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Growth estimates of generalized eigenfunctions and principle of limiting absorption

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1. Introduction

In this note we present a unified approach to growth estimates of generalized eigen functions and principle of limiting absorption for the Schrödinger operators. The results are applicable to short-range, long-range, oscillating long-range and exploding potentials.

As an example we consider the Schrödinger operator $L = -\Delta + c(x)$ with von Neumann-Wigner type potential

$$c(x) = \frac{c \sin br}{r} + c_2(x), \quad x \in \mathbf{R}^n,$$

where b, $c > 0$, $r = |x|$ and $c_2(x)$ is a real valued short-range potential: $c_2(x) = o(r^{-1-\delta})$ $(0 < \delta \leq 1)$. Obviously, L is selfadjoint and $\sigma_e(L) = [0, \infty)$. As for the growth estimates of generalized eigunctions

$$-\Delta u + c(x)u = \lambda u, \quad \lambda > 0, \tag{1}$$

the following results is known. Assume that the support of solution u is not compact.

Kato [1]: Let $\lambda > c^2/4$, where $c = \limsup_{r\to\infty} r|c(x)|$. Then for any $\epsilon > 0$

$$\lim_{r\to\infty} r^{c/\sqrt{\lambda}+\epsilon} \int_{S_r} \{|\partial_r u|^2 + |u|^2\} dS = \infty.$$

Thus, $(c^2, \infty) \subset \sigma_c(L)$ if L has a uique continuation property.

Mochizuki-Uchiyama [2]: Let $\lambda > bc/\gamma$ for $0 < \gamma \leq 2$. Then

$$\liminf_{r\to\infty} r^{\gamma/2} \int_{S_r} \{|\partial_r u|^2 + |u|^2\} dS > 0.$$

Thus, $\left(\frac{bc}{2},\infty\right) \subset \sigma_c(L)$ if L has a unique continuation property.

For solution of the stationary equation

$$-\Delta u + c(x)u - \zeta u = f(x), \quad \zeta \in \{\zeta \in C; \mathrm{Re}\zeta > 0, \pm\mathrm{Im}\zeta > 0\}, \tag{2}$$

we define the vector function $\theta = \theta(x,\zeta)$ by

$$\theta(x,\zeta) = \nabla u + \tilde{x}K(x,\zeta)u, \quad \tilde{x} = x/r,$$

where

$$K(x,\zeta) = -i\sqrt{k(x,\zeta)} + \frac{n-1}{2r} + \frac{\partial_r k(x,\zeta)}{4k(x,\zeta)}$$

$$\text{with} \quad k(x,\zeta) = \zeta - \eta(\zeta)\frac{c\sin br}{r}, \quad \eta(\zeta) = \frac{4\zeta}{4\zeta - b^2}.$$

This function is introduced in Mochizuki-Uchiyama [3] to define the radiation condition for (2) and to show, under the above results of [2], the principle of limiting absorption in

$$\left(\frac{b^2}{4} + \frac{bc}{\min\{2, 4\delta\}}, \infty\right).$$

Jäger-Rajto [4]: Let $|\lambda - b^2/4| > bc/2$. If solution u of (1) has no compact support, then

$$\liminf_{r\to\infty} \int_{S_r} |\theta(x, \lambda \pm i0)|^2 dS > 0.$$

Not only growth estimates of generalized eigenfunctions, this is directly applied to to show the principle of limiting absorption in

$$\left(0, \frac{b^2}{4} - \frac{bc}{\min\{2,4\delta\}}\right) \cup \left(\frac{b^2}{4} + \frac{bc}{\min\{2,4\delta\}}, \infty\right); \tag{3}$$

Mochizuki [5], [6]: Let I be any interval in this set and $0 < \epsilon_0 \leq 1$. We define

$$\Gamma_\pm = \Gamma_\pm(I, \epsilon_0) = \{\zeta = \lambda \pm i\epsilon; \lambda \in I, 0 < \epsilon < \epsilon_0\}.$$

For positive function $\xi = \xi(r)$ we define the weighted L^2-space $L^2_\xi = L^2_\xi(\mathbf{R}^n)$ with norm

$$\|f\|_\xi^2 = \int \xi(r)|f(x)|^2 dx.$$

Let $\mu = \mu(r) = (1+r)^{-1-\delta}$ and $\varphi = \varphi(r) = \delta^{-1}(1+r)^\delta$ The principle then is derived as follows: Let $R(\zeta) = (L-\zeta)^{-1}$, $\zeta \in \Gamma_\pm$, be the resolvent of L. Then $R(\zeta)$ continuously extended to $\overline{\Gamma_\pm}$ as an operator from $L^2_{\mu^{-1}}$ to L^2_μ, and we have

$$\sup_{\zeta\in\Gamma_\pm} \|R(\zeta)f\|_\mu \leq C\|f\|_{\mu^{-1}}, \quad C = C(\Gamma_\pm) > 0.$$

Moreover, $u = R(\zeta)f$ satisfies the radiation condition $\|\theta(\cdot, \lambda \pm i0)\|_{\varphi'} < \infty$.

This result is dissatisfactory in the sense that the set (3) vanishes if δ goes to 0.

One purpose of this talk is to improve (3) to the set independent of $\delta > 0$ as follows

$$\left(0, \frac{b^2}{4} - \frac{bc}{2}\right) \cup \left(\frac{b^2}{4} + \frac{bc}{2}, \infty\right).$$

Moreover, we can treat general second order elliptic operators in exterior domain which also cover some exploding potential $c(x) \to -\infty$ as $r \to \infty$.

Main tasks will be done under a modification of the radiation conditions.

2. Results

Let $\Omega \subset \mathbf{R}^n$ $(n \geq 2)$ be an exterior domain with smooth boundary $\partial\Omega$. We consider in Ω the boundary value problem

$$Lu - \zeta u = f(x) \ \text{ in } \Omega, \quad \mathcal{B}u = 0 \ \text{ on } \partial\Omega; \tag{4}$$

$$L = -\Delta_{a,b} + c(x) = -\sum_{j,k=1}^{n} \{\partial_j + ib_j(x)\}a_{jk}(x)\{\partial_k + ib_k(x)\} + c(x)$$

and $\mathcal{B}u|_{\partial\Omega} = 0$ is the Dirichlet or Robin boundary condition. Here $\zeta \in \mathbf{C}$, $\partial_j = \partial/\partial x_j$ and $i = \sqrt{-1}$. The coefficients are all real and sufficiently smooth, $A = (a_{jk}(x))$ is uniformly positive definite and $c(x) \geq -C(1 + r^\alpha)$ $(\alpha < 2)$. Then L determines a selfadjoint operator in $L^2(\Omega)$ with domain

$$\mathcal{D}(L) = \{u \in H^2_{\text{loc}}(\overline{\Omega}) \cap L^2(\Omega); -\Delta_{a,b}u + cu \in L^2(\Omega), \mathcal{B}u|_{\partial\Omega} = 0\}.$$

Let $\mu = \mu(r) > 0$ be a decreasing weight function verifying

$(\mu.1)$ $\qquad \mu(r) = o(r^{-1})$, decreasing and $\int_0^\infty \mu(r)dr < \infty$.

[Assumptions]

$(A.1)$ $\qquad \nabla^\ell\{a_{jm}(x) - \delta_{jm}\} = O(r^{-\ell+1}\mu)$ $(\ell = 0, 1, 2)$,

(oscillating long-range potentials) $c(x) = c_0(r) + c_1(x) + c_2(x)$ where

$(A.2)_o$ $\qquad \partial_r^\ell c_0(r) = O(r^{-1})$, $\partial_r^2 c_0(r) + ac_0(r) = O(\mu)$ for some $a \geq 0$,

$(A.3)_o$ $\qquad c_1(x) = O(r\mu)$, $\nabla c_1^\ell(x) = O(\mu)$ $(\ell = 1, 2)$,

$(A.4)_o$ $\qquad \nabla \times b(x)$, $c_2(x) = O(\mu)$.

(exploding potentials) $c(x) = c_0(r) + c_1(x) + c_2(x)$ where

$$(A.2)_e \qquad 1 \le -c_0(r) \le C(1+r^\alpha)\ (0<\alpha<2),\quad c_0(r) \to -\infty\ (r\to\infty),$$

$$-\frac{\beta}{r} \le \frac{\partial_r c_0(r))}{2c_0(r)} \le \frac{1}{r}\ \ (0<\beta<1),\quad \frac{\partial_r^2 c_0(r)}{c_0(r)} = O(r^{-1}),$$

$$(A.3)_e \qquad \frac{c_1(x)}{c_0(r)} = O(r\mu),\quad \frac{\nabla^\ell c_1(x)}{c_0(r)} = O(\mu)\ \ (\ell = 1,2),$$

$$(A.4)_e \qquad \frac{\nabla\times b(x)}{\sqrt{-c_0(r)}},\ \frac{c_2(x)}{\sqrt{-c_0(r)}} = O(\mu).$$

Remark 1. Oscillating long-range potential $c_0(r)$ is generalized to $c_0(x)$ if we require

$$\tilde\nabla\partial_r^\ell c_0(x) = O(\mu)\ \ (\ell=0,1),\ \text{where } \tilde\nabla = \nabla - \tilde x\partial_r.$$

This condition is satisfied e.g. by $c_0(x) = \dfrac{x_1\sin br}{r^2}$.

2. For general exploding potential $c(x) = \tilde c(x) + c_2(x)$ satisfying $(A.2)_e$, put

$$c_0(r) = \frac{1}{|S_1|}\int_{S_1}\tilde c(r\tilde x)dS_{\tilde x}.$$

Then $c_1(x) = \tilde c(x) - c_0(r)$ may verify $(A.3)_e$ under the additional assumption

$$\tilde\nabla\partial_r^\ell\tilde c(x) = O(r^{-\ell}\mu)\ \ (\ell=0,1).$$

For oscillating long-range potentials we choose an interval $I = [\lambda_1,\lambda_2]$ to satisfy

$$\lambda_1 > \frac{a}{4} + E^+ \ \text{ or }\ 0<\lambda_1<\lambda_2<\frac{a}{4} - E^-,\quad E^\pm = \limsup_{r\to\infty}\Big[\pm\frac{1}{2}r\partial_r c_0(x)\Big].$$

For exploding potentials I is any interval in $\mathbf{R}$. Put $\Gamma_\pm = \{\zeta = \lambda \pm i\epsilon; \lambda\in I, 0 < \epsilon \le \epsilon_0\}$. For $(x,\zeta)\in\Omega\times\overline{\Gamma_\pm}$ let

$$k(x,\zeta) = \frac{\zeta - \eta(\zeta)c_0(r) - c_1(x)}{\tilde x\cdot A\tilde x},\quad \eta(\zeta) = \frac{4\zeta}{4\zeta - a}$$

(in exploding case $\eta(\zeta)\equiv 1$). Then the following estimates hold for $(x,\zeta)\in\Omega'_{R_1}\times\overline{\Gamma}_\pm$ if R_1 is chosen sufficiently large.

$$(K.1) \qquad 0 < C_0 \le \mathrm{Re}k(x,\zeta) \le C(1+r^\alpha),\quad |\mathrm{Im}k(x,\zeta)| \le C|\mathrm{Im}\zeta|,$$

$$(K.2) \qquad -\frac{\beta}{r} \le \mathrm{Re}\frac{\partial_r k(x,\zeta)}{2k(x,\zeta)} \le \frac{1}{r} + O(\mu) \quad \text{for some } \beta \in (0,1),$$

$$(K.3) \qquad \frac{\nabla^{\ell+1} k(x,\zeta)}{k(x,\zeta)} = O(r^{-1}), \quad \frac{\tilde{\nabla}\partial_r^{\ell} k(x,\zeta)}{k(x,\zeta)} = O(\mu), \quad \ell = 0,1,$$

as $r \to \infty$ uniformly in $\zeta \in \Gamma_\pm$.

$$(K.4) \qquad c(x) - \zeta + \tilde{x} \cdot A\tilde{x}\Big\{k(x,\zeta) + \frac{\partial_r^2 k(x,\zeta)}{4k(x,\zeta)}\Big\} = O(\mu)$$

as $r \to \infty$ uniformly in $\zeta \in \overline{\Gamma}_\pm$.

For solution $u \in H^2_{\mathrm{loc}}$ of (4) let

$$K(x,\zeta) = -i\sqrt{k(x,\zeta)} + \frac{n-1}{2r} + \frac{\partial_r k(x,\zeta)}{4k(x,\zeta)}$$

and we define the vector function $\theta = \theta(x,\zeta)$ by

$$\theta(x,\zeta) = \nabla_b u + \tilde{x}K(x,\zeta)u \quad \text{where } \nabla_b = \nabla + ib(x).$$

Theorem 1 *Under the above Assumption, let $u \in H^2_{\mathrm{loc}}(\overline{\Omega})$ solves the eigenvalue problem*

$$-\Delta_{a,b}u + cu - \lambda u = 0 \quad \text{in } \Omega, \quad \mathcal{B}u = 0 \quad \text{on } \partial\Omega \tag{5}$$

with $\lambda \in I$. If the support of u is not compact, then it satisfies

$$\liminf_{t\to\infty} \int_{S_t} \frac{1}{\sqrt{k(x,\lambda)}} |\tilde{x} \cdot A\theta(x, \lambda \pm i0)|^2 dS > 0.$$

Assume that there exists a positive decreasing function $\mu_0(r) \le \mu(r)$ such that the functions

$$\varphi_0(r) = \Big(\int_r^\infty \mu_0(s)ds\Big)^{-1}, \quad \varphi(r) = \Big(\int_r^\infty \mu(s)ds\Big)^{-1}$$

satisfy for $r > R_1$

$$(\mu.2) \qquad \varphi_0'(r) \le \varphi'(r) \quad \text{and} \quad \frac{\varphi_0'(r)}{\varphi_0(r)} \le \frac{1}{r} + \min\Big\{0, \mathrm{Re}\frac{\partial_r k(x,\zeta)}{2k(x,\zeta)}\Big\}.$$

Definition 1 The solution of (4) is said to satisfy the radiation condition if

$$\int \mu_0(r)|\sqrt{k(x,\zeta)}||u(x,\zeta)|^2 dx < \infty, \quad \int \frac{\varphi_0'(r)}{|\sqrt{k(x,\zeta)}|} |\tilde{x} \cdot A\theta(x,\zeta)|^2 dx < \infty.$$

A solution of (4) which also satisfies the radiation condition is called a radiative solution.

Let $\zeta \in \Gamma_\pm$. Then the resolvent $R(\zeta) = (L-\zeta)^{-1}$ forms a bounded operator in $L^2(\Omega)$ which depends continuously on ζ. Moreover, if $f \in L^2_{(\mu_0|\sqrt{k}|)^{-1}}(\Omega)$, then $u = R(\zeta)f$ is shown to satisfy the above radiation condition.

Theorem 2 *Under the above Assumption, let $\zeta \in \Gamma_\pm$ and $f \in L^2_{(\mu_0|\sqrt{k}|)^{-1}}$. Then there exists $C = C(\Gamma_\pm) > 0$ such that*

$$\sup_{\zeta\in\Gamma_\pm} \|R(\zeta)f\|_{\mu_0|\sqrt{k}|} \le C\|f\|_{(\mu_0|\sqrt{k}|)^{-1}},$$

and as an operator from $L^2_{(\mu_0|\sqrt{k}|)^{-1}}$ to $L^2_{\mu_0|\sqrt{k}|}(\Omega)$, $R(\zeta)$ is extended continuously to $\overline{\Gamma}_\pm$. Moreover, $u = R(\lambda \pm i0)f$ becomes an (outgoing $(+)$ or incoming $(-)$) raditative solution of (4) *with $\zeta = \lambda$.*

Remark 3. In case of exploding potentials, similar results is obtained by Yamada [7] under slightly stringent conditions on the coefficients. In his case the radiation conditions are, as in the case of [3], defined by

$$\|u\|_{\mu|\sqrt{k}|} < \infty, \quad \|\tilde{x}\cdot\theta\|_{\varphi'} < \infty$$

3. A quadratic identity

For the sake of simplicity we restrict ourselves to the equation with $a_{jk}(x) = \delta_{jk}$:

$$-\Delta_b u + c(x)u - \zeta u = f(x) \quad \text{in } \mathbf{R}^n, \tag{6}$$

where $\Delta_b = \nabla_b \cdot \nabla_b$ with $\nabla_b = \nabla + ib(x)$.

For solution u of (6) we put

$$u_\sigma = e^\sigma u, \quad f_\sigma = e^\sigma f \quad \text{and} \quad \theta_\sigma = \nabla_b u_\sigma + \tilde{x}Ku_\sigma,$$

where $\sigma = \sigma(r)$ is a positive function of $r > 0$. (6) is rewritten as

$$-\nabla_b \cdot \theta_\sigma + (K + 2\sigma')\tilde{x}\cdot\theta_\sigma + q_{K,\sigma}u = f_\sigma, \tag{7}$$

$$q_{K,\sigma} = q_K + \sigma'' + \frac{n-1}{r}\sigma' - \sigma'^2 - 2K\sigma' \quad \text{with}$$

$$q_K = c(x) - \zeta + \partial_r K + \frac{n-1}{r}K - K^2.$$

For a smooth weight function $\Phi = \Phi(x) > 0$, let us consider the real part of the equation (6) multiplied by $\Phi\tilde{x}\cdot\overline{\theta_\sigma}$. The integrating by parts over $B_{R,t} = \{x \in \mathbf{R}^n; R < |x| < t\}$ give the following identity:

$$-\left[\int_{S_t} - \int_{S_R}\right]\Phi\left\{|\tilde{x}\cdot\theta_\sigma|^2 - \frac{1}{2}|\theta_\sigma|^2\right\}dS + \mathrm{Re}\int_{\Omega_{R,t}}\Phi\left[\frac{1}{r}\{|\theta_\sigma|^2 - |\tilde{x}\cdot\theta_\sigma|^2\}\right.$$
$$+\left(K - \frac{n-1}{2r}\right)|\theta_\sigma|^2 + 2\sigma'|\tilde{x}\cdot\theta_\sigma|^2 + \frac{\nabla\Phi}{\Phi}\cdot\theta_\sigma(\tilde{x}\cdot\overline{\theta_\sigma}) - \frac{\partial_r\Phi}{2\Phi}|\theta_\sigma|^2$$
$$\left.+\mathcal{B}(u_\sigma,\theta_\sigma) + (q_{K,\sigma} - q_K)u_\sigma(\tilde{x}\cdot\overline{\theta_\sigma})\right]dx = \mathrm{Re}\int_{\Omega_{R,t}}\Phi f_\sigma(\tilde{x}\cdot\overline{\theta_\sigma})dx, \qquad (8)$$

where

$$\mathcal{B}(u_\sigma,\theta_\sigma) = iu_\sigma(\nabla\times b)\cdot(\tilde{x}\times\overline{\theta_\sigma}) + u_\sigma(\tilde{\nabla}K\cdot\overline{\theta_\sigma}) + q_{K,\sigma}u_\sigma(\tilde{x}\cdot\overline{\theta_\sigma}).$$

Lemma 1 *Under the above Assumptions we have*

$$|\mathcal{B}(u_\sigma,\theta_\sigma)| = O(\mu)|k(x,\zeta)|^{1/2}|u_\sigma||\theta_\sigma| \quad \text{as } r\to\infty.$$

4. Outline of the proof of Theorem 1

We choose $0 < \delta < 1-\beta$ and put

$$\varphi_0(x,\lambda) = \frac{1}{\sqrt{k(x,\lambda)}}, \quad \varphi(x,\lambda) = \frac{r^{2-\delta}\sqrt{k_0(r,\lambda)}^{\,-2-\delta}}{\sqrt{k(x,\lambda)}},$$

where $k_0(r,\lambda) = \lambda - \eta(\lambda)c_0(r)$. Note that

$$\frac{\partial_r k(x,\lambda)}{k(x,\lambda)} - \frac{\partial_r k_0(r,\lambda)}{k_0(r,\lambda)} = O(\mu). \qquad (9)$$

We define the two functionals of solution u of the homogeneous equation (5).

$$F_0(t) = \int_{S_t}\varphi_0\left\{|\tilde{x}\cdot\theta|^2 - \frac{1}{2}|\theta|^2\right\}dS,$$
$$F_{\sigma,\tau}(t) = \int_{S_t}\varphi\left\{|\tilde{x}\cdot\theta_\sigma|^2 - \frac{1}{2}|\theta_\sigma|^2 + \frac{1}{2}(\sigma^2-\tau)|u_\sigma|^2\right\}dS$$

where $\sigma = \sigma(r)$ and $\tau = \tau(r)$ are positive smooth functions given later.

Lemma 2 *The weight functions* φ_0 *and* φ *verify*

$$\frac{\nabla\varphi_0}{\varphi_0} = -\frac{\partial_r k}{2k}\tilde{x} + O(\mu), \qquad (10)$$
$$\frac{\nabla\varphi}{\varphi} = \frac{2-\delta}{r} + (1-\delta)\frac{\partial_r k}{2k}\tilde{x} + O(\mu). \qquad (11)$$

Lemma 3 *u be a solution of* (5). *Then for each* $r > R_0$ *and* $\lambda \in I$ *we have*

$$\mathrm{Im}\Big[\int_{S_r} \tilde{x} \cdot \nabla_b u_\sigma \overline{u_\sigma} dS\Big] = 0.$$

Lemma 4 *Let* $r > R_1$. *Then for each solution u of* (5) *we have*

$$\int_{S_r} \varphi_0 k |u_\sigma|^2 dS \leq \int_{S_r} \varphi_0 |\tilde{x} \cdot \theta_\sigma|^2 dS,$$

Proof of Theorem 1, Part 1 In this part we require an additional assumption that there exists a sequence $r_k \to \infty$ such that $F_0(r_k) > 0$.

We choose $\Phi = \varphi_0, \zeta = \lambda \pm i0$, $f = 0$ and $\sigma = 0$ in identity (8). Then noting

$$\mathrm{Re}\Big(K - \frac{n-1}{2r}\Big) = \frac{\partial_r k}{4k},$$

(10) and Lemmas 1, 4 we have

$$\begin{aligned} \frac{d}{dt} F_0(t) &\geq \int_{S_t} \varphi_0 \Big[\Big(\frac{1}{r} + \frac{\partial_r k}{2k}\Big)(|\theta|^2 - |\tilde{x} \cdot \theta|^2) - O(\mu)|\theta|^2\Big] dS \\ &= \int_{S_1} \varphi_0 \Big[\Big(\frac{1}{r} + \frac{\partial_r k}{2k} - 2O(\mu)\Big)(|\theta|^2 - |\tilde{x} \cdot \theta|^2) \\ &\quad -2O(\mu)\Big\{|\tilde{x} \cdot \theta|^2 - \frac{1}{2}|\theta|^2\Big\}\Big] dS \geq -2O(\mu(t))F_0(t) \end{aligned}$$

for $t \geq R_1$ if $R_1 \geq R_0$ is chosen sufficiently large. By assumption there exists $r_n \geq R_1$ and hence we conclude

$$F_0(t) \geq e^{-C \int_{r_n}^{\infty} \mu(s) ds} F_0(r_n) > 0,$$

which proves Theorem 1 since we have

$$\int_{S_t} \frac{1}{\sqrt{k}} |\tilde{x} \cdot \theta|^2 dS \geq 2F_0(t).$$

Proof of Theorem 1, Part 2 We assume $F_0(t) \leq 0$ in $t > R_0$ and u does not have compact support.

We choose $\Phi = \varphi$, $\zeta = \lambda \pm i0$ and $f = 0$ in identity (8) added by the identity

$$\frac{1}{2}\Big[\int_{S_t} - \int_{S_R}\Big] \varphi(\sigma'^2 - \tau)|u_\sigma|^2 dS = \mathrm{Re} \int_{B_{R,t}} \varphi\Big[(\sigma'^2 - \tau) u_\sigma (\tilde{x} \cdot \overline{\theta_\sigma})$$

$$+(\sigma'^2-\tau)\Big(\frac{\nabla\varphi}{2\varphi}-\frac{\partial_r k}{4k}\Big)|u_\sigma|^2+\Big(\sigma'\sigma''-\frac{\tau'}{2}\Big)|u_\sigma|^2\Big]dx,$$

where $\tau=\tau(r)>0$, and differentiate both sides by t. Then we have

$$\frac{d}{dt}\int_{S_t}\varphi\Big\{|\tilde{x}\cdot\theta_\sigma|^2-\frac{1}{2}|\theta_\sigma|^2+\frac{1}{2}(\sigma'^2-\tau)|u_\sigma|^2\Big\}dS=\mathrm{Re}\int_{S_t}\varphi\Big[\frac{1}{r}\{|\theta_\sigma|^2-|\tilde{x}\cdot\theta_\sigma|^2\}$$
$$+\frac{\partial_r k}{4k}|\theta_\sigma|^2+2\sigma'|\tilde{x}\cdot\theta_\sigma|^2+\Big(\frac{\nabla\varphi}{\varphi}\Big)\cdot\Big\{\theta_\sigma(\tilde{x}\cdot\overline{\theta_\sigma})-\frac{1}{2}|\theta_\sigma|^2\Big\}$$
$$+\mathcal{B}(u_\sigma,\theta_\sigma)+\Big(\sigma''+\frac{n-1}{r}\sigma'-2\sigma'K\Big)u_\sigma(\tilde{x}\cdot\overline{\theta_\sigma})-\tau u_\sigma(\tilde{x}\cdot\overline{\theta_\sigma})$$
$$+(\sigma'^2-\tau)\Big(\frac{\nabla\varphi}{2\varphi}-\frac{\partial_r k}{4k}\Big)|u_\sigma|^2+\Big(\sigma'\sigma''-\frac{\tau'}{2}\Big)|u_\sigma|^2\Big]dS.$$

Here, by use of (11) we have

- $$\frac{1}{r}\{|\theta_\sigma|^2-|\tilde{x}\cdot\theta_\sigma|^2\}+\frac{\partial_r k}{4k}|\theta_\sigma|^2+\Big(\frac{\nabla\varphi}{\varphi}\Big)\cdot\Big\{\theta_\sigma(\tilde{x}\cdot\overline{\theta_\sigma})-\frac{1}{2}|\theta_\sigma|^2\Big\}$$
$$\geq\Big(\frac{1-\delta}{r}+(1-\delta)\frac{\partial_r k}{2k}\Big)|\tilde{x}\cdot\theta_\sigma|^2+\Big\{\frac{\delta}{2r}+\delta\frac{\partial_r k}{4k}-O(\mu)\Big\}|\theta_\sigma|^2,$$
- $$2\sigma'|\tilde{x}\cdot\theta_\sigma|^2+\mathrm{Re}\Big\{\Big(\sigma''-2\sigma'\frac{\partial_r k}{4k}+2\sigma' i\sqrt{k}\Big)u_\sigma(\tilde{x}\cdot\overline{\theta_\sigma})\Big\}$$
$$\geq 2\sigma'|\tilde{x}\cdot\theta_\sigma+i\sqrt{k}u_\sigma|^2+2\sigma'\mathrm{Im}\{\sqrt{k}u_\sigma\overline{(\tilde{x}\cdot\theta_\sigma+i\sqrt{k}u_\sigma)}\}$$
$$-\frac{\sigma'}{2}|\tilde{x}\cdot\theta_\sigma+i\sqrt{k}u_\sigma|^2-\frac{\sigma'}{2}\Big(\frac{\sigma''}{\sigma'}-\frac{\partial_r k}{2k}\Big)^2|u_\sigma|^2,$$
- $$-\tau\mathrm{Re}[u_\sigma(\tilde{x}\cdot\overline{\theta_\sigma})]+(\sigma'^2-\tau)\Big(\frac{\nabla\varphi}{2\varphi}-\frac{\partial_r k}{4k}\Big)|u_\sigma|^2+\Big(\sigma'\sigma''-\frac{\tau'}{2}\Big)|u_\sigma|^2$$
$$\geq-\frac{\sigma'}{2}|\tilde{x}\cdot\theta_\sigma+i\sqrt{k}u_\sigma|^2-\Big(\frac{\tau^2}{2\sigma'}+\frac{\tau'}{2}+\frac{C\tau}{r}\Big)|u_\sigma|^2$$
$$+\frac{\sigma'^2}{2}\Big(\frac{2-\delta}{r}-O(\mu)-\delta\frac{\partial_r k}{2k}+\frac{2\sigma''}{\sigma}\Big)|u_\sigma|^2$$

with $C>0$ chosen to satisfy $\dfrac{2-\delta}{r}-\delta\dfrac{\partial_r k}{2k}-O(\mu)\leq\dfrac{C}{r}$. Moreover, since

$$\mathrm{Re}\int_{S_t}\varphi\mathcal{B}(u_\sigma,\theta_\sigma)dS\leq\int_{S_t}\varphi O(\mu)|\theta_\sigma|^2dS$$

by Lemmas 1 and 4, it follows that

$$\frac{d}{dt}F_{\sigma,\tau}(t)\geq\int_{S_t}\varphi\Big[\Big\{\frac{1-\delta}{r}+(1-\delta)\frac{\partial_r k}{2k}\Big\}|\tilde{x}\cdot\theta_\sigma|^2+\Big\{\frac{\delta}{2r}+\delta\frac{\partial_r k}{4k}-2O(\mu)\Big\}|\theta_\sigma|^2$$

$$+\left\{\sigma'|\tilde{x}\cdot\theta_\sigma + i\sqrt{k}u_\sigma|^2 + 2\sigma'\mathrm{Im}\left[\sqrt{k}u_\sigma(\overline{\tilde{x}\cdot\theta_\sigma + i\sqrt{\kappa}u_\sigma})\right]\right\}$$
$$-\frac{\sigma'}{2}\left(\frac{\sigma''}{\sigma'} - \frac{\partial_r k}{2k}\right)^2|u_\sigma|^2 - \left(\frac{\tau^2}{2\sigma'} + \frac{C\tau}{r} + \frac{\tau'}{2}\right)|u_\sigma|^2$$
$$+\frac{\sigma'^2}{2}\left(\frac{2-\delta}{r} - O(\mu) - \delta\frac{\partial_r k}{2k} + \frac{2\sigma''}{\sigma'}\right)|u_\sigma|^2\Big]dS.$$

Now, let $m \geq 1$ and $\frac{1}{3} < \gamma < 1 - \delta$ (without loss of generality we can assume $\delta < \frac{2}{3}$ in Theorem 1) and choose $\sigma(r)$ and $\tau(r)$ as follows:

$$\sigma(r) = \frac{m}{1-\gamma}r^{1-\gamma}, \quad \tau(r) = r^{-2\gamma}\log r. \tag{12}$$

Then as $r \to \infty$,

$$-\frac{\sigma'}{2}\left(\frac{\sigma''}{\sigma'} - \frac{\partial_r k}{2k} - \frac{o(1)}{r}\right)^2 = mO(r^{-2-\gamma}),$$
$$\frac{\sigma'^2}{2}\left(\frac{2-\delta}{r} - \delta\frac{\partial_r k}{2k} - \frac{o(1)}{r} + \frac{2\sigma''}{\sigma'}\right)$$
$$\geq m^2\left\{2(1-\delta-\gamma) - o(1)\right\}r^{-1-2\gamma} > 0 \tag{13}$$

since $1 - \delta > \gamma$, and

$$-\left(\frac{\tau^2}{2\sigma'} + \frac{C\tau}{r} + \frac{\tau'}{2}\right) \geq -C_5\mu_1,$$

where $\mu_1 = r^{-3\gamma}(\log r)^2 \in L^1([R_1,\infty))$ and $C_5 > 0$ is independent of m and $r \geq R_4$.

Moreover, by Lemma 3

$$\mathrm{Im}\int_{S_t}\varphi\sqrt{k}u_\sigma(\overline{\tilde{x}\cdot\theta_\sigma + i\sqrt{\kappa}u_\sigma})dS$$
$$= t^{2-\delta}k_0(t,\lambda)^{(2-\delta)/2}\mathrm{Im}\int_{S_t}u_\sigma\tilde{x}\cdot\overline{\nabla u_\sigma}dS = 0. \tag{14}$$

Summarizing these results, we obtain the following: for any $m \geq 1$, there exists $R_5 \geq R_4$ such that

$$\frac{d}{dt}F_{\sigma,\tau}(t) \geq \int_{S_t}\varphi\left\{\left(\frac{1-\delta}{r} + (1-\delta)\frac{\partial_r k}{2k}\right)|\tilde{x}\cdot\theta_\sigma|^2 - C_5\mu_1|u_\sigma|^2\right\}dS \geq 0 \tag{15}$$

in $t \geq R_5$. Here we have used Lemma 4 again to show the last inequality.

By assumption that the support of u is not compact, R_5 can be chosen to satisfy

$$\int_{S_{R_5}}|u_\sigma|^2dS > 0.$$

Then as we see from (13), $F_{\sigma,\tau}(R_5)$ goes to ∞ as $m \to \infty$. We fix a large m satisfying $F_{\sigma,\tau}(R_5) > 0$ to conclude $F_{\sigma,\tau}(t) > 0$ for $t \geq R_5$.

Finally, we note the identity

$$F_{\sigma,\tau}(t) = e^{2\sigma} t^{2-\delta} k_0(r,\lambda)^{(2-\delta)/2} \Big\{ F_0(t) + \sigma' \mathrm{Re} \int_{S_t} \varphi_0 (\tilde{x} \cdot \nabla u) \overline{u} dS$$

$$+ \int_{S_t} \varphi_0 \Big(\sigma'^2 - \frac{1}{2}\tau + \sigma' \frac{n-1}{2t} + \sigma' \frac{\partial_r k}{4k} \Big) \int_{S(t)} \varphi_1 |u|^2 dS \Big\}$$

In this equation we use

$$\mathrm{Re} \int_{S_t} \varphi_0 (\tilde{x} \cdot \nabla u) \overline{u} dS - \frac{1}{2} \frac{d}{dt} \int_{S_t} \varphi_1 |u|^2 dS$$

$$= -\frac{1}{2} \int_{S_t} \Big\{ \partial_r \varphi_0 + \frac{n-1}{r} \varphi_1 \Big\} |u|^2 dS \leq \int_{S_t} O(r^{-1}) \varphi_0 |u|^2 dS,$$

and note that $F_0(t) \leq 0$ near infinity by assumption. Then since

$$\sigma'^2 - \frac{1}{2}\tau + \sigma' \frac{n-1}{2t} + \sigma' \frac{\partial_r k}{4k} + \sigma' O(t^{-1})$$

becomes negative when t goes large, it follows that

$$\frac{d}{dt} \int_{S_t} \varphi_0 |u|^2 dS > 0$$

for t large enough. This and Lemma 4 establish the conclusion of the Theorem. □

Remark 4. In case of general oscillating potential $c_0(x)$ in Remark 1, we have to divide the proof of *Part 2* in two steps. We choose

$$\varphi_1(x) = \frac{r^\delta \sqrt{k_1(x,\lambda)}^{2-\delta}}{\sqrt{k(x,\lambda)}}, \qquad \varphi(x) = \frac{r^{2-\delta} \sqrt{k_1(x,\lambda)}^{2-\delta}}{\sqrt{k(x,\lambda)}}$$

with $k_1(x,\lambda) = \lambda - \eta(\lambda) c_1(x)$, and define

$$F_1(t) = \int_{S_t} \varphi_1 \Big\{ |\tilde{x} \cdot \theta|^2 - \frac{1}{2} |\theta|^2 \Big\} dS,$$

$$F_{\sigma,\tau}(t) = \int_{S_t} \varphi \Big\{ |\tilde{x} \cdot \theta_\sigma|^2 - \frac{1}{2} |\theta_\sigma|^2 + \frac{1}{2} (\sigma^2 - \tau) |u_\sigma|^2 \Big\} dS.$$

Step 1 $F_0(t) \leq 0$ in $t > R_0$ and u does not have compact support, on the other hand, there exists a sequence $r_p \to \infty$ such that $F_1(r_p) > 0$.

Step 2 $F_1(t) \le 0$ in $T > R_0$ and u does not have compact support.

In the proof of *Step 1* the inequality

$$\int_{S_t} \varphi_1 |\tilde{x} \cdot \theta|^2 dS \le (1 + O(r^{-1})) \int_{S_t} \varphi_1 \{|\theta|^2 - |\tilde{x} \cdot \theta|^2\} dS$$

which follows from the assumtion $F_0(t) \le 0$ plays an important role. On the other hand, in the proof of *Step 2* equation (14) is not expected to hold. Instead, we have

$$2\sigma' \int_{S_t} \varphi \Big\{ \mathrm{Im}\left[\pm\sqrt{k}(\tilde{x} \cdot A\tilde{x}) u_\sigma \tilde{x} \cdot A\overline{\theta_{\sigma,1}} \right] + \frac{1}{2} |\tilde{x} \cdot A\theta_{\sigma,1}|^2 \Big\} dS$$
$$\ge -C \int_{S_t} \varphi \sigma' r^{-2} |u_\sigma|^2 dS = -Cm \int_{S_t} \varphi r^{-2-\gamma} |u|^2 dS$$

since $\varphi\sqrt{k} = r^{2-\delta}\lambda^{(2-\delta)/2}\{1 + O(r^{-1})\}$. Thus, this term can be absorbed in the term corresponding to (13).

5. Outline of the proof of Theorem 2

We choose the weight function $\mu = \mu(r)$ to satisfy (μ.1) and also the following: There exists $\mu_0(r)$ verifying also (μ.1) such that

$$(\mu.2) \qquad \mu(r) \le \mu_0(r)$$

and if we put

$$\varphi(r) = \left(\int_r^\infty \mu(\tau) d\tau \right)^{-1} \quad \text{and} \quad \varphi_0(r) = \left(\int_r^\infty \mu_0(\tau) d\tau \right)^{-1}, \tag{16}$$

then it satisfies for $r \ge R_0$

$$(\mu.3) \qquad \varphi_0'(r) \le \varphi'(r) \quad \text{and} \quad \frac{1}{r} - \frac{\varphi_0'(r)}{\varphi_0(r)} \ge \max\Big\{0, -\mathrm{Re}\frac{\partial_r k}{2k}\Big\}.$$

Remark 5. If $\mu = r^{-1-\delta}$ $(0 < \delta \le 1)$ for $r > R_0$, then $\varphi = \delta r^\delta$ and $\varphi' = \delta^2 r^{-1+\delta}$. In this case ($\mu$.3) is verified from ($K$.2) if we choose $\mu_0 = r^{-1-\tilde{\delta}}$ with $0 < \tilde{\delta} \le \min\{\delta, 1 - \beta\}$.

If $\mu = r^{-1}(\log r)^{-1-\delta}$ $(0 < \delta \le 1)$, then $\varphi = \delta(\log r)^\delta$ and $\varphi' = \delta^2 r^{-1}(\log r)^{-1+\delta}$. Thus, we have $\dfrac{\varphi'}{\varphi} = o(r^{-1})$ and (μ.3) is satisfied by $\mu_0(r) = \mu(r)$.

Lemma 5 *We have for any* $R > 0$,

$$\frac{\varphi_0'(r)}{\varphi_0(r)} = \mu_0(r)\varphi_0(r) \notin L^1([R, \infty)).$$

Proof By definition $\varphi_0(r) \to \infty$ as $r \to \infty$. So, the assertion holds since we have

$$\int_R^r \frac{\varphi_0'(s)}{\varphi_0(s)} ds = \log\left\{\frac{\varphi_0(r)}{\varphi_0(R)}\right\} \to \infty \quad \text{as } r \to \infty. \qquad \square$$

Lemma 6 *Let u be a radiative solution of* (6).

(*i*) *If* $\mathrm{Im}\zeta \neq 0$, *then we have* $u \in L^2(\Omega)$ *and*

$$|\mathrm{Im}\zeta| \|u\| \leq \|f\|.$$

(*ii*) *There exists $C > 0$ such that for any $R \geq R_0$ and $\zeta \in \Gamma_\pm$,*

$$\int_{B_R'} \mu_0 |\sqrt{k}| |u|^2 dx \leq C \varphi_0(R)^{-1} \left\{ \|\tilde{x} \cdot \theta\|^2_{\varphi_0' \mathcal{H}^{-1}, B_R'} + \|u\|^2_{\mu_0 \mathcal{H}} + \|f\|^2_{(\mu_0 \mathcal{H})^{-1}} \right\}$$

Proof By the Green formula

$$\mathrm{Im} \int_{B_r} f \bar{u} dx = -\mathrm{Im} \int_{S_r} (\tilde{x} \cdot \nabla_b u) \bar{u} dS - \mathrm{Im}\zeta \int_{B_r} |u|^2 dx.$$

This is rewritten as

$$\mathrm{Im}\zeta \int_{B_r} |u|^2 dx - \int_{S_r} \mathrm{Im} K |u|^2 dS = -\mathrm{Im}\left[\int_{S_r} \tilde{x} \cdot \theta \bar{u} dS + \int_{B_r} f \bar{u} dx\right].$$

Note here that $\mathrm{Im}\zeta$ and $-\mathrm{Im}K$ has the same sign when r is large, say for $r \geq R$. $\square$

Lemma 7 *Let u be a radiative solution of* (6). *Then there exists $C = C(\Gamma_\pm) > 0$ such that*

$$\int_{B_{R+1}'} \varphi_0' |\sqrt{k}|^{-1} |\theta|^2 dx \leq C \left\{ \|u\|^2_{\mu_0 |\sqrt{k}|, B_R'} + \|f\|^2_{(\mu_0 |\sqrt{k}|)^{-1}, B_R'} \right\}.$$

Proof In the quadratic identity (8) with $\sigma = 0$ we choose

$$\Phi = \frac{\chi \varphi_0(r)}{|k(x,\zeta)|^{1/2}}. \tag{17}$$

where $\chi = \chi(r)$ is smooth and satisfy $\chi(r) = 0$ $(r < R)$ and $= 1$ $((r > R+1)$. Then

$$-\left[\int_{S_t} - \int_{S_R}\right] \Phi \left\{ |\tilde{x} \cdot \theta|^2 - \frac{1}{2}|\theta|^2 \right\} dS + \mathrm{Re} \int_{B_{R,t}} \Phi \left[\frac{1}{r} \{ |\theta|^2 - |\tilde{x} \cdot \theta|^2 \} \right.$$
$$\left. + \left(-i\sqrt{k} + \frac{\partial_r k}{4k} \right) |\theta|^2 + \left(\frac{\varphi_0'}{\varphi_0} - \mathrm{Re}\frac{\partial_r k}{2k} + \frac{\chi'}{\chi} \right) \left\{ |\tilde{x} \cdot \theta|^2 - \frac{1}{2}|\theta|^2 \right\} \right.$$

$$-\mathrm{Re}\frac{\tilde{\nabla}k}{2k}\cdot\theta(\tilde{x}\cdot\overline{\theta})+u(\tilde{\nabla}K\cdot\overline{\theta})+q_K u(\tilde{x}\cdot\overline{\theta})\Big]dx=\mathrm{Re}\int_{B_{R,t}}\Phi f_\sigma(\tilde{x}\cdot\overline{\theta})dx.$$

Since

$$\frac{1}{r}\{|\theta|^2-|\tilde{x}\cdot\theta|^2\}+\mathrm{Re}\frac{\partial_r k}{4k}|\theta|^2+\left(\frac{\varphi_0'}{\varphi_0}-\mathrm{Re}\frac{\partial_r k}{2k}\right)\left\{|\tilde{x}\cdot\theta|^2-\frac{1}{2}|\theta|^2\right\}$$

$$=\left(\frac{1}{r}-\frac{\varphi_0'}{\varphi_0}+\mathrm{Re}\frac{\partial_r k}{2k}\right)\{|\theta|^2-|\tilde{x}\cdot\theta|^2\}+\frac{\varphi_0'}{2\varphi_0}|\theta|^2, \tag{18}$$

it follows that

$$\int_{S_t}\Phi\left\{|\tilde{x}\cdot\theta|^2-\frac{1}{2}|\theta|^2\right\}dS\geq\mathrm{Re}\int_{B_{R,t}}\Phi\left[\left\{\frac{2\varphi_0'}{\varphi_0}-C\mu\right\}|\theta|^2\right.$$

$$\left.-C_1\mu|\theta|^2-C_2\mu|\sqrt{k}||u||\tilde{x}\cdot\theta|-|f||\theta|\right]dx$$

$$+\int_{B_{R,R+1}}\chi'\varphi_0|\sqrt{k}|^{-1}\left\{|\tilde{x}\cdot\theta|^2-\frac{1}{2}|\theta|^2\right\}dx.$$

Note the identity $\varphi_0'=\mu_0\varphi_0^2$. Then the Schwarz inequality and letting $t\to\infty$ show the desired assertion. □

We need one more lemma, which is not obvious for exploding potentials.

Lemma 8 *For $\zeta\in\Gamma_\pm$ and $f\in L^2_{(\mu_0\mathcal{H})^{-1}}(\mathbf{R}^n)$ let $u=R(\zeta)f$. Then u satisfies the radiation condition.*

Now, as is given in Eidus [8], the Theorem 2 is proved as follows.

Let $\{\zeta_j,f_j\}\subset\Gamma_\pm\times L^2_{\mu_1^{-1}}$ converges to $\{\zeta_0,f_0\}$ as $j\to\infty$. Since the other case is easier, we assume that $\zeta_0=\lambda\pm i0$, $\lambda\in I$. Let $u_j=R(\zeta_j)f_k$.

(i) Each u_j satisfies the radiation conditions: *by Lemma 8.*

(ii) $\{u_k\}$ is pre-compact in $L^2_{\mu_0|\sqrt{k}|}$ if it is bounded in the same space, and every accumulation $u_0\in L^2_{\mu_0|\sqrt{k}|}$ satisfies the radiation conditions: *by Rellich compactness criterion, Lemmas 6 (ii) and 7.*

(iii) The boundedness $\{u_j\}$ is proved by contradiction.

In fact, assume that there exists a subsequence, which we also write $\{u_j\}$, such that $\|u_j\|_{\mu_0|\sqrt{k}|}\to\infty$ as $j\to\infty$. Put $v_j=u_j/\|u_j\|_{\mu_2}$. Then as is explained above, $\{\zeta_j,v_j\}$ has a convergent subsequence, and if we denote the limit by $\{\lambda_0\pm i0,v_0\}$, then it satisfies the eigenvalue problem (5) with $\lambda=\lambda_0$ and also

$$\|v_0\|_{\mu_0|\sqrt{k}|}=1,\ \|\partial_r v_0+K_\pm v_0\|_{\varphi_0'|\sqrt{k}|^{-1}}<\infty, \tag{19}$$

where $K_\pm = K(x, \lambda_0 \pm i0)$. The second inequality implies

$$\liminf_{r\to\infty} \int_{S(r)} \sqrt{k}^{-1} |\partial_r v_0 + K_\pm v_0|^2 dS = 0$$

since $\varphi_0'(r) \notin L^1([R,\infty))$ for any $R > 0$ by Lemma 5. Comparing this with Theorem 1, we see that v_0 has a compact support in $x \in \mathbf{R}^n$. Hence, $v_0 \equiv 0$ by the unique continuation property for solutions to (5). But this contradicts to the first equation of (19).

(iv) If we apply Theorem 1 once more, then $\{u_j\}$ itself is shown to converge. □

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