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VARIANTS OF THE GROUND AXIOM

TOSHIMICHI USUBA

Definition 0.1. The *Ground Axiom* GA is the assertion that the universe V does not have a proper ground model.

It is known that GA is a first order assertion. Let us say that a transitive model $W \subseteq V$ of ZFC is called a *groud* if there is a poset $\mathbb{P} \in W$ and a $(W, \mathbb{P})$-generic G with $V = W[G]$ ($V = W$ is possible).

Fact 0.2 (Reitz [2], Fuchs-Hamkins-Reitz [1]). *There is a first order formula* $\varphi(x, y)$ *such that:*

(1) For every set r*, the class* $W_r = \{x : \varphi(x, r)\}$ *is a groud of* V*.*
(2) For every ground W *of* V*, there is* r *with* $W = W_r$*.*

Then GA is the assertion that $\forall r (V = W_r)$.
We consider the following variant of GA, which is suggested by Reitz [2]:

Definition 0.3. Let Γ be a class of posets (e.g., c.c.c. posets, proper posets). GA_Γ is the assertion that the universe V does not have a proper ground W such that there is $\mathbb{P} \in \Gamma^W$ and a $(W, \mathbb{P})$-generic G with $V = W[G]$.

Note that if Γ is a parameter free definable class, then GA_Γ is a first order assertion as well.
In the paper we will consider the following classes of posets:

(1) c.c.c.,
(2) productively c.c.c, where the poset $\mathbb{P}$ is *productively c.c.c.* if for every c.c.c. poset $\mathbb{Q}$, the product poset $\mathbb{P} \times \mathbb{Q}$ is c.c.c.,
(3) proper,
(4) semi-proper,
(5) ω_1-stationary preserving, where the poset $\mathbb{P}$ is ω_1*-stationary preserving* if for every stationary subset S of ω_1, the forcing with $\mathbb{P}$ preserves the stationarity of S.
(6) ω_1-preserving, where the poset $\mathbb{P}$ is ω_1*-preserving* if the forcing with $\mathbb{P}$ preserves the cardinality of ω_1^V.

We prove the following:

Theorem 0.4. *The following are consistent:*

(1) $\mathsf{GA}_{\omega_1\text{-stat. pres.}} + \neg \mathsf{GA}_{\omega_1\text{-pres.}}$.

(2) $\mathsf{GA}_{\text{semi-proper}} + \neg\mathsf{GA}_{\omega_1\text{-stat. pres.}}$.
(3) $\mathsf{GA}_{\text{proper}} + \neg\mathsf{GA}_{\text{semi-proper}}$ *(under some large cardinal assumption).*
(4) $\mathsf{GA}_{\text{c.c.c.}} + \neg\mathsf{GA}_{\text{proper}}$.
(5) $\mathsf{GA}_{\text{prod. c.c.c.}} + \neg\mathsf{GA}_{\text{c.c.c.}}$.

1. Separating ω_1-stationary preserving, semi-proper, and proper

We use the following facts which are due to Shelah ([3]):

Fact 1.1. *(1) Namba forcing is ω_1-stationary preserving, and forces $\mathrm{cf}(\omega_2^V) = \omega$.*
(2) If CH holds, then Namba forcing does not add new reals.
(3) The following are equivalent:
(a) Namba forcing is semi-proper.
(b) The strong Chang's conjecture holds.
(c) There is a semi-proper forcing $\mathbb{P}$ which forces $\mathrm{cf}(\omega_2^V) = \omega$.
Here, the strong Chang's conjecture *is the assertion that for every sufficiently large regular θ, every countable $M \prec H_\theta$, and every $\gamma < \omega_2$, there is a countable $N \prec H_\theta$ such that $M \subseteq N$, $M \cap \omega_1 = N \cap \omega_1$, and $\sup(N \cap \omega_2) > \gamma$.*

Note that, if one of (a)–(c) in the fact holds, then Chang's conjecture holds, and $0^\#$ exists.

We start the proof. First we prove the consistency of $\mathsf{GA}_{\text{semi-proper}} + \neg\mathsf{GA}_{\omega_1\text{-stat. pres.}}$.

Suppose $V = L$. Let $\mathbb{P}$ be a Namba forcing notion, and let G be $(V, \mathbb{P})$-generic. Let $c \subseteq \omega_2^V$ be a generic cofinal subset of order-type ω. We see that $L[c]$ is a required model[1]. $L[c]$ is an ω_1-stationary preserving forcing extension of L, hence $\mathsf{GA}_{\omega_1\text{-stat. pres.}}$ fails in $L[c]$. To show that $\mathsf{GA}_{\text{semi-proper}}$ holds in $L[c]$, take a ground $W \subseteq L[c]$, a poset $\mathbb{Q} \in W$ which is semi-proper in W, and a $(W, \mathbb{Q})$-generic G with $L[c] = W[G]$. We see that $c \in W$, hence $W = L[c]$. Since $L \subseteq W \subseteq L[c]$, we have $\omega_1 = \omega_1^W = \omega_1^L = \omega_1^{L[c]}$. Moreover, since CH holds in L, we have $\mathcal{P}(\omega) \cap L = \mathcal{P}(\omega) \cap L[c] = \mathcal{P}(\omega) \cap W$.

Claim 1.2. *In W, $\mathrm{cf}(\omega_2^L) = \omega$.*

Proof. If $\mathrm{cf}^W(\omega_2^L) = \omega_2^L$, then $\omega_2^L = \omega_2^W$. Hence, in W, $\mathbb{Q}$ is a semi-proper forcing notion which forces $\mathrm{cf}(\omega_2^W) = \omega$. By Fact 1.1, we have that $0^\#$ exists in W, hence so does in $L[c]$. This is impossible because $L[c]$ is a set-forcing extension of L.

Next suppose $\mathrm{cf}^W(\omega_2^L) = \omega_1$. Because $\mathrm{cf}(\omega_2^L) = \omega$ in $L[c]$, we have that $\mathrm{cf}(\omega_1) = \omega$ in $L[c]$, hence ω_1 is collapsed. This is impossible. □

In W, take a club $C = \{x_i : i < \omega_1\}$ in $[\omega_2^L]^\omega$. In $L[c]$, C is a club in $[\omega_2^L]^\omega$. $c \subseteq \omega_2^L$ is countable, so there is some $i < \omega_1$ with $c \subseteq x_i$. Because x_i is countable in W and

[1] The author does not know if $L[c] = L[G]$. However, since $L \subseteq L[c] \subseteq L[G]$, we have that $L[c]$ is an ω_1-stationary preserving forcing extension of L.

$\mathcal{P}(\omega)\cap L = \mathcal{P}(\omega)\cap L[c] = \mathcal{P}(\omega)\cap W$, we have $\mathcal{P}(x)\cap L[c] = \mathcal{P}(x)\cap W$, and $c \in W$. This completes the proof of the consistency of $\mathsf{GA}_{\text{semi-proper}} + \neg\mathsf{GA}_{\omega_1\text{-stat. pres.}}$.

Next we prove $\mathsf{GA}_{\text{proper}} + \neg\mathsf{GA}_{\text{semi-proper}}$, but our proof needs some large cardinal assumption.

The *mantle* $\mathbf{M}$ is the class $\bigcap_r W_r$. GA is equivalent to the assertion $V = \mathbf{M}$. It is known that the mantle is a model of ZFC (Usuba [4]).

Suppose V satisfies GA, and there exists a measurable cardinal κ. This is consistent assuming the existence of a measurable cardinal.

Let $\mathbb{P}$ be a Prikry forcing notion associated with a normal measure over κ. Let G be a $(V,\mathbb{P})$-generic filter, and c a generic cofinal sequence in κ of order type ω. It is known that $V[c] = V[G]$. We see that $V[c]$ is a required model.

Prikry forcing is semi-proper, hence $\mathsf{GA}_{\text{semi-proper}}$ fails in $V[c]$. In order to see that $\mathsf{GA}_{\text{proper}}$ holds in $V[c]$, take a ground $W \subseteq V[c]$, a poset $\mathbb{Q} \in W$ which is proper in W, and a $(W,\mathbb{Q})$-generic H with $V[c] = W[H]$. We see that $c \in W$. V satisfies GA, hence V is equal to its mantle $\mathbf{M}$. Note that the mantle is forcing invariant ([4]). W is a ground of $V[c]$, hence we have that $V = \mathbf{M} \subseteq W$. Because Prikry forcing does not add new reals, we have $\mathcal{P}(\omega)\cap V = \mathcal{P}(\omega)\cap V[c] = \mathcal{P}(\omega)\cap W$. $V[c]$ is a proper forcing extension of W, hence there is $x \in W$ which is countable in W and $c \subseteq w$. We know $\mathcal{P}(\omega)\cap V[c] = \mathcal{P}(\omega)\cap W$, so $\mathcal{P}(x)\cap V[c] = \mathcal{P}(x)\cap W$ and we can conclude that $c \in W$. Finally, since $V = \mathbf{M} \subseteq W \subseteq V[c]$, we have $W = V[c]$.

Note 1.3.
(1) The same proof shows the consistency of $\mathsf{GA}_{\omega\text{-covering}}+\neg\mathsf{GA}_{\text{semi-proper}}$, where a poset $\mathbb{P}$ satisfies the *ω-covering property* if for every $(V,\mathbb{P})$-generic G and every countable set $x \in [V]^\omega \cap V[G]$, there is $y \in V$ which is countable in V and $x \subseteq y$.
(2) The author does not know the exact consistency strengths of $\mathsf{GA}_{\text{proper}} + \neg\mathsf{GA}_{\text{semi-proper}}$ and $\mathsf{GA}_{\omega\text{-covering}} + \neg\mathsf{GA}_{\text{semi-proper}}$.

2. Separating proper, c.c.c., and productively c.c.c.

To proceed our proofs, we will use the approximation property.

Definition 2.1. Let $\mathbb{P}$ be a poset, and κ a cardinal. We say that $\mathbb{P}$ satisfies the *κ-approximation property* if for every $(V,\mathbb{P})$-generic G and every set $A \in V[G]$ of ordinals, if $A \cap x \in V$ for every $x \in V$ with $|x| < \kappa$ in V, then $A \in V$.

Fact 2.2 (Usuba [5]). *Let κ be a regular uncountable cardinal, and $\mathbb{P}$ be a κ-c.c. poset. Suppose that, for every κ-Suslin tree T, we have $\Vdash_{\mathbb{P}}$ "T has no cofinal branch". Then $\mathbb{P}$ satisfies the κ-approximation property.*

The following is immediate from the above fact, but (2) would be a kind of folklore.

Corollary 2.3. *Let κ be a regular uncountable cardinal, and $\mathbb{P}$ a κ-c.c. poset.*

(1) If there is no κ-Suslin tree, or the product poset $\mathbb{P} \times \mathbb{P}$ is κ-c.c., then $\mathbb{P}$ satisfies the κ-approximation property.

(2) If $\mathbb{P}$ is non-trivial, then the forcing with $\mathbb{P}$ must add new subset of κ.

Now we prove the consistency of $\mathsf{GA}_{\text{c.c.c.}} + \neg\mathsf{GA}_{\text{proper}}$. Suppose $V = L$. Let $\mathbb{P}$ be any non-trivial ω_2-closed forcing notion. Take a $(V, \mathbb{P})$-generic, and work in $V[G]$. We check that $V[G]$ is a model of $\mathsf{GA}_{\text{c.c.c.}} + \neg\mathsf{GA}_{\text{proper}}$. Clearly ω_2-closed forcing is proper, hence we have that $\mathsf{GA}_{\text{proper}}$ fails in $V[G]$. Next take a ground $W \subseteq V[G]$, a poset $\mathbb{Q} \in W$ which is c.c.c. in W, and a $(W, \mathbb{Q})$-generic H with $V[G] = W[H]$. If $H \notin W$, by Corollary 2.3, there is a subset $x \subseteq \omega_1$ which is in $W[H]$ but not in W. However, since $V = L \subseteq W$ and $V[G]$ is an ω_2-closed forcing extension of V, we have $\mathcal{P}(\omega_1) \cap L = \mathcal{P}(\omega_1) \cap V[G] = \mathcal{P}(\omega_1) \cap W$. Hence $x \in W$, this is a contradiction. Thus $H \in W$, and we have $V[G] = W$.

Next we see $\mathsf{GA}_{\text{prod. c.c.c.}} + \neg\mathsf{GA}_{\text{c.c.c.}}$. Suppose $V = L$, and fix a Suslin tree T. We may assume that T is of the form $\langle \omega_1, \leq_T \rangle$. Let $\mathbb{P}$ be a c.c.c. forcing T with $\geq_T$. Let B be a $(V, \mathbb{P})$-generic branch of T, and we see that $V[B]$ is a required model. $V[B]$ is a c.c.c. forcing extension of V, so $\mathsf{GA}_{\text{c.c.c.}}$ fails. Suppose $W \subseteq V[B]$ is a ground such that $V[B]$ is a forcing extension of W via productively c.c.c. poset $\mathbb{Q} \in W$. By Corollary 2.3, $\mathbb{Q}$ satisfies the ω_1-approximation property in W. Now, since $T \in L \subseteq W$ and B is a cofinal branch of T, we have that $B \cap x \in W$ for every countable set $x \in W$. Hence $B \in W$ by the ω_1-approximation property of $\mathbb{Q}$, and $V[B] = L[B] \subseteq W$.

Question 2.4. *How are* GA *for classes of other variants of c.c.c. posets? For instance, is* $\mathsf{GA}_{\text{property K}} + \neg\mathsf{GA}_{\text{prod. c.c.c.}}$ *consistent?*

3. Separationg ω_1-stationary preserving and ω_1-preserving

In this section we prove the consistency of $\mathsf{GA}_{\omega_1\text{-stat. pres.}} + \neg\mathsf{GA}_{\omega_1\text{-pres.}}$.

First, for a given subset of $[\omega_1]^{<\omega_1}$, we define a poset such that, in the generic extension, the subset of $[\omega_1]^{<\omega_1}$ is coded by disjoint stationary subsets.

Suppose CH, and fix a surjection $\pi : \omega_1 \to [\omega_1]^{<\omega_1}$. Fix disjoint stationary subsets $\vec{S} = \langle S_\alpha : \alpha < \omega_1 \rangle$ of ω_1. Fix a non-empty set $X \subseteq [\omega_1]^{<\omega_1}$.

Definition 3.1. $\mathbb{C} = \mathbb{C}(\vec{S}, X)$ is the poset consists of bounded closed subsets p of ω_1 such that for every $\alpha < \omega_1$, if $\pi(\alpha) \notin X$ then $p \cap S_\alpha = \emptyset$. For $p, q \in \mathbb{C}$, define $p \leq q$ if p is an end-extension of q.

Note that $\mathbb{C} \subseteq [\omega_1]^{<\omega_1}$.

Lemma 3.2. $|\mathbb{C}| = \omega_1$*, hence has the* ω_2*-.c.c.*

Lemma 3.3. *For every* $p \in \mathbb{C}$ *and* $\gamma < \omega_1$*, there is* $q \leq p$ *with* $\max(q) > \gamma$*.*

Proof. Fix $\alpha < \omega_1$ with $\pi(\alpha) \in X$. Take $\delta \in S_\alpha$ with $\max(p), \gamma < \delta$, and set $q = p \cup \{\delta\}$. We have $q \in \mathbb{C}$, $\max(q) > \gamma$, and $q \leq p$. □

Let θ be a sufficiently large regular cardinal. The following is immediate from the definition.

Lemma 3.4. *Let $M \prec H_\theta$ be countable containing all relevant objects. Let $\langle p_n : n < \omega \rangle$ be a descending sequence in $\mathbb{C}$ such that $p_n \in M$ and for every dense open $D \in M$ in $\mathbb{C}$, there is $n < \omega$ with $p_n \in D \cap M$. Let $p^* = \bigcup_{n<\omega} p_n \cup \{\sup(M \cap \omega_1)\}$. If $M \cap \omega_1 \notin \bigcup_{\alpha<\omega_1} S_\alpha$, or $M \cap \omega_1 \in S_\alpha$ for some α with $\pi(\alpha) \in X$, then $p^* \in \mathbb{C}$ and $p^* \leq p_n$ for every $n < \omega$.*

Now the following is immediate from the above lemma.

Lemma 3.5. *(1) $\mathbb{C}$ is σ-Bare.*

(2) Let C be $(V, \mathbb{C})$-generic.

- *(a) If $\pi(\alpha) \in X$, then S_α is stationary in ω_1 in $V[C]$.*
- *(b) If $S \subseteq \omega_1 \setminus \bigcup_{\alpha<\omega_1} S_\alpha$ is stationary in ω_1 in V, so is in $V[C]$.*
- *(c) For $\alpha < \omega_1$, $\pi(\alpha) \in X$ if and only if S_α is stationary in ω_1 in $V[C]$.*

Next we consider the iteration of $\mathbb{C}(\vec{S}, X)$ of length ω. Fix pairwise disjoint stationary subsets $\langle S_{n,\alpha} : n < \omega, \alpha < \omega_1 \rangle$ of ω_1 such that $\omega_1 \setminus \bigcup_{n<\omega,\alpha<\omega_1} S_{n,\alpha}$ is stationary in ω_1. For $n < \omega$, let $\vec{S}_n = \langle S_{n,\alpha} : \alpha < \omega_1 \rangle$.

Define a countable support iteration $\langle \mathbb{P}_n, \dot{\mathbb{Q}}_m : n, m < \omega \rangle$ as follows:

(1) $\mathbb{P}_0$ is the trivial poset, and $\dot{\mathbb{Q}}_0$ is the $\mathbb{P}_0$-name for the poset $\mathbb{C}(\vec{S}_0, \{\emptyset\})$.

(2) $\mathbb{P}_{n+1} = \mathbb{P}_n * \dot{\mathbb{Q}}_n$.

(3) $\Vdash_{\mathbb{P}_{n+1}}$ "$\dot{\mathbb{Q}}_{n+1} = \mathbb{C}(\vec{S}_{n+1}, \dot{C}_n)$", where $\dot{C}_n$ is a canonical name for a $(V^{\mathbb{P}_n}, \mathbb{Q}_n)$-generic filter.

Recall that, for a stationary subset $E \subseteq \omega_1$, a poset $\mathbb{P}$ is *E-complete* if: Let $M \prec H_\theta$ be countable such that $M \cap \omega_1 \in E$ and M contains all relevant objects. Let $\langle p_n : n < \omega \rangle$ be a descending sequence in $\mathbb{P}$ such that for every dense open $D \in M$ in $\mathbb{P}$, there is $n < \omega$ with $p_n \in D \cap M$. Then $\langle p_n : n < \omega \rangle$ has a lower bound.

Fact 3.6 (Shelah [3]). *(1) If a poset $\mathbb{P}$ is E-complete, then $\mathbb{P}$ is σ-Baire, and for every stationary subset $E' \subseteq E$, $\mathbb{P}$ preserves the stationarity of E'.*

(2) Every countable support iteration of E-complete forcings is E-complete.

Lemma 3.4 shows that $\mathbb{P}_\omega$ is $(\omega_1 \setminus \bigcup_{n<\omega,\alpha<\omega_1} S_{n,\alpha})$-complete, hence $\mathbb{P}_\omega$ is σ-Baire. Moreover, for each $n < \omega$, $\mathbb{P}_{n+1}$ is $S_{n+1,\alpha}$-complete for every $\alpha < \omega_1$. In $V^{\mathbb{P}_{n+1}}$, let C_n be a $(V^{\mathbb{P}_n}, \mathbb{Q}_n)$-generic filter. Then for every $\alpha < \omega_1$, if $\pi(\alpha) \in C_n$ then $\mathbb{Q}_{n+1}$ is $S_{n+1,\alpha}$-complete, hence preserves the stationarity of $S_{n+1,\alpha}$. Furthermore, if $\pi(\alpha) \in C_n$, then $\mathbb{P}_\omega / \mathbb{P}_{n+2}$ is $S_{n+1,\alpha}$-complete by Lemma 3.4 again. These observations show the following:

Lemma 3.7. *Let G be $(V, \mathbb{P}_\omega)$-generic. In $V[G]$, for $n < \omega$, let $G_n = G \cap \mathbb{P}_n$ and C_n be $(V[G_n], \mathbb{Q}_n)$-generic induced by G. Then, for every $n < \omega$ and $\alpha < \omega_1$, $\pi(\alpha) \in C_n \iff S_{n+1,\alpha}$ is stationary in ω_1 in $V[G]$.*

Now we construct a model of $\mathsf{GA}_{\omega_1\text{-stat. pres.}} + \neg\mathsf{GA}_{\omega_1\text{-pres.}}$. Suppose $V = L$. Fix pairwise disjoint stationary subsets $\langle S_{n,\alpha} : n < \omega, \alpha < \omega_1 \rangle$ of ω_1 such that $\omega_1 \setminus \bigcup_{n<\omega,\alpha<\omega_1} S_{n,\alpha}$ is stationary in ω_1, and fix a surjection $\pi : \omega_1 \to [\omega_1]^{<\omega_1}$. Take a poset $\mathbb{P}_\omega$ using $\langle S_{n,\alpha} : n < \omega, \alpha < \omega_1 \rangle$. Take a $(V, \mathbb{P}_\omega)$-generic G, and work in $V[G]$. We show that $V[G]$ is a model of $\mathsf{GA}_{\omega_1\text{-stat. pres.}} + \neg\mathsf{GA}_{\omega_1\text{-pres.}}$. Clearly $\mathsf{GA}_{\omega_1\text{-pres.}}$ fails. To see that $\mathsf{GA}_{\omega_1\text{-stat. pres.}}$, take a ground $W \subseteq V[G]$ such that $V[G]$ is an ω_1-stationary preserving forcing extension of W. Note that $\langle S_{n,\alpha} : n < \omega, \alpha < \omega_1 \rangle, \pi \in W$. For $n < \omega$, let C_n be the $(V[G_n], \mathbb{Q}_n)$-generic filter induced by G. Then, in $V[G]$, $\pi(\alpha) \in C_n \iff S_{n+1,\alpha}$ is stationary in ω_1 in $V[G]$. Because $\vec{S}_n \in W$ and $V[G]$ is an ω_1-stationary preserving forcing extension of W, we have that $\{\pi(\alpha) : S_{n,\alpha}$ is stationary in ω_1 in $W\} = C_n \in W$. Hence we have $\langle C_n : n < \omega \rangle \in W$. G can be constructed in W using $\langle C_n : n < \omega \rangle$, thus we have $G \in W$, and $W = V[G]$. This completes the proof.

Question 3.8. *Is $\mathsf{GA}_{\omega_1\text{-pres.}} + \neg\mathsf{GA}$ consistent?*

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(T. Usuba) Faculty of Fundamental Science and Engineering, Waseda University, Okubo 3-4-1, Shinjyuku, Tokyo, 169-8555 Japan

E-mail address: usuba@waseda.jp