n-\alpha p+\nu p)/p} S_q(I_{\alpha,\ell} f, r) < \infty$$

for all $f \in M^{p,\nu}(\boldsymbol{B})$.

THEOREM 4.3. *Suppose* $n - n/p + \ell < \nu < n - n/p + \ell + 1$. *If* $(n - \alpha p - 1)/(p(n-1)) < 1/q \le 1/p$, *then*

$$\liminf_{r\to 0+} r^{(n-\alpha p+\nu p)/p} S_q(I_{\alpha,\ell} f, r) = 0$$

for all $f \in M_0^{p,\nu}(\boldsymbol{B})$.

Part II

Boundary growth properties

5 Superharmonic functions on $\boldsymbol{B}$

The Riesz kernel is written as

$$|x-y|^{\alpha-n} = \sum_{\ell} (1-|y|)^{\ell} \phi_{\alpha,\ell}(x, \tilde{y}),$$

where $\tilde{y} = y/|y|$ and

$$\phi_{\alpha,\ell}(x,\tilde{y}) = \sum_{\ell/2 \le k \le \ell} a_{\alpha,\ell,k} |x-\tilde{y}|^{\alpha-n-2k} (x\cdot\tilde{y} - 1)^{2k-\ell}.$$

In fact, consider the Taylor expansion of

$$|x-y|^{\alpha-n} = |x-\tilde{y}+t\tilde{y}|^{\alpha-n}$$

and set $t = 1 - |y|$.

Now define

$$K_{\alpha,m}(x,y) = \frac{1}{(n-\alpha)\sigma_n} \begin{cases} |x-y|^{\alpha-n} & (y \in B(0,1/2)); \\ |x-y|^{\alpha-n} - \sum_{\ell=0}^{m} (1-|y|)^{\ell} \phi_{\alpha,\ell}(x,\tilde{y}) & (y \in \boldsymbol{B} \setminus B(0,1/2)). \end{cases}$$

The following properties for $K_{2,m}$ are fundamental.

LEMMA **5.1** (cf. [9, Lemma 2.2]). *(1)* $\Delta K_{2,m}(\cdot, y) = \delta_y$ *when* $n > 2$*;*

(2) $|K_{\alpha,m}(x,y)| \le C|x-y|^{\alpha-n-m-1}(1-|y|)^{m+1}$ *when* $1-|y| \le (\sqrt{2}-1)|x-\tilde{y}|$.

We show Riesz decomposition for superharmonic functions on $\boldsymbol{B}$.

THEOREM **5.2.** *If* u *is superharmonic in* $\boldsymbol{B}$ *and*

$$\liminf_{r\to 1} (1-r)^a S(u,r) > -\infty,$$

then

$$u(x) = \int_{\boldsymbol{B}} K_{2,m}(x,y)\, d\mu(y) + h_0(x),$$

where h_0 *is harmonic in* $\boldsymbol{B}$ *and* m *is an integer greater than* a *.*

Set

$$C(0,r) = B(0, r+(1-r)/2) \setminus B(0, r-(1-r)/2)$$

for $0 < r < 1$.

Denote by $\tilde{M}^{p,\nu}(\boldsymbol{B})$ the family of all functions $f \in L^1_{\mathrm{loc}}(\boldsymbol{B})$ such that

$$\|f\|_{\tilde{M}^{p,\nu}(\boldsymbol{B})} = \sup_{0<r<1} (1-r)^{\nu} \|f\|_{L^p(C(0,r))} < \infty.$$

Now we give a continuation of the results by Gardiner [13].

THEOREM **5.3.** *Let* $1 \le q < \infty$ *and suppose* $\|F\|_{\tilde{M}^{p,\nu}(\boldsymbol{B})} \le 1$ *with* $F(y) = (1-|y|)f(y)$*. Then there exists a constant* $C > 0$ *such that*

(1) *if* $n+m-1-\alpha p < (n-1)/q \le n+m-\alpha p$*, then*

$$\liminf_{r\to 1} (1-r)^{(n-\alpha p+\nu)/p-(n-1)/q} S_q(K_{\alpha,m} f, r) \le C;$$

(2) *if* $n-\alpha p+m < (n-1)/q < n+m+1-\alpha p$*, then*

$$\sup_{1/2<r<1} (1-r)^{(n-\alpha p+\nu)/p-(n-1)/q} S_q(K_{\alpha,m} f, r) \le C.$$

6 Isolated singularities for monotone functions in the sense of Lebesgue [18, 1907]

A continuous function u on a domain D is said to be monotone in the sense of Lebesgue [18] if for every subdomain G, $\overline{G} \subset D$,

$$\max_{\bar{G}} u = \max_{\partial G} u \qquad \text{and} \qquad \min_{\bar{G}} u = \min_{\partial G} u;$$

see Heinonen-Kilpeläinen- Martio [15], Koskela-Manfredi-Villamor [17], Manfredi-Villamor [20, 21], the author [22, 23], the author-Shimomura [24, 25], Villamor-Li [29] , Vuorinen [30, 31] .

THEOREM **6.1.** *Suppose $n-1<p<\nu+n$. Let u be a function on $\boldsymbol{B}\setminus\{0\}$ which is monotone in the sense of Lebesgue and satisfies*

$$\sup_{0<r<1} r^{\nu}\int_{A(0,r)}|\nabla u(x)|^{p}\,dx\le 1\quad (p>n-1).$$

Then

$$\sup_{x\in\boldsymbol{B}} r^{(n-p-\nu)/p}|u(x)|\le C<\infty.$$

For monotone functions in sense of Lebesgue, the following is a crucial tool.

LEMMA **6.2** (cf. [20], [21], [23]). *If u is monotone in $B(x_0,2r)$ in the sense of Lebesgue and $p_1>n-1$, then $\forall x,y\in B(x_0,r)$*

$$|u(x)-u(y)|^{p_1}\le Cr^{p_1-n}\int_{B(x_0,2r)}|\nabla u(z)|^{p_1}dz. \tag{6.1}$$

EXAMPLE **6.3.** For $\beta>0$, consider $u(x)=|x|^{-\beta}$. Then

- u is monotone in $\boldsymbol{B}\setminus\{0\}$ in the sense of Lebesgue ;
- $|\nabla u(x)|\le C|x|^{-\beta-1}$.

If $-(\beta+1)p+\nu+n\ge 0$,

$$\sup_{0<r<1} r^{\nu}\int_{A(0,r)}|\nabla u(x)|^{p}\,dx<\infty.$$

Hence, letting $\beta=(n-p+\nu)/p\ge 0$, we find

$$\lim_{x\to 0}|x|^{(n-p+\nu)/p}u(x)=1.$$

Finally we show boundary growth for monotone functions on $\boldsymbol{B}$ in the sense of Lebesgue.

THEOREM **6.4.** *Suppose $n-1<p<\nu+n$, $p<q<\infty$. Let u be a function on $\boldsymbol{B}$ which is monotone in the sense of Lebesgue and satisfies*

$$\sup_{0<r<1}(1-r)^{\nu}\int_{C(0,r)}|\nabla u(x)|^{p}\,dx\le 1.$$

If $(n-1)/q<(n-p+\nu)/p$, then

$$\sup_{0<r<1}(1-r)^{(n-p-\nu)/p-(n-1)/q}S_q(u,r)\le C<\infty.$$

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