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Some problems in forcing theory: large continuum and generalized cardinal invariants

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Abstract

We present a number of open problems in forcing theory. Our focus is on two areas: models where the continuum $\mathfrak{c} = 2^{\omega}$ is larger than $\aleph_2$, and models where the generalized continuum 2^{κ} for regular uncountable κ is blown up.

Introduction

We present a number of open problems in set theory. Our list has a strongly personal flavor, and we do not strive towards completeness in any of the areas we are touching upon. While it is not excluded that a ZFC result is lurking behind some of our questions, we believe that almost all of them will eventually lead to consistency results and are – thus – problems in forcing theory. In fact, many of these problems may require the development of novel forcing techniques, and our focus is on two areas where the use of new methods seems essential, namely

- models in which the continuum is larger than $\aleph_2$, see Sections 1 and 2 (such models are often difficult to construct because the powerful method of countable support iteration csi is not available),
- models in which the generalized continuum 2^{κ} for regular uncountable κ is increased, see Section 3 (such models are not well-understood yet because we don't have a good preservation theory for the iterations).

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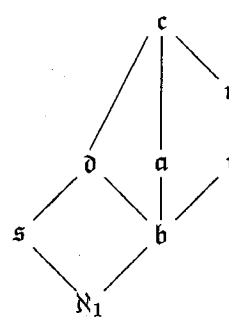
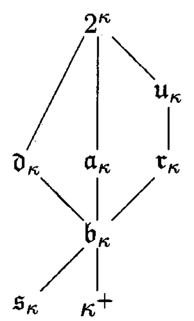
Most (though not all) of our problems involve cardinal invariants and we begin by reviewing some basic definitions. Fix a regular cardinal κ.

- The *unbounding number* $\mathfrak{b}_\kappa$ is the smallest size of a family $\mathcal{F} \subseteq \kappa^\kappa$ such that for all $g \in \kappa^\kappa$ there is $f \in \mathcal{F}$ which is cofinally often above g;
- the *dominating number* $\mathfrak{d}_\kappa$ is the smallest size of a family $\mathcal{F} \subseteq \kappa^\kappa$ such that for all $g \in \kappa^\kappa$ there is $f \in \mathcal{F}$ which is eventually above g;
- *splitting number* $\mathfrak{s}_\kappa$ is the smallest size of a family $\mathcal{A} \subseteq [\kappa]^\kappa$ such that for all $X \in [\kappa]^\kappa$ there is $A \in \mathcal{A}$ such that $|X \cap A| = |X \setminus A| = \kappa$;
- the *unreaping number* $\mathfrak{r}_\kappa$ is the smallest size of a family $\mathcal{A} \subseteq [\kappa]^\kappa$ such that for all $X \in [\kappa]^\kappa$ there is $A \in \mathcal{A}$ such that either $|A \setminus X| < \kappa$ or $|A \cap X| < \kappa$;
- the *ultrafilter number* $\mathfrak{u}_\kappa$ is the smallest size of a base of a uniform ultrafilter on κ;
- the *almost disjointness number* $\mathfrak{a}_\kappa$ is the smallest size of a maximal almost disjoint family in $[\kappa]^\kappa$ of size $\geq \kappa$.

If $\kappa = \omega$, we omit the subscript. The unbounding and dominating numbers are dual to each other, and so are the splitting and unreaping numbers. The order relationship between these cardinals is given by van Douwen's diagram:





For proofs of the inequalities in the left-hand diagram, which is for $\kappa = \omega$, see [Bl]. The corresponding inequalities for regular uncountable κ have exactly the same proofs. The only difference is the position of $\mathfrak{s}_\kappa$. In fact, $\mathfrak{s}_\kappa$ can be smaller than κ (more explicitly, $\mathfrak{s}_\kappa \geq \kappa$ iff κ is strongly inaccessible [Za1], $\mathfrak{s}_\kappa \geq \kappa^+$ iff κ is weakly compact [Za1], and the exact consistency strength of, say, $\mathfrak{s}_\kappa = \kappa^{++}$ is $o(\kappa) = \kappa^{++}$ [Za1, BG]) and $\mathfrak{s}_\kappa \leq \mathfrak{b}_\kappa$ holds in ZFC [RS1] (we will come back to this result in Section 3), while $\mathfrak{s} > \mathfrak{b}$ is consistent [Sh2] (see also Section 2). The left-hand diagram is complete in the sense that for any two cardinal invariants $\mathfrak{x}$ and $\mathfrak{y}$, if $\mathfrak{x}$ is not below $\mathfrak{y}$ in the diagram, then $\mathfrak{x} > \mathfrak{y}$ is consistent. This is not known for the right-hand diagram, see e.g. Problems 25 and 27 below.

For the classical Cantor and Baire spaces, 2^ω and ω^ω, cardinal invariants of the meager and the null ideals have been investigated thoroughly. Let $\mathcal{I}$ be a non-trivial ideal on a set X; that is, we assume $X \notin \mathcal{I}$ and all singletons belong to $\mathcal{I}$. Define

- the *additivity*: $\mathsf{add}(\mathcal{I}) = \min\{|\mathcal{J}| \colon \mathcal{J} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{J} \notin \mathcal{I}\}$;
- the *covering number*: $\mathsf{cov}(\mathcal{I}) = \min\{|\mathcal{J}| \colon \mathcal{J} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{J} = X\}$;
- the *uniformity*: $\mathsf{non}(\mathcal{I}) = \min\{|Y| \colon Y \subset X \text{ and } Y \notin \mathcal{I}\}$;
- the *cofinality*: $\mathsf{cof}(\mathcal{I}) = \min\{|\mathcal{J}| \colon \mathcal{J} \subseteq \mathcal{I} \text{ and } \forall I \in \mathcal{I}\, \exists J \in \mathcal{J}\ (I \subseteq J)\}$.

Additivity and cofinality are dual to each other, and so are the covering number and uniformity. If we let $\mathcal{M}$ and $\mathcal{N}$ denote the ideals of meager and null sets on the real numbers, respectively, then the order relationship between their cardinal invariants – as well as $\mathfrak{b}$ and $\mathfrak{d}$ – can be displayed in Cichoń's diagram:

$$
\begin{tikzcd}
 & \mathsf{cov}(\mathcal{N}) \arrow[r, dash] \arrow[dd, dash] & \mathsf{non}(\mathcal{M}) \arrow[r, dash] \arrow[d, dash] & \mathsf{cof}(\mathcal{M}) \arrow[r, dash] \arrow[d, dash] & \mathsf{cof}(\mathcal{N}) \arrow[r, dash] \arrow[dd, dash] & \mathfrak{c} \\
 & & \mathfrak{b} \arrow[r, dash] \arrow[d, dash] & \mathfrak{d} \arrow[d, dash] & & \\
\aleph_1 \arrow[r, dash] & \mathsf{add}(\mathcal{N}) \arrow[r, dash] & \mathsf{add}(\mathcal{M}) \arrow[r, dash] & \mathsf{cov}(\mathcal{M}) \arrow[r, dash] & \mathsf{non}(\mathcal{N}) &
\end{tikzcd}
$$

Additionally, $\mathsf{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathsf{cov}(\mathcal{M})\}$ and, dually, $\mathsf{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathsf{non}(\mathcal{M})\}$ hold. The proofs of all the inequalities can be found in either [BJ, Chapter 2] or [Bl]. We shall see in Section 3 that almost all the proofs in the middle part of the diagram can be redone for uncountable regular κ and the generalized Cantor and Baire spaces, 2^κ and κ^κ. Cichoń's diagram is complete in the strong sense that any assignment of the values $\aleph_1$ and $\aleph_2$ which does not contradict the diagram (and the two equalities mentioned above) is consistent. This is proved with csi [BJ, Chapter 7].

1 Large continuum

When Laver [Lav] had established the consistency of the Borel conjecture BC, with an ω_2-length countable support iteration (csi) of either Laver or Mathias forcing over a model of CH, the question of the consistency of BC with $\mathfrak{c} > \aleph_2$ soon arose. For indeed, csi only allows for a continuum of size at most $\aleph_2$. There was even some hope that this was a test problem whose solution might shed some light on new forcing techniques for large continuum. This, however, turned out not to be the case: Judah, Shelah, and Woodin [JSW] (see also [BJ, Theorem 8.3.7]) proved that BC still holds when many random reals are added (by the measure algebra) over the Laver or the Mathias models.

For several classes of ultrafilters on the natural numbers, the consistency of the non-existence of ultrafilters in the class was established by a csi of proper forcing. This is so because Cohen reals which naturally arise in finite support iterations (fsi) of ccc forcing can be used to produce all sorts of ultrafilters with strong combinatorial properties. The most prominent examples are Ramsey ultrafilters, P-points, Q-points, and nowhere dense ultrafilters, shown to consistently not exist by Kunen [Ku] (see also [Je, Theorem 91]), Shelah (see [Sh1], [Wi], or [BJ, Theorem 4.4.7]), Miller [Mi1], and Shelah [Sh3], respectively. Recall here that a free ultrafilter $\mathcal{U} \subseteq [\omega]^\omega$ is

- a *P-point* if given any countable $\mathcal{A} \subseteq \mathcal{U}$ there is $B \in \mathcal{U}$ with $B \subseteq^* A$ for all $A \in \mathcal{A}$;
- a *Q-point* if given any partition $(X_n : n \in \omega)$ of ω into finite sets, there is $A \in \mathcal{U}$ with $|X_n \cap A| \leq 1$ for all n;
- *Ramsey* if given any partition $(X_n : n \in \omega)$ of ω, either $X_n \in \mathcal{U}$ for some n or there is $A \in \mathcal{U}$ with $|X_n \cap A| \leq 1$ for all n, iff $\mathcal{U}$ is both a P-point and a Q-point;
- *nowhere dense* if for all functions $f : \omega \to \mathbb{Q}$ there is $A \in \mathcal{U}$ such that $f[A]$ is nowhere dense.

Since $\mathcal{U}$ is a P-point iff for all functions $f : \omega \to \mathbb{Q}$ there is $A \in \mathcal{U}$ such that $f[A]$ is a converging sequence (possibly converging to $\pm\infty$), every P-point is nowhere dense.

Kunen's model for no Ramsey ultrafilters is the random model (and thus, the continuum can be of arbitrary size), while the other models are obtained by csi. Models for no Q-points are the Laver and Mathias models, that is, the classical BC models, and, indeed, Judah and Shelah [JS] (see also [BJ, Theorem 4.6.7]) proved that adding random reals over either model still gives a model with no Q-points. What happens to P-points after adding random reals is much less clear. Cohen [Co] claimed there is a P-point in the standard random model (the model obtained by adding random reals over a model of CH), but his argument is flawed (as pointed out by Guzmán and Hrušák).

Problem 1. *Assume CH and add any number of random reals. Is there a P-point?*

We strongly conjecture this is correct because it does hold for some CH models.

Observation 1 (Kunen). *Add first ω_1 Cohen reals and then any number of random reals. Then there is a P-point.*

Proof sketch. Denote the ground model by V. Let $(c_\alpha : \alpha < \omega_1)$ be the Cohen reals. Let $\bar{r}$ be the random sequence. Put $W_\alpha = V[c_\beta : \beta < \alpha][\bar{r}]$ for $\alpha \leq \omega_1$. The W_α are intermediate models of the final extension W_{ω_1}, and – since random forcing is ω^ω-bounding – the c_α are still unbounded over W_α (though not Cohen anymore). It is relatively straightforward to prove that this implies that if $\mathcal{U}_\alpha$ is an ultrafilter in W_α, then there is an ultrafilter $\mathcal{U}_{\alpha+1}$ in $W_{\alpha+1}$ containing $\mathcal{U}_\alpha$ such that for any countable $\mathcal{A} \subseteq \mathcal{U}_\alpha$, $\mathcal{A} \in W_\alpha$, there is $B \in \mathcal{U}_{\alpha+1}$ such that $B \subseteq^* A$ for every $A \in \mathcal{A}$. More explicitly, given $\mathcal{A} = (A_n : n \in \omega) \subseteq \mathcal{U}_\alpha$ from W_α let $B_\mathcal{A} = \bigcup_n (\bigcap_{i \leq n} A_i \cap c_\alpha(n))$ and check that $\mathcal{U}_\alpha$ together with all the $B_\mathcal{A}$ still forms a filter base in $W_{\alpha+1}$. Thus, a P-point can be constructed in ω_1 many steps. □

It's less clear what happens if we add random reals over models with no P-points.

Problem 2. *Let V be a model with no P-points, like the model obtained by a csi of Grigorieff forcing or the Silver model, and add any number of random reals over V. Is there still no P-point?*

Again, we conjecture this to be true. The classical model for no P-points [Sh1, BJ] is obtained by a csi of Grigorieff forcing. Very recently, Chodounský and Guzmán [CG] established that there are no P-points in Silver models, namely, in the models obtained by adding Silver reals with either a csi or a countable support product (csp). Since the latter allows for a continuum of arbitrary size, the consistency of no P-points with large continuum follows. We still do not know:

Problem 3. *Is it consistent that $\mathfrak{c} > \aleph_2$ and there are no nowhere dense ultrafilters?*

We even don't know whether this holds in Silver models. Adding random reals over a model for no nowhere dense ultrafilter is of no use because random reals force *generic existence* of nowhere dense ultrafilters [Br2] (the statement that every filter base of size $< \mathfrak{c}$ can be extended to a nowhere dense ultrafilter).

A classical problem [Mi4, Problem 9.1] asks:

Problem 4 (Miller). *Is it consistent that there are simultaneously no P-points and no Q-points?*

The point is that $\mathfrak{c} > \aleph_2$ must hold in such a model because $\mathfrak{d} = \aleph_1$ implies existence of Q-points while $\mathfrak{d} = \mathfrak{c}$ implies existence (even generic existence) of P-points (see [BJ, Theorems 4.4.5 and 4.6.6]). Since $\mathfrak{d} = \aleph_1$ in all known models for no P-points, a simpler problem may be:

Problem 5. *Is it consistent that $\mathfrak{d} \geq \aleph_2$ and there are no P-points?*

All known models for no Q-points have $\mathfrak{b} = \mathfrak{d} = \aleph_2$ or $\mathfrak{b} = \aleph_1 < \mathfrak{d} = \aleph_2$ [Mi3]. While this is no obstruction to solving Problem 4, the model for no Q-points obtained by adding random reals over either the Laver or the Mathias models does have P-points, by the argument of Observation 1. Also we may ask:

Problem 6. *Is it consistent that $\mathfrak{d} \geq \aleph_3$ (or even $\mathfrak{b} \geq \aleph_3$) and there are no Q-points?*

Similarly, BC implies $\mathfrak{b} \geq \aleph_2$ and in all known models of BC, $\mathfrak{b} = \mathfrak{d} = \aleph_2$, so we may ask:

Problem 7. *Is it consistent that $\mathfrak{b} \geq \aleph_3$ and the Borel conjecture holds?*

Like BC, a number of interesting consistency results have been obtained by csi of definable proper forcing notions, such as Sacks, Silver, Miller, Laver, Mathias, and other forcing notions, and in several cases it is unknown whether consistency with $\mathfrak{c} > \aleph_2$ can be obtained. A typical problem is:

Problem 8 (Miller [Mi2]). *Is it consistent that $\mathfrak{c} > \aleph_2$ and every set of reals of size $\mathfrak{c}$ maps continuously onto the unit interval?*

Miller [Mi2] proved that this holds in the Sacks model, that is, the model obtained by a csi of Sacks forcing, but fails after adding Sacks reals with a csp. In fact, this statement follows from one of the covering property axioms CPA introduced by Ciesielski and Pawlikowski [CP] (see, in particular, [CP, Section 1.1]) which capture to a large extent the combinatorics of the csi Sacks model. Similar axioms have been considered for countable support iterations of other forcing notions, see also Zapletal's work [Za2, Subsection 6.1.1]. Thus the ultimate question in this direction may well be:

Problem 9. *Are there natural analogs of the covering property axioms CPA which are consistent with the continuum being larger than $\aleph_2$?*

Basically we are asking here to what extent there are Sacks-like or Laver-like models in which the continuum is large. One approach to solve this kind of problem might be to try to generalize Neeman's method [Ne] to the context of countable sequences of models and countable support.

2 A plethora of cardinal invariants

In the previous section we already looked at situations where we not only want the continuum to be large while a certain combinatorial statement holds but where we also require a cardinal invariant to assume a specific value, see e.g. Problems 5, 6, and 7. Pushing this further, we may consider situations where several cardinal invariants assume distinct values.

First consider the three cardinals $\mathfrak{b}$, $\mathfrak{a}$, and $\mathfrak{s}$. Their relationship has been investigated in a series of articles. $\mathfrak{b}$ is below $\mathfrak{a}$ (see Introduction), but the consistency of $\mathfrak{b} < \mathfrak{a}$ is not straightforward. In fact, the combinatorial principle $\diamondsuit(\mathfrak{b})$ which is a bit stronger than $\mathfrak{b} = \aleph_1$ already implies $\mathfrak{a} = \aleph_1$ [MHD]. Shelah [Sh2] established the consistency of $\aleph_1 = \mathfrak{b} < \mathfrak{a} = \mathfrak{c} = \aleph_2$, and we generalized this to regular κ and κ^+ instead of $\aleph_1$ and $\aleph_2$, respectively [Br1]. Using templates or ultrapowers [Sh5], the gap between $\mathfrak{b}$ (and even $\mathfrak{d}$) and $\mathfrak{a}$ can be made arbitrarily large (see also [Br3]). The order relationship of $\mathfrak{s}$ and $\mathfrak{b}$ (or $\mathfrak{a}$, for that matter) is independent. $\mathfrak{s} < \mathfrak{b}$ is easier and holds in Hechler's model, while $\mathfrak{b} < \mathfrak{s}$ (and even $\mathfrak{a} < \mathfrak{s}$) was established by Shelah [Sh2]. In fact, his model for $\mathfrak{b} < \mathfrak{a}$ mentioned above is a modification of the latter model, and $\mathfrak{s}$ is large in both models. In particular, the following is still open (see [Br4, Problem 1]):

Problem 10 (Brendle and Raghavan [BR]). *Is it consistent that $\mathfrak{b} = \mathfrak{s} = \aleph_1$ and $\mathfrak{a} > \aleph_1$?*

Of course, in template models $\max\{\mathfrak{b}, \mathfrak{s}\}$ is strictly smaller than $\mathfrak{a}$, but then $\mathfrak{b}$ is at least $\aleph_2$. In fact, with templates, models for $\mathfrak{s} < \mathfrak{b} < \mathfrak{a}$ can be constructed, see [Br3] for the case $\mathfrak{s} = \aleph_1$ and [Me2, FM] for the general case $\mathfrak{s} > \aleph_1$ ([Me2] uses a measurable and [FM] is on the basis of ZFC alone). Having the three cardinals distinct in another order is a much harder problem.

Matrix iterations (that is, two-dimensional systems of partial orders and complete embeddings between them), originally introduced by Blass and Shelah [BS] to prove that $\mathfrak{u} < \mathfrak{d}$ may consistently assume arbitrary regular values, have been used by Fischer and the author [BF] to show the consistency of $\mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \mathfrak{c} = \lambda$ for arbitrary regular κ and λ and, assuming the consistency of the existence of a measurable cardinal μ, of $\mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{a} = \mathfrak{c} = \lambda$ where both κ and λ are larger than μ. (The reason for the use of the measurable is that in one direction in the matrix iteration we take ultrapowers of partial orders to increase $\mathfrak{a}$.)

Problem 11 (Brendle and Fischer [BF]). *Show the consistency of $\mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{a} = \mathfrak{c} = \lambda$ for arbitrary regular κ and λ on the basis of ZFC alone!*

In seminal work, Raghavan and Shelah [RS2] have recently proved the consistency $\mathfrak{b} < \mathfrak{s} < \mathfrak{a}$ using Boolean ultrapowers of partial orders and assuming the consistency of the existence of a supercompact cardinal. However, we still do not know:

Problem 12 (Brendle and Fischer [BF]). *Is it consistent that $\mathfrak{b} < \mathfrak{a} < \mathfrak{s}$?*

A natural approach for this would be to use a three-dimensional matrix iteration, adding κ Cohen reals in one direction as a witness for $\mathfrak{b}$ and then iterating by taking ultrapowers for λ steps, and by forcing with Mathias forcing with a carefully constructed ultrafilter for μ steps, in the other two directions. The main problem with this approach is that we do not know how to show the completeness of all the embeddings in this three-dimensional setting. Notice that while such an approach would still need a measurable, it should also reduce the consistency strength of the Raghavan-Shelah result $\mathfrak{b} < \mathfrak{s} < \mathfrak{a}$ to a measurable.

So far, three-dimensional matrix iterations have been used only once, in recent and important work of Fischer, Friedman, Mejía, and Montoya [FFMM] which deals with models in which many cardinal invariants in Cichoń's diagram simultaneously assume distinct values. However, the situation they are dealing with is rather special, and it seems doubtful whether it can be used for the above problems. Still open in this context is:

Problem 13 (Folklore). *Is it consistent that eight of the cardinals in Cichoń's diagram assume distinct values? Can we additionally have that $\aleph_1 < \mathsf{add}(\mathcal{N})$ and $\mathsf{cof}(\mathcal{N}) < \mathfrak{c}$?*

Note that while there are ten cardinals in Cichoń's diagram, eight is the maximum number which could be simultaneously distinct because of the equalities $\mathsf{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathsf{cov}(\mathcal{M})\}$ and $\mathsf{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathsf{non}(\mathcal{M})\}$.

Roughly speaking, two approaches have been used to obtain models with many different cardinals: fsi of ccc forcing, with the best results obtained by matrix-style iterations (see, e.g., the work [FFMM] quoted above which has six distinct values), and csp of proper forcing (see, e.g., [FGKS] with four distinct values, which we will discuss below). To solve Problem 13, using the former method seems to be much more feasible because products have to be ω^ω-bounding (otherwise they collapse $\mathfrak{c}$ because they will embed the countable support product of Cohen forcing) and thus force $\mathfrak{d} = \aleph_1$.

An old folklore result says that four cardinals on the left-hand side of Cichoń's can be separated: $\aleph_1 < \mathsf{add}(\mathcal{N}) < \mathsf{cov}(\mathcal{N}) < \mathsf{add}(\mathcal{M}) = \mathfrak{b} < \mathsf{non}(\mathcal{M}) < \mathfrak{c}$ is consistent ($\star$): first blow up the continuum and then do an fsi of ccc forcing of cofinality ν going cofinally often through eventually different reals forcing to obtain $\mathsf{non}(\mathcal{M}) = \nu$, and through subalgebras of size $< \kappa$ ($< \lambda$, $< \mu$, respectively) of amoeba forcing (random forcing, Hechler forcing, resp.) for obtaining $\mathsf{add}(\mathcal{N}) = \kappa$ ($\mathsf{cov}(\mathcal{N}) = \lambda$, $\mathfrak{b} = \mu$, respectively) where $\kappa < \lambda < \mu < \nu$ are regular cardinals. In this model $\mathsf{cov}(\mathcal{M}) = \mathsf{non}(\mathcal{M})$, and Goldstern, Mejía, and Shelah [GMS] proved that $\aleph_1 < \mathsf{add}(\mathcal{N}) < \mathsf{cov}(\mathcal{N}) < \mathsf{add}(\mathcal{M}) = \mathfrak{b} < \mathsf{non}(\mathcal{M}) < \mathsf{cov}(\mathcal{M}) = \mathfrak{c}$ is also consistent. This is much harder because subforcings of eventually different reals forcing may add dominating reals. It is not known whether this can be pushed further to also get $\mathsf{cov}(\mathcal{M}) < \mathfrak{c}$:

Problem 14 (Goldstern, Mejía, and Shelah [GMS]). *Is* $\mathfrak{b} < \mathsf{non}(\mathcal{M}) < \mathsf{cov}(\mathcal{M}) < \mathfrak{c}$ *consistent?*

Since subforcings of random forcing may also add dominating reals we can ask as well for:

Problem 15 (Goldstern, Mejía, and Shelah [GMS]). *Is* $\mathfrak{b} < \mathsf{cov}(\mathcal{N}) < \mathsf{non}(\mathcal{M}) < \mathfrak{c}$ *(or* $< \mathsf{cov}(\mathcal{M})$*) consistent?*

Note in this context that Shelah's model for $cf(\mathsf{cov}(\mathcal{N})) = \omega$ [Sh4] necessarily satisfies $\mathfrak{b} < \mathsf{cov}(\mathcal{N}) < \mathsf{non}(\mathcal{M})$ (see [BJ, Theorem 5.1.17]) so that his technique may be relevant.

Concerning the right-hand side of Cichoń's diagram, things get more difficult: the consistency dual to ($\star$) above is not known because it is unclear what "going through subalgebras of ..." should be replaced with. The most promising choice is a multi-dimensional matrix iteration.

Problem 16 (Mejía [Me1]). *Is it consistent that* $\mathsf{cov}(\mathcal{M}) < \mathfrak{d} < \mathsf{non}(\mathcal{N}) < \mathsf{cof}(\mathcal{N})$*?*

In a product-like construction, Fischer, Goldstern, Kellner, and Shelah [FGKS] proved the consistency of $\mathsf{cov}(\mathcal{N}) = \mathfrak{d} = \aleph_1 < \mathsf{non}(\mathcal{M}) < \mathsf{non}(\mathcal{N}) < \mathsf{cof}(\mathcal{N}) < \mathfrak{c}$ and of $\mathsf{cov}(\mathcal{N}) = \mathfrak{d} = \aleph_1 < \mathsf{non}(\mathcal{N}) < \mathsf{non}(\mathcal{M}) < \mathsf{cof}(\mathcal{N}) < \mathfrak{c}$. While $\mathfrak{d} = \aleph_1$ is a drawback of such constructions, their advantage is that the cardinals on the left are not necessarily smaller than those on the right as is the case with fsi (because of the Cohen reals).

Apart from csp (and csi for the special case $\mathfrak{c} = \aleph_2$), models in which some cardinals on the left-hand side are larger than some on the right-hand side are obtained by random extensions. For example, letting $\kappa \leq \lambda \leq \mu \leq \nu$ with κ and λ regular uncountable, $\mathsf{cof}([\mu]^\omega) = \mu$ and $\nu^\omega = \nu$, we can force $\aleph_1 = \mathsf{non}(\mathcal{N}) \leq \mathfrak{b} = \kappa \leq \mathfrak{d} = \lambda \leq \mathsf{cov}(\mathcal{N}) = \mathsf{cof}(\mathcal{N}) = \mu \leq \mathfrak{c} = \nu$: first force $\mathfrak{b} = \kappa$, $\mathfrak{d} = \mathsf{cof}(\mathcal{N}) = \lambda$, and $\mathfrak{c} = \nu$ (this can be done with a matrix iteration, see e.g. [Me1]), and then add μ random reals with the measure algebra.

One of the drawbacks of random extensions is that they always force $\mathsf{non}(\mathcal{N}) = \aleph_1$. To address this problem and build models in which $\mathsf{non}(\mathcal{N})$ and $\mathsf{cov}(\mathcal{M})$ are larger but still smaller than some cardinals on the left, we developed shattered iterations [Br6]. They can used to show, e.g., the consistency of $\aleph_1 < \mathsf{cov}(\mathcal{M}) = \mathsf{non}(\mathcal{N}) = \kappa < \mathsf{cov}(\mathcal{N}) = \mathsf{non}(\mathcal{M}) = \lambda < \mathfrak{c} = \nu$ where κ and λ are regular and $\nu^\omega = \nu$. It is unclear, however, how $\mathfrak{b}$ can be pushed up in this context:

Problem 17 (Brendle [Br6]). *Is it consistent that* $\mathfrak{b} > \mathsf{cov}(\mathcal{M}) \geq \aleph_2$*? Can we even obtain* $\mathfrak{c} > \mathfrak{b}$*?*

The dual statement $\mathfrak{d} < \mathsf{non}(\mathcal{M}) < \mathfrak{c}$ is consistent, as remarked above (if $\mathfrak{d} = \aleph_1$, [FGKS] shows this, and for arbitrary $\mathfrak{d}$, use a random extension).

Let us note that there are a plethora of similar problems about models with many distinct cardinal invariants. We just singled out a number of test problems about $\mathfrak{b}$, $\mathfrak{a}$, and $\mathfrak{s}$, and about cardinals in Cichoń's diagram research on which has gotten a lot of attention during the past couple of years. To finish this section, let us mention one more interesting problem addressed by Raghavan and Shelah:

Problem 18 (Raghavan and Shelah [RS2]). *Is it consistent that* $\mathfrak{b} < \mathfrak{s} < \mathfrak{d}$*?*

3 Generalized cardinal invariants

In this section κ is always a regular uncountable cardinal. We consider cardinal invariants describing the combinatorial structure of the *generalized Baire space* κ^κ or the *generalized Cantor space* 2^κ. We start with problems arising from a generalized version of Cichoń's diagram investigated in work of Brooke-Taylor, Friedman, Montoya, and the author [BBFM].

While it is unclear how to generalize the null ideal to this context, there is a natural generalization of the meager ideal. For $s \in 2^{<\kappa}$, let $[s] = \{f \in 2^\kappa : s \subseteq f\}$, and consider the topology on 2^κ whose basic open (even clopen) sets are sets of the form $[s]$. This is the *$< \kappa$-box topology.* Say a subset of 2^κ is *κ-meager* if it is a union of at most κ many nowhere dense sets in this topology. Denote the ideal of κ-meager sets by $\mathcal{M}_\kappa$. The analog of the Baire Category Theorem holds, that is, no non-empty open set is κ-meager.

Proposition 2. (a) (Landver [Lan]) *If* $2^{<\kappa} > \kappa$, *then* $\mathsf{add}(\mathcal{M}_\kappa) = \mathsf{cov}(\mathcal{M}_\kappa) = \kappa^+$.

(b) (Blass, Hyttinen, and Zhang [BHZ]) $\mathsf{non}(\mathcal{M}_\kappa) \geq 2^{<\kappa}$.

(c) [Br5] $\mathsf{cof}(\mathcal{M}_\kappa) > 2^{<\kappa}$.

Proof sketch. (c) Let $\tilde{\sigma} \in 2^{<\kappa}$ and $\{\sigma_\gamma : \gamma < 2^{<\kappa}\} \subseteq 2^{<\kappa}$ be such that all σ_γ are pairwise incompatible and incompatible with $\tilde{\sigma}$. Let length$(\sigma_\gamma) = \zeta_\gamma$ for $\gamma < 2^{<\kappa}$. Fix a function $g : 2^{<\kappa} \to \kappa$. We will build a nowhere dense tree $T_g \subseteq 2^{<\kappa}$, as follows.

Recursively define sets C_g^α, $\alpha < \kappa$, such that

- all C_g^α are antichains in $2^{<\kappa}$,
- if $\alpha < \beta$ and $\tau \in C_g^\beta$, then there is $\sigma \in C_g^\alpha$ with $\sigma \subsetneq \tau$,
- if $\sigma \in C_g^\alpha$ then there is $\tau \supsetneq \sigma$ incompatible with all elements of $C_g^{\alpha+1}$.

Suppose we are at stage α and the C_g^β for $\beta < \alpha$ have been produced. If α is a limit ordinal, we put σ into C_g^α if there is a strictly increasing sequence of $\sigma^\beta \in C_g^\beta$, $\beta < \alpha$, such that $\sigma = \bigcup_{\beta<\alpha} \sigma^\beta$. If $\alpha = \beta + 1$ is successor, fix $\sigma \in C_g^\beta$. Assume length$(\sigma) = \zeta$. Put $\tau \supsetneq \sigma$ into C_g^α if there is $\gamma < 2^{<\kappa}$ such that $\sigma\hat{\ }\sigma_\gamma \subseteq \tau$ and length$(\tau) = \zeta + \zeta_\gamma + g(\gamma)$. Unfixing σ, let C_g^α be the set of such τ's. Notice that $\tau = \sigma\hat{\ }\tilde{\sigma}$ extends σ and is incompatible with all elements of C_g^α. Hence all the clauses are satisfied.

Let T_g be the nowhere dense tree generated by the C_g^α's, that is, $\tau \in T_g$ if there are $\alpha < \kappa$ and $\sigma \in C_g^\alpha$ such that $\tau \subseteq \sigma$.

Now assume $A \in \mathcal{M}_\kappa$, $A = \bigcup_{\alpha<\kappa} A_\alpha$, where the A_α form an increasing sequence of nowhere dense sets. Let $h^A : 2^{<\kappa} \to 2^{<\kappa}$ be such that for all $\alpha < \kappa$ and all $\sigma \in 2^\alpha$, we have $\sigma \subseteq h^A(\sigma)$ and $A_\alpha \cap [h^A(\sigma)] = \emptyset$. Next fix $\sigma \in 2^{<\kappa}$ with length$(\sigma) = \zeta$. Define $f_\sigma^A : 2^{<\kappa} \to \kappa$ such that length$(h^A(\sigma\hat{\ }\sigma_\gamma)) = \zeta + \zeta_\gamma + f_\sigma^A(\gamma)$ for all $\gamma < 2^{<\kappa}$.

If $\mathcal{A} \subseteq \mathcal{M}_\kappa$ is of size at most $2^{<\kappa}$, choose $g : 2^{<\kappa} \to \kappa$ such that no f_σ^A, $A \in \mathcal{A}$ and $\sigma \in 2^{<\kappa}$, bounds g. Fix $A \in \mathcal{A}$. It is easy to recursively construct $x \in [T_g] \setminus A$. Thus $\mathcal{A}$ is not a basis of $\mathcal{M}_\kappa$ and $\mathsf{cof}(\mathcal{M}_\kappa) > 2^{<\kappa}$ follows. □

In particular, while $\mathsf{add}(\mathcal{M}_\kappa)$, $\mathsf{cov}(\mathcal{M}_\kappa)$, and $\mathsf{non}(\mathcal{M}_\kappa)$ lie between κ^+ and 2^κ in ZFC, $\mathsf{cof}(\mathcal{M}_\kappa)$ is strictly larger than 2^κ assuming $2^{<\kappa} = 2^\kappa$. On the other hand, if $2^{<\kappa} = \kappa$, then $\mathsf{cof}(\mathcal{M}_\kappa) \leq 2^\kappa$ (because clearly $\mathsf{cof}(\mathcal{M}_\kappa) \leq 2^{2^{<\kappa}}$ in ZFC).

Concerning the middle part of Cichoń's diagram, the following hold for regular κ (this generalizes results well-known for ω, see the Introduction and [BJ, Chapter 2]):

- $\mathfrak{b}_\kappa \leq \mathsf{non}(\mathcal{M}_\kappa)$ and, dually, $\mathsf{cov}(\mathcal{M}_\kappa) \leq \mathfrak{d}_\kappa$ [BBFM, Observation 17],
- $\mathsf{add}(\mathcal{M}_\kappa) \leq \mathfrak{b}_\kappa$ and, dually, $\mathfrak{d}_\kappa \leq \mathsf{cof}(\mathcal{M}_\kappa)$ (see [BBFM, Corollary 28] for the case κ is strongly inaccessible and [Br5] for the general case),
- $\mathsf{add}(\mathcal{M}_\kappa) \geq \min\{\mathfrak{b}_\kappa, \mathsf{cov}(\mathcal{M}_\kappa)\}$ and, dually, if $2^{<\kappa} = \kappa$ then $\mathsf{cof}(\mathcal{M}_\kappa) \leq \max\{\mathfrak{d}_\kappa, \mathsf{non}(\mathcal{M}_\kappa)\}$ [BBFM, Corollary 31].

By the second item, the inequalities in the third item are actually equalities. Note that the assumption for the second part of the third item is necessary because if $2^{<\kappa} = 2^\kappa$, then $\mathfrak{d}_\kappa, \mathsf{non}(\mathcal{M}_\kappa) \leq 2^\kappa < \mathsf{cof}(\mathcal{M}_\kappa)$ by Proposition 2.

In the degenerate case $2^{<\kappa} > \kappa$, one can rather freely monkey around with these cardinals except for the equality $\mathsf{cov}(\mathcal{M}_\kappa) = \kappa^+$. For example, $\kappa^+ = \mathsf{add}(\mathcal{M}_\kappa) = \mathsf{cov}(\mathcal{M}_\kappa) < \mathfrak{b}_\kappa = \kappa^{++} < \mathfrak{d}_\kappa = \kappa^{+++} < \mathsf{non}(\mathcal{M}_\kappa) = 2^\omega = 2^\kappa = \kappa^{+4} < \mathsf{cof}(\mathcal{M}_\kappa) = 2^{\kappa^{+4}} = \kappa^{+5}$ is consistent: start with a model of GCH, first force $\mathfrak{b}_\kappa = \kappa^{++}$ and $\mathfrak{d}_\kappa = 2^\kappa = \kappa^{+++}$ [CS], and then add κ^{+4} Cohen reals.

For the much more interesting case $2^{<\kappa} = \kappa$, models in which these cardinals assume distinct values are much harder to construct, and for a number of consistency results known for $\kappa = \omega$ it is unclear how to generalize them to uncountable κ. The Cohen model can be generalized for obtaining the consistency of $\mathsf{add}(\mathcal{M}_\kappa) = \mathsf{non}(\mathcal{M}_\kappa) = \kappa^+ < \mathsf{cov}(\mathcal{M}_\kappa) = \mathsf{cof}(\mathcal{M}_\kappa) = 2^\kappa$. Shelah [Sh6] proved the consistency of $\mathsf{cov}(\mathcal{M}_\kappa) < \mathfrak{d}_\kappa$ for a supercompact cardinal κ. Since his model is a "dual model" we conjecture [BBFM, Question 20]:

Problem 19 (Brendle, Brooke-Taylor, Friedman, and Montoya). *Is $\mathfrak{b}_\kappa < \mathsf{non}(\mathcal{M}_\kappa)$ consistent for supercompact κ?*

It is known [Lag] that there is no hope to generalize the consistency proofs of $\mathsf{add}(\mathcal{M}) < \mathfrak{b}$ using csi of Mathias or Laver or similar forcing notions to uncountable κ, and we therefore conjecture [BBFM, Question 84]:

Problem 20 (Brendle, Brooke-Taylor, Friedman, and Montoya). *Assume $2^{<\kappa} = \kappa$. Does $\mathsf{add}(\mathcal{M}_\kappa) = \mathfrak{b}_\kappa$ and $\mathfrak{d}_\kappa = \mathsf{cof}(\mathcal{M}_\kappa)$?*

Similarly, for successor cardinals it is unclear how to separate $\mathfrak{b}_\kappa$ and $\mathsf{non}(\mathcal{M}_\kappa)$ [BBFM, Question 24]:

Problem 21 (Brendle, Brooke-Taylor, Friedman, and Montoya). *Assume κ is successor with $2^{<\kappa} = \kappa$. Does $\mathfrak{b}_\kappa = \mathsf{non}(\mathcal{M}_\kappa)$ and $\mathsf{cov}(\mathcal{M}_\kappa) = \mathfrak{d}_\kappa$?*

There is another way to look at this:

- $\mathfrak{b}_\kappa(\neq^*)$ is the least size of a family $\mathcal{F} \subseteq \kappa^\kappa$ such that for all $g \in \kappa^\kappa$ there is $f \in \mathcal{F}$ such that f and g agree cofinally often;
- dually, $\mathfrak{d}_\kappa(\neq^*)$ is the least size of $\mathcal{F} \subseteq \kappa^\kappa$ such that for all $g \in \kappa^\kappa$ there is $f \in \mathcal{F}$ such that f and g eventually disagree.

It is easy to see that $\mathfrak{b}_\kappa \leq \mathfrak{b}_\kappa(\neq^*) \leq \mathsf{non}(\mathcal{M}_\kappa)$ and $\mathsf{cov}(\mathcal{M}_\kappa) \leq \mathfrak{d}_\kappa(\neq^*) \leq \mathfrak{d}_\kappa$ always hold. It is well-known [BJ, Theorems 2.4.1 and 2.4.7] that for $\kappa = \omega$, $\mathfrak{b}(\neq^*) = \mathsf{non}(\mathcal{M})$ and $\mathfrak{d}(\neq^*) = \mathsf{cov}(\mathcal{M})$ and this generalizes to the case κ is strongly inaccessible [BBFM, Corollary 19]. On the other hand, if κ is successor, then $\mathfrak{b}_\kappa(\neq^*) = \mathfrak{b}_\kappa$ (Hyttinen [Hy]) and if we additionally assume $2^{<\kappa} = \kappa$, then $\mathfrak{d}_\kappa(\neq^*) = \mathfrak{d}_\kappa$ (Matet and Shelah [MS]). It is unclear whether the assumption is $2^{<\kappa} = \kappa$ is necessary:

Problem 22 (Matet and Shelah [MS]). *Is it consistent that κ is a successor cardinal and $\mathfrak{d}_\kappa(\neq^*) < \mathfrak{d}_\kappa$?*

It may well be that $\mathfrak{b}_\kappa(\neq^*) = \mathsf{non}(\mathcal{M}_\kappa)$ and $\mathfrak{d}_\kappa(\neq^*) = \mathsf{cov}(\mathcal{M}_\kappa)$ hold whenever $2^{<\kappa} = \kappa$, and this would solve Problem 21.

While there have been attempts to generalize the null ideal to inaccessible cardinals κ (see e.g. [Sh7] and [FL]), we shall not pursue this here but rather look at cardinals which generalize combinatorial characterizations of measure invariants. Fix a function $h \in \kappa^\kappa$ with $\lim_{\alpha \to \kappa} h(\alpha) = \kappa$. A function φ with $\mathrm{dom}(\varphi) = \kappa$ and $\varphi(\alpha) \in [\kappa]^{|h(\alpha)|}$ for all $\alpha < \kappa$ is called an *h-slalom.*

- $\mathfrak{b}_h(\in^*)$ is the least size of a family $\mathcal{F} \subseteq \kappa^\kappa$ such that for all h-slaloms φ, there is $f \in \mathcal{F}$ such that $f(\alpha) \notin \varphi(\alpha)$ for cofinally many α;
- dually, $\mathfrak{d}_h(\in^*)$ is the least size of a set Φ of h-slaloms such that for all $f \in \kappa^\kappa$ there is $\varphi \in \Phi$ such that $f(\alpha) \in \varphi(\alpha)$ eventually holds.

Then $\mathfrak{b}_h(\in^*) \leq \mathfrak{b}_\kappa$ and $\mathfrak{d}_h(\in^*) \geq \mathfrak{d}_\kappa$, and for successor cardinals equality trivially holds. So these cardinals are only of interest for inaccessible κ. For $\kappa = \omega$, $\mathfrak{b}_h(\in^*) = \mathsf{add}(\mathcal{N})$ and $\mathfrak{d}_h(\in^*) = \mathsf{cof}(\mathcal{N})$ hold for all h [BJ, Theorem 2.3.9]. In view of this, the inequalities $\mathfrak{b}_h(\in^*) \leq \mathsf{add}(\mathcal{M}_\kappa)$ and $\mathsf{cof}(\mathcal{M}_\kappa) \leq \mathfrak{d}_h(\in^*)$ for strongly inaccessible κ [BBFM, Theorem 40] can be seen as a generalization of the classical Bartoszyński-Raisonnier-Stern Theorem.

On the consistency side, the consistency of $\mathfrak{b}_h(\in^*) < \mathsf{add}(\mathcal{M}_\kappa)$ and of $\mathsf{cof}(\mathcal{M}_\kappa) < \mathfrak{d}_h(\in^*)$ for strongly inaccessible κ is known [BBFM, Theorem 60]. Also, in κ-Sacks models, i.e., in models obtained over a model of GCH by either an iteration of length κ^{++} of κ-Sacks forcing with support of size κ, or by a κ-support product of κ-Sacks forcing, $\mathfrak{d}_h(\in^*) = \kappa^+ < \mathfrak{d}_{id}(\in^*) = 2^\kappa$ where h is the power set function $h(\alpha) = 2^\alpha$ while id is the identity function. Thus, unlike for ω, the different $\mathfrak{d}_h(\in^*)$ may assume distinct values for strongly inaccessible κ. Therefore we conjecture [BBFM, Question 71]:

Problem 23 (Brendle, Brooke-Taylor, Friedman, and Montoya). *Is $\mathfrak{b}_{id}(\in^*) < \mathfrak{b}_h(\in^*)$ consistent where h is the power set function?*

The main problem in this context is that we do not have preservation theorems for iterations with supports of size $< \kappa$ or of size κ corresponding to the preservation theorems for fsi and csi. The point is that the natural attempt to generalize the latter breaks down in limit stages of cofinality less than κ (a situation which does not occur for $\kappa = \omega$).

Problem 24. *Develop preservation theory for $< \kappa$-support iterations of κ^+-cc forcing and for κ-support iterations of generalizations of classical definable proper forcing notions like κ-Sacks forcing etc.!*

We now turn to problems on some other generalized cardinals. Raghavan and Shelah [RS1] proved $\mathfrak{s}_\kappa \leq \mathfrak{b}_\kappa$ for regular uncountable κ. It is unclear, however, whether the dual statement holds:

Problem 25 (Raghavan and Shelah [RS1]). *Does $\mathfrak{d}_\kappa \leq \mathfrak{r}_\kappa$?*

Using the method of Boolean ultrapowers of partial orders already mentioned in Section 2, Raghavan and Shelah [RS2] recently established the consistency of $\mathfrak{d}_\kappa < \mathfrak{a}_\kappa$ for arbitrary uncountable regular κ assuming the consistency of the existence of a supercompact cardinal.

Problem 26 (Raghavan and Shelah [RS2]). *Can the consistency of $\mathfrak{b}_\kappa < \mathfrak{a}_\kappa$ (or even $\mathfrak{d}_\kappa < \mathfrak{a}_\kappa$) for uncountable regular κ be established on the basis of the consistency of ZFC alone?*

We recall in this context that in case $\kappa = \omega$ such proofs (on the basis of ZFC) are done via templates [Sh5] (see also [Br3]), but it is unclear how to generalize the template framework to $\kappa > \omega$. We also note that Raghavan and Shelah [RS2] established that if $\mathfrak{b}_\kappa = \kappa^+$ then $\mathfrak{a}_\kappa = \kappa^+$, thus improving an earlier result of Blass, Hyttinen, and Zhang [BHZ] who had proved this with $\mathfrak{b}_\kappa$ replaced by $\mathfrak{d}_\kappa$.

Garti and Shelah [GS], building on earlier work of Džamonja and Shelah [DS], proved that for supercompact κ, $\mathfrak{u}_\kappa < 2^\kappa$ is consistent. It is unknown, however, whether such a consistency can be proved for "small" cardinals [Mi4, Problem 8.5].

Problem 27 (Kunen). *Is it consistent that $\mathfrak{u}_{\omega_1} < 2^{\omega_1}$?*

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