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BOUNDEDNESS OF FUNCTIONS OF SCHRÖDINGER OPERATORS ON OPEN SETS

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ABSTRACT. This paper is a resume of the paper "Boundedness of spectral multipliers for Schrödinger operators on open sets" by Iwabuchi, Matsuyama and Taniguchi. The purpose is to overview the results in the paper, namely, L^p-estimates and gradient estimates for functions of Schrödinger operators on an arbitrary open set of d-dimensional Euclidean space.

1. INTRODUCTION

This paper is a resume of Iwabuchi, Matsuyama and Taniguchi [15].

Let Ω be an open set of $\mathbb{R}^d$, where $d \geq 1$. We consider the Schrödinger operator

$$H_V = -\Delta + V(x) = -\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} + V(x) \quad \text{on } L^2(\Omega)$$

with the Dirichlet boundary condition, where $V(x)$ is a real-valued measurable function on Ω. When H_V is self-adjoint on $L^2(\Omega)$, an operator $\varphi(H_V)$ can be defined on $L^2(\Omega)$ by

$$\varphi(H_V) := \int_{-\infty}^{\infty} \varphi(\lambda)\, dE_{H_V}(\lambda)$$

for any Borel measurable function φ on $\mathbb{R}$, where $\{E_{H_V}(\lambda)\}_{\lambda\in\mathbb{R}}$ is the spectral resolution of the identity for H_V. This paper is devoted to proving L^p-boundedness of $\varphi(H_V)$ for $1 \leq p \leq \infty$, and uniform L^p-estimates for $\varphi(\theta H_V)$ with respect to a parameter $\theta > 0$. If $H_V = -\Delta$ on $\mathbb{R}^d$, then $\varphi(-\Delta)$ is a Fourier multiplier, whose L^p-boundedness is well-known. In this sense, $\varphi(H_V)$ is a generalization of Fourier multiplier, and it is expected that its L^p-boundedness is a fundamental rule in studying function spaces and PDEs on domains (see [1, 4, 6, 9, 14, 16, 19, 23]).

Let us introduce some notations used in this paper. We denote by $\mathcal{B}(X, Y)$ the space of all bounded linear operators from a Banach space X to another one Y. When $X = Y$, we denote by $\mathcal{B}(X) = \mathcal{B}(X, X)$. We use the notation $\mathcal{D}(T)$ for the domain of an operator T. We denote by $\mathcal{S}(\mathbb{R})$ the space of rapidly decreasing functions on $\mathbb{R}$. We denote by χ_E the characteristic function of a measurable set E. For a self-adjoint

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operator T on a Hilbert space, we denote by $\sigma(T)$ the spectrum of T. We define the inner product $\langle\cdot,\cdot\rangle$ of $L^2(\Omega)$ by

$$\langle u,v\rangle := \int_\Omega u(x)\overline{v(x)}\,dx, \quad u,v\in L^2(\Omega).$$

2. Main results

In this section we state the results. For this purpose, we suppose that the potential V satisfies the following condition:

Assumption A. *V is a real-valued measurable function on Ω such that*

$$V=V_+-V_-,\quad V_\pm\geq 0,\quad V_+\in L^1_{\mathrm{loc}}(\Omega)\quad \text{and}\quad V_-\in K_d(\Omega),$$

where $K_d(\Omega)$ is the Kato class of potentials.

Following Simon (see [22, Section A.2]), let us give the definition of $K_d(\Omega)$ as follows:

Definition (Kato class of potentials). *We say that V belongs to the class $K_d(\Omega)$ if*

$$\begin{cases} \displaystyle\lim_{r\to 0}\sup_{x\in\Omega}\int_{\Omega\cap\{|x-y|<r\}}\frac{|V(y)|}{|x-y|^{d-2}}\,dy=0 & \text{for } d\geq 3,\\ \displaystyle\lim_{r\to 0}\sup_{x\in\Omega}\int_{\Omega\cap\{|x-y|<r\}}\log(|x-y|^{-1})|V(y)|\,dy=0 & \text{for } d=2,\\ \displaystyle\sup_{x\in\Omega}\int_{\Omega\cap\{|x-y|<1\}}|V(y)|\,dy<\infty & \text{for } d=1.\end{cases}$$

If V satisfies assumption A, then it is well-known that $-\Delta+V$ has a self-adjoint realization on $L^2(\Omega)$, and we denote by H_V its realization with the domain

$$\mathcal{D}(H_V)=\left\{u\in H_0^1(\Omega)\;\middle|\;\sqrt{V_+}u\in L^2(\Omega),\ H_Vu\in L^2(\Omega)\right\},$$

where $H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with $H^1(\Omega)$-norm. Moreover H_V is semi-bounded, and the infimum of $\sigma(H_V)$ is finite. Hence

$$\langle H_Vu,u\rangle\geq \inf\sigma(H_V)\|u\|^2_{L^2(\Omega)}$$

for any $u\in\mathcal{D}(H_V)$. For the details, see appendix A.

We shall prove the following:

Theorem 2.1. *Let $\varphi\in\mathcal{S}(\mathbb{R})$. Suppose that the potential V satisfies assumption A. Then $\varphi(H_V)$ is extended to a bounded linear operator on $L^p(\Omega)$ for any $1\leq p\leq\infty$. Furthermore, the following assertions hold:*

(i) *There exists a constant $C>0$ such that*

$$\|\varphi(\theta H_V)\|_{\mathcal{B}(L^p(\Omega))}\leq C \tag{2.1}$$

for any $0<\theta\leq 1$.

(ii) *Assume further that* V_- *satisfies*

$$\begin{cases} \sup_{x\in\Omega} \int_\Omega \frac{V_-(y)}{|x-y|^{d-2}}\,dy < \frac{\pi^{\frac{d}{2}}}{\Gamma(d/2-1)} & \text{if } d \geq 3, \\ V_- = 0 & \text{if } d = 1, 2. \end{cases} \tag{2.2}$$

Then the estimate (2.1) *holds for any* $\theta > 0$.

Corollary 2.2. *Let* $\varphi \in \mathcal{S}(\mathbb{R})$. *Suppose that the potential* V *satisfies assumption* A. *Let* m *be a non-negative integer, and let* $1 \leq p \leq q \leq \infty$. *Then* $H_V^m \varphi(H_V)$ *is extended to a bounded linear operator from* $L^p(\Omega)$ *to* $L^q(\Omega)$. *Furthermore, the following assertions hold:*

(i) *There exists a constant* $C > 0$ *such that*

$$\|H_V^m \varphi(\theta H_V)\|_{\mathcal{B}(L^p(\Omega),L^q(\Omega))} \leq C\theta^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})-m} \tag{2.3}$$

for any $0 < \theta \leq 1$.

(ii) *Assume further that* V_- *satisfies* (2.2). *Then the estimate* (2.3) *holds for any* $\theta > 0$.

Remark. We note that the potential like

$$V(x) \simeq -c|x|^{-2} \quad \text{as } |x| \to \infty, \quad c > 0 \tag{2.4}$$

is excluded from assumption (2.2) on V. The potentials such as (2.4) are very interesting. However, the uniform boundedness (2.1) for any $\theta > 0$ in Theorem 2.1 would not be generally obtained, since

$$\lim_{t\to\infty} \|e^{-tH_V}\|_{L^p \to L^p} = \infty$$

for some $p \neq 2$ which was proved in [12, 13].

Furthermore, we show the following result on gradient estimates for $\varphi(H_V)$.

Theorem 2.3. *Let* $\varphi \in \mathcal{S}(\mathbb{R})$. *Suppose that the potential* V *satisfies assumption* A. *Then* $\varphi(H_V)$ *is extended to a bounded linear operator from* $L^p(\Omega)$ *to* $W^{1,p}(\Omega)$ *for any* $1 \leq p \leq 2$. *Furthermore, the following assertions hold:*

(i) *There exists a constant* $C > 0$ *such that*

$$\|\nabla \varphi(\theta H_V)\|_{\mathcal{B}(L^p(\Omega))} \leq C\theta^{-\frac{1}{2}} \tag{2.5}$$

for any $0 < \theta \leq 1$.

(ii) *Assume further that* V_- *satisfies* (2.2). *Then the estimate* (2.5) *holds for any* $\theta > 0$.

Let us give some known results on Theorems 2.1 and 2.3. When $\Omega = \mathbb{R}^d$, there are many results on L^p-estimates for $\varphi(\theta H_V)$ under the assumption that the potential is non-negative on $\mathbb{R}^d$ (see, e.g., [9, 11, 23]). On the other hand, when the potentials are admitted to be negative, several results are known; Jensen and Nakamura dealt with the Schrödinger operator with potential whose negative part is of Kato class (see [16, 17]), and then D'Ancona and Pierfelice also dealt with the same type of potentials satisfying (2.2) (see [4]). Theorem 2.1 is a generalization of the results on L^p-estimates for $\varphi(\theta H_V)$ in [4, 16, 17]. We mention the results in the more general setting. There are several studies on L^p-estimates for more general operators $\varphi(L)$, where L is a non-negative self-adjoint operator having the property that the integral kernel of semigroup $\{e^{-tL}\}_{t>0}$ has a Gaussian upper bound (see [5, 10, 19, 20]). Among other things, there is a result on the estimates involving a parameter $\theta > 0$; Duong, Ouhabaz and Sikora proved uniform L^p-estimates for $\varphi(\theta L)$ with respect to $\theta > 0$, where $\varphi \in H^s(\mathbb{R})$ with compact support for some $s > d/2$ (see [5]).

As to Theorem 2.3, the problem is closely related to L^p-boundedness of operators ∇e^{-tH_V} and $\nabla H_V^{-1/2}$. When V is non-negative, the results of [3, 20] imply the estimate (2.5) for $p \le 2$. On the other hand, the situation of the case $p > 2$ is more complicated (see [2, 3, 7, 18, 20, 21]).

One of crucial tools to prove Theorem 2.1 is Gaussian upper bounds for semigroup e^{-tH_V}. In this paper we derive Gaussian upper bounds under assumption on V in Theorem 2.1. To prove Theorem 2.1, we use amalgam spaces on Ω, and show the estimates for the resolvent of H_V and $\varphi(H_V)$. This idea comes from Jensen and Nakamura [16, 17]. Furthermore, this paper reveals that the gradient estiamtes (2.5) in Theorem 2.3 is derived in a similar way to Theorem 2.1.

This paper is organized as follows. In §3, we state the result on Gaussian upper bounds for e^{-tH_V}, and the outline of its proof. In §4, we prepare two lemmas to prove Theorem 2.1. In §5, the proofs of Theorem 2.1 and Corollary 2.2 are given. In §6, we give the outline of proof of Theorem 2.3. In appendix A, we mention self-adjointness of H_V.

3. Gaussian upper bounds for e^{-tH_V}

In this section we shall prove pointwise estimates for the kernel $K(t,x,y)$ of semigroup $\{e^{-tH_V}\}_{t>0}$ generated by H_V. These estimates are fundamental tools in proving Theorems 2.1 and 2.3. Throughout this section we use the following notation:

$$\gamma_d := \frac{\pi^{\frac{d}{2}}}{\Gamma(d/2-1)} \quad \text{for } d \ge 3,$$

and

$$\|V\|_{K_d(\Omega)} := \sup_{x\in\Omega} \int_\Omega \frac{V(y)}{|x-y|^{d-2}}\,dy \quad \text{for } d \ge 3.$$

Then we have the following:

Theorem 3.1. *Suppose that the potential V satisfies assumption* A. *Then the following assertions hold:*

(i) *There exist two constants $\omega \ge -\inf\sigma(H_V)$ and $C > 0$ such that*

$$0 \le K(t,x,y) \le Ct^{-\frac{d}{2}} e^{\omega t} e^{-\frac{|x-y|^2}{8t}} \quad a.e.\ x,y \in \Omega \tag{3.1}$$

for any $t > 0$.

(ii) *Assume further that V_- satisfies* (2.2). *Then there exists a constant $C = C_{d,V} > 0$ such that*

$$0 \le K(t,x,y) \le Ct^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{8t}} \quad a.e.\ x,y \in \Omega \tag{3.2}$$

for any $t > 0$. Here the constant C in (3.2) is written as

$$C = C_{d,V} = \begin{cases} \dfrac{(2\pi)^{-\frac{d}{2}}}{1 - \|V_-\|_{K_d(\Omega)}/\gamma_d} & \text{if } d \geq 3, \\ (4\pi)^{-\frac{d}{2}} & \text{if } d = 1, 2. \end{cases}$$

In the rest of this section, let us state the outline of proof of Theorem 3.1.

The following lemma is crucial in the proof of Theorem 3.1.

Lemma 3.2. *Suppose that the potential V satisfies assumption A. Let $\tilde{V}$ and $\tilde{V}_-$ be the zero extensions of V and V_- to $\mathbb{R}^d$, respectively. Let $\tilde{H}_{\tilde{V}}$ and $\tilde{H}_{\tilde{V}_-}$ be the self-adjoint extensions of $-\Delta + \tilde{V}$ and $-\Delta - \tilde{V}_-$ on $L^2(\mathbb{R}^d)$, respectively. Then for any non-negative function $f \in L^2(\Omega)$, the following estimates hold:*

$$\left(e^{-tH_V} f\right)(x) \geq 0 \quad a.e.\, x \in \Omega, \tag{3.3}$$

$$\left(e^{-tH_V} f\right)(x) \leq \left(e^{-t\tilde{H}_{\tilde{V}}} \tilde{f}\right)(x) \quad a.e.\, x \in \Omega, \tag{3.4}$$

$$\left(e^{-t\tilde{H}_{\tilde{V}}} \tilde{f}\right)(x) \leq \left(e^{-t\tilde{H}_{\tilde{V}_-}} \tilde{f}\right)(x) \quad a.e.\, x \in \Omega \tag{3.5}$$

for any $t > 0$, where $\tilde{f}$ is the zero extension of f to $\mathbb{R}^d$.

For the details of proof of Lemma 3.2, see [15, section 3]. Let us turn to proof of Theorem 3.1.

Proof of Theorem 3.1. We adopt a sequence $\{j_\varepsilon(x)\}_{\varepsilon>0}$ of functions on $\mathbb{R}^d$ defined by letting

$$j_\varepsilon(x) := \frac{1}{\varepsilon^d} j\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^d,$$

where

$$j(x) = \begin{cases} A_d\, e^{-\frac{1}{1-|x|^2}} & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

with

$$A_d := \left(\int_{|x|<1} e^{-\frac{1}{1-|x|^2}}\, dx \right)^{-1}.$$

As is well-known, the sequence $\{j_\varepsilon(x)\}_{\varepsilon>0}$ enjoys the following property:

$$j_\varepsilon(\cdot - y) \to \delta_y \quad \text{in } \mathcal{S}'(\mathbb{R}^d) \text{ as } \varepsilon \to 0, \tag{3.6}$$

where δ_y is the Dirac delta function at $y \in \Omega$. Let $y \in \Omega$ be fixed, and let $\tilde{K}(t, x, y)$ be the kernel of $e^{-t\tilde{H}_{\tilde{V}_-}}$, where we denote by $\tilde{H}_{\tilde{V}_-}$ the self-adjoint extension of $-\Delta - \tilde{V}_-$ on $L^2(\mathbb{R}^d)$. Taking $\varepsilon > 0$ sufficiently small so that $\operatorname{supp} j_\varepsilon(\cdot - y) \subset \Omega$, and applying (3.3)–(3.5) from Lemma 3.2 to both f and $\tilde{f}$ replaced by $j_\varepsilon(\cdot - y)$, we get

$$0 \leq \int_\Omega K(t, x, z) j_\varepsilon(z - y) dz \leq \int_{\mathbb{R}^d} \tilde{K}(t, x, y) j_\varepsilon(z - y) dz \quad \text{a.e. } x \in \Omega.$$

Noting (3.6) and taking the limit of the previous inequality as $\varepsilon \to 0$, we get

$$0 \leq K(t, x, y) \leq \tilde{K}(t, x, y) \quad \text{a.e. } x, y \in \Omega$$

for any $t > 0$. Finally, by using the pointwise estimates:

$$\tilde{K}(t,x,y) \le Ct^{-d/2}e^{\omega t}e^{-\frac{|x-y|^2}{8t}} \quad \text{a.e. } x, y \in \Omega$$

for any $t > 0$ (see Proposition B.6.7 in [22]), we obtain the estimate (3.1), as desired. Thus the assertion (i) is proved.

Finally, we prove the assertion (ii). We recall Proposition 5.1 in [4] that if $d \ge 3$, then

$$\tilde{K}(t,x,y) \le \frac{(2\pi t)^{-d/2}}{1 - \|\tilde{V}_-\|_{K_d(\mathbb{R}^d)}/\gamma_d} e^{-\frac{|x-y|^2}{8t}} \left(= \frac{(2\pi t)^{-d/2}}{1 - \|V_-\|_{K_d(\Omega)}/\gamma_d} e^{-\frac{|x-y|^2}{8t}} \right)$$

for a.e. $x, y \in \Omega$ and any $t > 0$. When $d = 1, 2$, we have

$$\tilde{K}(t,x,y) \le (4\pi t)^{-d/2}e^{-\frac{|x-y|^2}{4t}} \quad \text{a.e. } x, y \in \Omega$$

for any $t > 0$. By the above estimates, we conclude (3.2). The proof of Theorem 3.1 is finished. □

4. Key lemmas

In this section we shall give outlines of proof of the estimates for the resolvent of H_V and $\varphi(H_V)$ in amalgam spaces. These lemmas play an crucial role in the proof of Theorem 2.1.

Following Fournier and Stewart (see [8]), let us give the definition of scaled amalgam spaces on Ω as follows.

Definition (Amalgam spaces). *Let $1 \le p, q \le \infty$ and $\theta > 0$. The space $l^p(L^q)_\theta$ is defined by letting*

$$l^p(L^q)_\theta = l^p(L^q)_\theta(\Omega) := \left\{ f \in L^q_{\mathrm{loc}}(\overline{\Omega}) \,\middle|\, \sum_{n \in \mathbb{Z}^d} \|f\|^p_{L^q(C_\theta(n))} < \infty \right\}$$

with norm

$$\|f\|_{l^p(L^q)_\theta} = \begin{cases} \left(\sum_{n \in \mathbb{Z}^d} \|f\|^p_{L^q(C_\theta(n))} \right)^{\frac{1}{p}} & \text{for } 1 \le p < \infty, \\ \sup_{n \in \mathbb{Z}^d} \|f\|_{L^q(C_\theta(n))} & \text{for } p = \infty, \end{cases}$$

where $C_\theta(n)$ is the cube centered at $\theta^{1/2}n$ ($n \in \mathbb{Z}^d$) with side length $\theta^{1/2}$:

$$C_\theta(n) = \left\{ x = (x_1, x_2, \cdots, x_d) \in \Omega \,\middle|\, \max_{j=1,\cdots,d} |x_j - \theta^{\frac{1}{2}} n_j| \le \frac{\theta^{\frac{1}{2}}}{2} \right\}.$$

Let us give a remark on the properties of $l^p(L^q)_\theta$-spaces. The spaces $l^p(L^q)_\theta$ are complete with respect to the norm $\|\cdot\|_{l^p(L^q)_\theta}$, and have the property that

$$l^p(L^q)_\theta \hookrightarrow L^p(\Omega) \cap L^q(\Omega)$$

for any $\theta > 0$, provided $1 \le p \le q \le \infty$.

4.1. **Estimates for $(H_V - z)^{-\beta}$.**

Lemma 4.1. *Let $1 \le p \le q \le \infty$, and β be such that*

$$\beta > \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right).$$

Suppose that the potential V satisfies assumption A. *Let $z \in \mathbb{C}$ with*

$$\mathrm{Re}(z) < \min\{-\omega, 0\},$$

where ω is the constant as in Theorem 3.1. Then $(H_V - z)^{-\beta}$ is extended to a bounded linear operator from $L^p(\Omega)$ to $l^p(L^q)_\theta$ with $\theta = 1$. Furthermore, the following assertions hold:

(i) *There exists a constant C depending on d, p, q, β and z such that*

$$\left\|(\theta H_V - z)^{-\beta}\right\|_{\mathcal{B}(L^p(\Omega), L^q(\Omega))} \le C\theta^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}, \tag{4.1}$$

$$\left\|(\theta H_V - z)^{-\beta}\right\|_{\mathcal{B}(L^p(\Omega), l^p(L^q)_\theta)} \le C\theta^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \tag{4.2}$$

for any $0 < \theta \le 1$.

(ii) *Assume further that V_- satisfies* (2.2). *Let $z \in \mathbb{C}$ be such that*

$$\mathrm{Re}(z) < 0.$$

Then the estimates (4.1) *and* (4.2) *hold for any $\theta > 0$.*

Outline of proof of Lemma 4.1. The following formula is well known: For any $M > -\inf \sigma(H_V)$ and $\beta > 0$,

$$(H_V + M)^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{-Mt} e^{-tH_V}\, dt$$

(see, e.g., (A9) in page 449 of Simon [22]). Combining this formula with Theorem 3.1, we can prove Lemma 4.1 along the argument of proof of Theorem 4.1 in [15] (see also [17]). □

4.2. **Estimates for $\varphi(H_V)$.**

Lemma 4.2. *Suppose that V satisfies assumption A. Then the following assertions hold:*

(i) *Then there exists a constant $C > 0$ such that*

$$\|\varphi(\theta H_V)\|_{\mathcal{B}(l^1(L^2)_\theta)} \le C \tag{4.3}$$

for any $0 < \theta \le 1$.

(ii) *Assume further that V_- satisfies* (2.2). *Then the estimate* (4.3) *holds for any $\theta > 0$.*

Let us state the outline of proof of Lemma 4.2. For this purpose, let us introduce a family $\mathscr{A}_\alpha$ of operators.

Definition. *Let $\alpha > 0$ and $\theta > 0$. We say that $L \in \mathscr{A}_\alpha (= \mathscr{A}_{\alpha,\theta})$ if $L \in \mathcal{B}(L^2(\Omega))$ and*

$$|||L|||_\alpha := \sup_{n \in \mathbb{Z}^d} \left\| |\cdot - \theta^{\frac{1}{2}} n|^\alpha L \chi_{C_\theta(n)} \right\|_{\mathcal{B}(L^2(\Omega))} < \infty.$$

To give the proof of Lemma 4.2, let us prepare the following two lemmas. The following lemma show a sufficient condition for L^2-functions to be bounded in $l^1(L^2)_\theta$.

Lemma 4.3. *Let $\theta > 0$, and let $L \in \mathscr{A}_\alpha$ for some $\alpha > d/2$. Then there exists a constant $C > 0$ depending only on α and d such that*

$$\|Lf\|_{l^1(L^2)_\theta} \le C\left(\|L\|_{\mathcal{B}(L^2(\Omega))} + \theta^{-\frac{d}{4}}|||L|||_\alpha^{\frac{d}{2\alpha}}\|L\|_{\mathcal{B}(L^2(\Omega))}^{1-\frac{d}{2\alpha}}\right)\|f\|_{l^1(L^2)_\theta}$$

for any $f \in l^1(L^2)_\theta$.

For the details on the proof of Lemma 4.3, see Lemma 6.2 in [15].

The following lemma states that $\varphi(H_V)$ belongs to $\mathscr{A}_\alpha$ for any $\alpha > 0$.

Lemma 4.4. *Let $\varphi \in \mathcal{S}(\mathbb{R})$ and $\alpha > 0$. Suppose that the potential V satisfies assumption* A. *Then the following assertions hold:*

(i) *The operator $\varphi(\theta H_V)$ belongs to $\mathscr{A}_\alpha$ for any $0 < \theta \le 1$. Furthermore, there exist a constant $C > 0$ such that*

$$|||\varphi(\theta H_V)|||_\alpha \le C\theta^{\frac{\alpha}{2}} \tag{4.4}$$

for any $0 < \theta \le 1$.

(ii) *Assume further that V_- satisfies* (2.2). *Then the same assertion as in the assertion* (i) *holds for any $\theta > 0$.*

For the details on the proof of Lemma 4.4, see Lemma 6.3 in [15].

Proof of Lemma 4.2. We prove only the assertion (i), since the proof of (ii) is the same as (i). Let $0 < \theta \le 1$. By Lemma 4.4, the operator $\varphi(\theta H_V)$ belongs to $\mathscr{A}_\alpha$ for any $\alpha > 0$. Choosing $\alpha > d/2$, and applying Lemma 4.3 to $\varphi(\theta H_V)$, we estimate

$$\begin{aligned}&\|\varphi(\theta H_V)f\|_{l^1(L^2)_\theta}\\ \le& C\left(\|\varphi(\theta H_V)\|_{\mathcal{B}(L^2(\Omega))} + \theta^{-\frac{d}{4}}|||\varphi(\theta H_V)|||_\alpha^{\frac{d}{2\alpha}}\|\varphi(\theta H_V)\|_{\mathcal{B}(L^2(\Omega))}^{1-\frac{d}{2\alpha}}\right)\|f\|_{l^1(L^2)_\theta}\end{aligned}$$

for any $f \in l^1(L^2)_\theta$. Hence, noting from (4.4) in Lemma 4.4 that

$$\|\varphi(\theta H_V)\|_{\mathcal{B}(L^2(\Omega))} \le C$$

and

$$|||\varphi(\theta H_V)|||_\alpha^{\frac{d}{2\alpha}} \le C\theta^{\frac{d}{4}},$$

we conclude (4.3). Thus the proof of Lemma 4.4 is finished. □

5. Proof of Theorem 2.1 and Corollary 2.2

In this section we prove Theorem 2.1 and Corollary 2.2. First let us turn to proof of Theorem 2.1.

Proof of Theorem 2.1. First we prove the assertion (i). Let $0 < \theta \le 1$. It suffices to show L^1-estimate for $\varphi(\theta H_V)$. In fact, if L^1-estimate is proved, then L^∞-estimate is also obtained by duality argument, and hence, the Riesz-Thorin interpolation theorem allows us to conclude L^p-estimates (2.1) for $1 \le p \le \infty$.

Let us proceed the proof of L^1-estimate. Going back to the definition of $l^1(L^2)_\theta$, we estimate

$$\begin{aligned}\|\varphi(\theta H_V)f\|_{L^1(\Omega)} &= \sum_{n\in\mathbb{Z}^d} \|\varphi(\theta H_V)f\|_{L^1(C_\theta(n))}\\ &\leq \sum_{n\in\mathbb{Z}^d} |C_\theta(n)|^{\frac{1}{2}} \|\varphi(\theta H_V)f\|_{L^2(C_\theta(n))}\\ &\leq \theta^{\frac{d}{4}} \|\varphi(\theta H_V)f\|_{l^1(L^2)_\theta},\end{aligned} \tag{5.1}$$

where we used the inequality:

$$|C_\theta(n)|^{\frac{1}{2}} \leq \theta^{\frac{d}{4}}.$$

Here, given a positive real number β, we choose $\tilde{\varphi} \in \mathcal{S}(\mathbb{R})$ as

$$\tilde{\varphi}(\lambda) = (\lambda + M)^\beta \varphi(\lambda) \quad \text{for } \lambda \in \sigma(H_V),$$

where M is a real number such that

$$M > \max\{\omega, 0\},$$

where ω is the constant in Theorem 3.1. Then, using assertions (i) in Lemmas 4.1 and 4.2, we estimate

$$\begin{aligned}\|\varphi(\theta H_V)f\|_{l^1(L^2)_\theta} &= \left\|\varphi(\theta H_V)(\theta H_V + M)^\beta(\theta H_V + M)^{-\beta} f\right\|_{l^1(L^2)_\theta}\\ &= \left\|\tilde{\varphi}(\theta H_V)(\theta H_V + M)^{-\beta} f\right\|_{l^1(L^2)_\theta}\\ &\leq C \left\|(\theta H_V + M)^{-\beta} f\right\|_{l^1(L^2)_\theta}\\ &\leq C\theta^{-\frac{d}{4}} \|f\|_{L^1(\Omega)}.\end{aligned}$$

Therefore, combining the estimates (5.1) and the above estimate, we conclude that

$$\|\varphi(\theta H_V)f\|_{L^1(\Omega)} \leq C\|f\|_{L^1(\Omega)}$$

for any $0 < \theta \leq 1$ and $f \in L^1(\Omega)$.

The assertion (ii) is proved in the same way as assertion (i) by using assertions (ii) in Lemmas 4.1 and 4.2 instead of assertions (i) in Lemmas 4.1, respectively. The proof of Theorem 2.1 is complete. □

In the rest of this section, we prove Corollary 2.2.

Proof of Corollary 2.2. We prove only the assertion (i), since the assertion (ii) is proved in the same way as assertion (i). Let $0 < \theta \leq 1$. Let M be a real number such that

$$M > \max\{\omega, 0\},$$

where ω is the constant in Theorem 3.1. Given $m \in \mathbb{N} \cup \{0\}$ and $\beta \in \mathbb{R}$ satisfying

$$\beta > \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right),$$

we choose $\tilde{\varphi} \in \mathcal{S}(\mathbb{R})$ as

$$\tilde{\varphi}(\lambda) = \lambda^m(\lambda + M)^\beta \varphi(\lambda) \quad \text{for } \lambda \in \sigma(H_V).$$

By using Lemma 4.1 and Theorem 2.1, we estimate

$$\begin{aligned}
&\|H_V^m\varphi(\theta H_V)\|_{\mathcal{B}(L^p(\Omega),L^q(\Omega))}\\
=&\|H_V^m\varphi(\theta H_V)(\theta H_V+M)^{\beta}(\theta H_V+M)^{-\beta}\|_{\mathcal{B}(L^p(\Omega),L^q(\Omega))}\\
\le&\theta^{-m}\|\tilde{\varphi}(\theta H_V)\|_{\mathcal{B}(L^q(\Omega))}\|(\theta H_V+M)^{-\beta}\|_{\mathcal{B}(L^p(\Omega),L^q(\Omega))}\\
\le&C\theta^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})-m}.
\end{aligned}$$

The proof of Corollary 2.2 is complete. □

6. Proof of Theorem 2.3

In this section we state the outline of proof of Theorem 2.3. For this purpose, we prepare the following lemma.

Lemma 6.1. *Suppose that V satisfies assumption A. Then the following assertions hold:*

(i) *Then there exists a constant $C>0$ such that*

$$\|\nabla\varphi(\theta H_V)\|_{\mathcal{B}(l^1(L^2)_\theta)}\le C\theta^{-\frac{1}{2}} \tag{6.1}$$

for any $0<\theta\le 1$.

(ii) *Assume further that V_- satisfies (2.2). Then the estimate (6.1) holds for any $\theta>0$.*

Lemma 6.1 follows from Lemma 4.3 and the following lemma in the same way as Lemma 4.2.

Lemma 6.2. *Let $\varphi\in\mathcal{S}(\mathbb{R})$. Suppose that the potential V satisfies assumption* A. *Let $\alpha>0$. Then the following assertions hold:*

(i) *The operator $\nabla\varphi(\theta H_V)$ belongs to $\mathscr{A}_\alpha$ for any $0<\theta\le 1$. Furthermore, there exist a constant $C>0$ such that*

$$|||\nabla\varphi(\theta H_V)|||_\alpha\le C\theta^{\frac{\alpha}{2}-\frac{1}{2}}$$

for any $0<\theta\le 1$.

(ii) *Assume further that V_- satisfies (2.2). Then the same assertion as in the assertion* (i) *holds for any $\theta>0$.*

For the details on the proof of Lemma 6.2, see Lemma 7.1 in [15].

Proof of Theorem 2.3. We prove only the assertion (i), since the proof of (ii) is similar to (i). Let $0<\theta\le 1$. First we prove L^2-estimate:

$$\|\nabla\varphi(\theta H_V)\|_{\mathcal{B}(L^2(\Omega))}\le C\theta^{-\frac{1}{2}}, \tag{6.2}$$

where the constant C is independent of θ. Since $\varphi(\theta H_V)f \in \mathcal{D}(H_V)$ for any $f \in L^2(\Omega)$, we estimate

$$
\begin{aligned}
&\|\nabla\varphi(\theta H_V)f\|^2_{L^2(\Omega)}\\
&=\int_\Omega \left(\nabla\varphi(\theta H_V)f\cdot\nabla\varphi(\theta H_V)f + V|\varphi(\theta H_V)f|^2 - V|\varphi(\theta H_V)f|^2\right)dx\\
&=\langle H_V\varphi(\theta H_V)f, \varphi(\theta H_V)f\rangle + \int_\Omega (V_- - V_+)|\varphi(\theta H_V)f|^2\,dx \qquad (6.3)\\
&\le\langle H_V\varphi(\theta H_V)f, \varphi(\theta H_V)f\rangle + \int_\Omega V_-|\varphi(\theta H_V)f|^2\,dx\\
&=: I + II.
\end{aligned}
$$

Then, by using Corollary 2.2, we estimate I as

$$
\begin{aligned}
I &\le \|H_V\varphi(\theta H_V)f\|_{L^2(\Omega)}\|\varphi(\theta H_V)f\|_{L^2(\Omega)}\\
&\le C\theta^{-1}\|f\|^2_{L^2(\Omega)}.
\end{aligned}
\qquad (6.4)
$$

As to the second term II, by using the inequality (A.1) in Lemma A.2, we have

$$
II \le \varepsilon\|\nabla\varphi(\theta H_V)f\|^2_{L^2(\Omega)} + b_\varepsilon\|\varphi(\theta H_V)f\|^2_{L^2(\Omega)}
$$

for any $\varepsilon > 0$. Noting that $\theta^{-1} > 1$, and using (2.1) in Theorem 2.1, we get

$$
b_\varepsilon\|\varphi(\theta H_V)f\|^2_{L^2(\Omega)} \le Cb_\varepsilon\theta^{-1}\|f\|^2_{L^2(\Omega)};
$$

whence

$$
II \le \varepsilon\|\nabla\varphi(\theta H_V)f\|^2_{L^2(\Omega)} + Cb_\varepsilon\theta^{-1}\|f\|^2_{L^2(\Omega)}. \qquad (6.5)
$$

Here we choose ε as $0 < \varepsilon < 1$. Then, combining the estimates (6.3)–(6.5), we obtain the estimate (6.2). In the case of assertion (ii), using the inequality (A.2) instead of (A.1), and proceeding the similar argument to the above, we obtain the estimate (6.2) for any $\theta > 0$.

Hence, if L^1-estimate

$$
\|\nabla\varphi(\theta H_V)\|_{\mathcal{B}(L^1(\Omega))} \le C\theta^{-\frac{1}{2}} \qquad (6.6)
$$

is proved, then we conclude the assertion (i) by the interpolation theorem. Therefore it sufficient to show L^1-estimate (6.6).

L^1-estimate (6.6) is proved in the same way as Theorem 2.1. In fact, letting $f \in L^1(\Omega)$, and going back to the definition of $l^1(L^2)_\theta$, we estimate

$$
\begin{aligned}
\|\nabla\varphi(\theta H_V)f\|_{L^1(\Omega)} &= \sum_{n\in\mathbb{Z}^d}\|\nabla\varphi(\theta H_V)f\|_{L^1(C_\theta(n))}\\
&\le \sum_{n\in\mathbb{Z}^d}|C_\theta(n)|^{1/2}\|\nabla\varphi(\theta H_V)f\|_{L^2(C_\theta(n))} \qquad (6.7)\\
&\le \theta^{\frac{d}{4}}\|\nabla\varphi(\theta H_V)f\|_{l^1(L^2)_\theta}.
\end{aligned}
$$

Here, given $\beta > 0$, we choose $\tilde{\varphi} \in \mathcal{S}(\mathbb{R})$ as

$$
\tilde{\varphi}(\lambda) = (\lambda + M)^\beta\varphi(\lambda) \quad \text{for } \lambda \in \sigma(H_V),
$$

where M is a real number such that

$$
M > \max\{\omega, 0\},
$$

where ω is the constant in Theorem 3.1. Then, using assertions (i) in Lemmas 4.1 and 6.1, we estimate

$$\begin{aligned}\|\nabla\varphi(\theta H_V)f\|_{l^1(L^2)_\theta} &= \left\|\nabla\varphi(\theta H_V)(\theta H_V+M)^\beta(\theta H_V+M)^{-\beta}f\right\|_{l^1(L^2)_\theta}\\ &= \left\|\nabla\tilde{\varphi}(\theta H_V)(\theta H_V+M)^{-\beta}f\right\|_{l^1(L^2)_\theta}\\ &\le C\theta^{-\frac{1}{2}}\left\|(\theta H_V+M)^{-\beta}f\right\|_{l^1(L^2)_\theta}\\ &\le C\theta^{-\frac{d}{4}-\frac{1}{2}}\|f\|_{L^1(\Omega)}.\end{aligned}$$

Therefore, combining the estimates (5.1) and the above estimate, we conclude that

$$\|\nabla\varphi(\theta H_V)f\|_{L^1(\Omega)} \le C\theta^{-\frac{1}{2}}\|f\|_{L^1(\Omega)}.$$

Thus the assertion (i) is proved. The proof of Theorem 2.3 is finished. □

Appendix A. (Self-adjointness of Schrödinger operators)

In this appendix we mention self-adjointness of Schrödinger operators with the Dirichlet boundary condition under assumption.

The self-adjointness of H_V is assured by the following proposition.

Proposition A.1. *Suppose that the potential V satisfies assumption* A. *Then the following assertions hold:*

(i) *There exists a unique semi-bounded self-adjoint operator H_V on $L^2(\Omega)$ with domain*

$$\mathcal{D}(H_V) = \left\{u \in H_0^1(\Omega) \mid \sqrt{V_+}u \in L^2(\Omega),\ H_V u \in L^2(\Omega)\right\}$$

such that

$$\langle H_V u, v\rangle = \int_\Omega \nabla u(x)\cdot\overline{\nabla v(x)}\,dx + \int_\Omega V(x)u(x)\overline{v(x)}\,dx$$

for any $u \in \mathcal{D}(H_V)$ and $v \in H_0^1(\Omega)$ with $\sqrt{V_+}v \in L^2(\Omega)$.

(ii) *Assume further that V_- satisfies*

$$\begin{cases}\displaystyle\sup_{x\in\Omega}\int_\Omega \frac{V_-(y)}{|x-y|^{d-2}}\,dy < \frac{4\pi^{\frac{d}{2}}}{\Gamma(d/2-1)} & \text{if } d \ge 3,\\ V_- = 0 & \text{if } d = 1,2.\end{cases}$$

Then H_V is non-negative on $L^2(\Omega)$.

Proposition A.1 is proved by using the theory of quadratic forms and the following lemma. The following lemma states that potentials of Kato class are relatively form bounded with respect to the Dirichlet Laplacian $-\Delta$.

Lemma A.2. *Suppose that the potential V belongs to $K_d(\Omega)$. Then the following assertions hold:*

(i) *For any $\varepsilon > 0$, there exists a constant $b_\varepsilon > 0$ such that*

$$\int_\Omega V(x)|u(x)|^2\,dx \le \varepsilon\|\nabla u\|^2_{L^2(\Omega)} + b_\varepsilon\|u\|^2_{L^2(\Omega)} \tag{A.1}$$

for any $u \in H_0^1(\Omega)$.

(ii) *Let* $d \geq 3$. *Assume further that* V *satisfies*

$$\|V\|_{K_d(\Omega)} := \sup_{x\in\Omega} \int_\Omega \frac{V(y)}{|x-y|^{d-2}}\, dy < \infty.$$

Then

$$\int_\Omega V(x)|u(x)|^2\, dx \leq \frac{\Gamma(d/2-1)\|V\|_{K_d(\Omega)}}{4\pi^{\frac{d}{2}}} \|\nabla u\|^2_{L^2(\Omega)} \tag{A.2}$$

for any $u \in H_0^1(\Omega)$.

For the more details on Proposition A.1 and Lemma A.2, we refer to [15, Section 2].

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