数理解析研究所講究録
第2046巻 2017年 1-22

An exterior nonlinear elliptic problem with a dynamical boundary condition

大阪府立大学・学術研究院　川上竜樹
Tatsuki Kawakami
Department of Mathematical Sciences, Osaka Prefecture University

1 Introduction

This is a survey article of the forthcoming paper [8]. We consider the problem

$$(1.1)\qquad \begin{cases} -\Delta u=u^p, \quad u\geq 0, & x\in\Omega,\ t>0,\\ \partial_t u+\partial_\nu u=0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=\varphi(x)\geq 0, & x\in\partial\Omega, \end{cases}$$

where $N\geq 2$, $\Omega\subset\mathbb{R}^N$, Δ is the N-dimensional Laplacian (in x), ν is the exterior normal vector on $\partial\Omega$, $\partial_t:=\partial/\partial t$, $\partial_\nu:=\partial/\partial\nu$, $p>1$, and φ is a nonnegative measurable function on $\partial\Omega$. For the half space, namely, $\Omega=\mathbb{R}^N_+$, Fila, Ishige and the author of this paper studied in [5, 6, 7] the existence and nonexistence of solutions to (1.1). They introduced a definition of a solution by the use of an integral identity and obtained the following:

(i) If $1<p\leq(N+1)/(N-1)$, $N>1$ and $\varphi\not\equiv 0$, then problem (1.1) possesses no local-in-time solutions.

(ii) Let $p>(N+1)/(N-1)$, $N>1$ and let $\varphi(x)=\mu(1+|x|)^{-2/(p-1)}$ on $\partial\mathbb{R}^N_+$ with $\mu>0$. If μ is sufficiently large, then problem (1.1) possesses no local-in-time solutions. On the other hand, if μ is sufficiently small, then a solution of (1.1) exists globally in time.

(iii) The following statements are equivalent:

  (a) Problem (1.1) has a local-in-time solution;

  (b) Problem (1.1) has a global-in-time solution;

(c) Problem

$$(1.2) \qquad -\Delta v = v^p, \quad v \geq 0 \quad \text{in} \quad \mathbb{R}^N_+, \qquad v(x) = \varphi(x) \quad \text{on} \quad \partial\mathbb{R}^N_+,$$

has a solution.

Furthermore, if $u = u(x,t)$ and $v = v(x)$ are minimal solutions of (1.1) and (1.2), respectively, then

$$(1.3) \qquad u(x', x_N, t) = v(x', x_N + t)$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \geq 0$ and $t > 0$.

Unfortunately, the arguments in [5, 6, 7] are available only if the domain is a half-space and are not applicable to other domains. Indeed, the definition of a solution in [5, 6, 7] is useful only for $\mathbb{R}^N_+$ and we cannot expect property (1.3) for other domains.

In this paper we focus on the exterior domain

$$\Omega := \{x \in \mathbb{R}^N : |x| > 1\}, \qquad N > 2,$$

and study the existence and nonexistence of solutions of (1.1). We introduce a definition of a solution of (1.1) using an integral equation and obtain results of a similar type as in (i), (ii) and (iii). However, there are some significant differences. The critical exponent $(N+1)/(N-1)$ in (i), (ii) is replaced by $N/(N-2)$ for problem (1.1) and the algebraic decay rate $t^{-(N-1)}$ of small solutions of (1.1) with $\Omega = \mathbb{R}^N_+$ (see [5]) is replaced by the exponential rate $e^{-(N-2)t}$ for problem (1.1). These rates are the decay rates of the Poisson kernels on the respective domains.

As far as we know, the only unbounded domain treated before is the half-space $\mathbb{R}^N_+$ ([1, 4, 5, 6, 7, 10, 11]). The main motivation of this paper is to study the effects of a change of geometry.

We begin with introducing a definition of solutions of the following elliptic problem

$$(1.4) \qquad \begin{cases} -\Delta u = F(t,u), \quad u \geq 0, & x \in \Omega, \ t > 0, \\ \partial_t u + \partial_\nu u = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = \varphi(x) \geq 0, & x \in \partial\Omega, \end{cases}$$

where F is nonnegative continuous function in $(0,\infty) \times [0,\infty)$. We introduce some notation first. Let $P = P(x,y)$ be the Poisson kernel on $B = B(0,1) := \{x \in \mathbb{R}^N : |x| < 1\}$, that is

$$P(x,y) := c_N \frac{1 - |x|^2}{|x-y|^N}, \qquad x \in B, \quad y \in \partial B,$$

where c_N is a constant to be chosen such that $\|P(x,\cdot)\|_{L^1(\partial B)} = 1$ for $x \in B$ (see (2.28) in [9]). Then $P = P(x,y)$ satisfies as a function of x

$$(1.5) \qquad -\Delta_x P = 0 \quad \text{in} \quad B, \qquad P(x,y) = \delta_y \quad \text{on} \quad \partial B,$$

where δ_y is the Dirac function on $\partial B = \partial\Omega$ at y. We denote by $K = K(x,y)$ the Kelvin transform of P as a function of x with respect to B, that is

$$K(x,y) := |x|^{-(N-2)} P\left(\frac{x}{|x|^2}, y\right), \qquad x \in \overline{\Omega}, \quad y \in \partial\Omega.$$

Set

$$\mathcal{K}(x,y,t) := K(e^t x, y), \qquad x \in \overline{\Omega}, \quad y \in \partial\Omega, \quad t \geq 0. \tag{1.6}$$

Then it follows from (1.5) that $\mathcal{K} = \mathcal{K}(x,y,t)$ as a function of x and t satisfies

$$\begin{cases} -\Delta_x \mathcal{K} = 0 & \text{in} \quad \Omega \times (0,\infty), \\ \partial_t \mathcal{K} + \partial_\nu \mathcal{K} = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ \mathcal{K}(\cdot, y, 0) = \delta_y & \text{on} \quad \partial\Omega. \end{cases}$$

For any nonnegative measurable function φ on $\partial\Omega$ and $t > 0$, we define

$$[S(t)\varphi](x) := \int_{\partial\Omega} \mathcal{K}(x,y,t)\varphi(y)\, d\sigma_y \equiv \int_{\partial\Omega} K(e^t x, y)\varphi(y)\, d\sigma_y, \qquad x \in \overline{\Omega}.$$

Let G be the Green function for the Laplace equation on Ω with the homogeneous Dirichlet boundary condition, that is

$$G(x,y) := \frac{c_N}{N-2}\left(|x-y|^{-(N-2)} - \big||x|(y - x_*)\big|^{-(N-2)}\right) \tag{1.7}$$

for x, $y \in \Omega$ with $x \neq y$, where $x_* := x/|x|^2$ for $x \in \Omega$.

Now we formulate our definition of a solution of (1.4).

Definition 1.1 *Let φ be a nonnegative measurable function on $\partial\Omega$ and $0 < T \leq \infty$.*
(i) *Let u and u^b be nonnegative measurable functions in $\Omega \times (0,T)$ and $\partial\Omega \times (0,T)$, respectively. Then we say that $U = (u, u^b)$ is a solution of* (1.4) *in $\Omega \times (0,T)$ if*

$$\begin{aligned} u(x,t) = & \int_{\partial\Omega} \mathcal{K}(x,y,t)\varphi(y)\, d\sigma_y + \int_\Omega G(x,y) F(t, u(y,t))\, dy \\ & + \int_0^t \int_{\partial\Omega} \mathcal{K}(x,y,t-s) \left\{ \int_\Omega K(z,y) F(s, u(z,s))\, dz \right\} d\sigma_y\, ds < \infty \end{aligned} \tag{1.8}$$

for almost all $x \in \Omega$ and $t \in (0,T)$ and

$$\begin{aligned} u^b(x',t) = & \int_{\partial\Omega} \mathcal{K}(x',y,t)\varphi(y)\, d\sigma_y \\ & + \int_0^t \int_{\partial\Omega} \mathcal{K}(x',y,t-s) \left\{ \int_\Omega K(z,y) F(s, u(z,s))\, dz \right\} d\sigma_y\, ds < \infty \end{aligned} \tag{1.9}$$

for almost all $x' \in \partial\Omega$ *and* $t \in (0,T)$. *If* u *and* u^b *satisfy* (1.8) *and* (1.9) *with* $=$ *replaced by* $\geq$, *then we say that* $U = (u, u^b)$ *is a supersolution of* (1.4).
(ii) *Let* $U = (u, u^b)$ *be a solution of* (1.4) *in* $\Omega \times (0,T)$. *Then we say that* U *is a minimal solution of* (1.4) *in* $\Omega \times (0,T)$ *if*

$$\begin{aligned} u(x,t) &\leq w(x,t) \quad \text{for almost all } x \in \Omega \text{ and } t \in (0,T), \\ u^b(x',t) &\leq w^b(x',t) \quad \text{for almost all } x' \in \partial\Omega \text{ and } t \in (0,T), \end{aligned}$$

for any solution $W = (w, w^b)$ *of* (1.4) *in* $\Omega \times (0,T)$.

Definition 1.1 is motivated by the definition of a solution of (1.1) for the case $\Omega = \mathbb{R}^N_+$, which was introduced in [5]. However, the derivation of integral equations (1.8) and (1.9) depends on Ω and it is different from the one in $\mathbb{R}^N_+$.

Remark 1.1 *Let us remark that if* $F \equiv 0$ *and* $\varphi \equiv c$, *where* $c \geq 0$, *then the solution given by Definition* 1.1 *is*

$$u_c(x,t) := c(e^t|x|)^{-(N-2)}, \qquad x \in \Omega, \quad t \geq 0, \tag{1.10}$$

(*see* (2.1)), *while the constant function c also satisfies* (1.4) *in the classical sense.*

Similarly to Definition 1.1, we define a solution of the elliptic problem

$$-\Delta v = f(v), \quad v \geq 0 \quad \text{in} \quad \Omega, \qquad v(x) = \varphi(x) \quad \text{on} \quad \partial\Omega, \tag{1.11}$$

where f is a nonnegative continuous function in $[0, \infty)$.

Definition 1.2 *Let* φ *be a nonnegative measurable function on* $\partial\Omega$.
(i) *Let* v *be a nonnegative measurable function in* Ω. *Then we say that* v *is a solution of* (1.11) *in* Ω *if*

$$v(x) = \int_{\partial\Omega} K(x,y)\varphi(y)\, d\sigma_y + \int_\Omega G(x,y) f(v(y))\, dy < \infty \tag{1.12}$$

for almost all $x \in \Omega$. *If* v *satisfies* (1.12) *with* $=$ *replaced by* $\geq$, *then we say that* v *is a supersolution of* (1.11).
(ii) *Let* v *be a solution of* (1.1) *in* Ω. *We say that* v *is a minimal solution of* (1.11) *in* Ω *if*

$$v(x) \leq w(x) \quad \text{for almost all } x \in \Omega,$$

for any solution w *of* (1.11) *in* Ω.
(iii) *Let* $v \in C^2(\Omega)$ *and* $v \geq 0$ *in* Ω. *We say that* v *is a classical supersolution of* (1.11) *if* v *satisfies*

$$\begin{cases} -\Delta v \geq f(v) & \text{in } \Omega, \\ \lim_{h \to +0} \min_{|x|=1} \{v(e^h x) - [S(h)\varphi_k](x)\} \geq 0 & \text{for any } k > 0, \end{cases}$$

where $\varphi_k := \min\{\varphi, k\}$.

Obviously, minimal solutions of (1.4) and (1.11) are uniquely determined, respectively.

Now we state the main results of this paper for problem (1.1). We first give a sufficient condition for the solution of (1.1) to exist globally in time.

Theorem 1.1 *Assume that*

$$p > p_* := \frac{N}{N-2}.$$

Then there exists $\delta > 0$ *such that, if* $\varphi \in L^\infty(\partial\Omega)$ *and* $\|\varphi\|_{L^\infty(\partial\Omega)} < \delta$, *then problem* (1.1) *possesses a global-in-time minimal solution* $U = (u, u^b)$ *satisfying*

$$\operatorname*{ess\,sup}_{t>0} \left[e^{(N-2)t} \|u^b(\cdot,t)\|_{L^\infty(\partial\Omega)} + e^{(N-2)t} \operatorname*{ess\,sup}_{x\in\Omega} |x|^{N-2} |u(x,t)| \right] < \infty. \tag{1.13}$$

The solution u_c given by (1.10) shows that the decay rates in (1.13) are optimal because then all integrals in (1.8) are nonnegative and the first one is bigger than or equal to u_c if $\varphi \geq c$.

In the second theorem we show that local solvability of problem (1.1) is equivalent to global solvability of problem (1.1). See also Theorem 4.1 and Corollary 4.1.

Theorem 1.2 *Assume that* $p > 1$. *Let* φ *be a nonnegative measurable function on* $\partial\Omega$. *Then the following are equivalent:*

- *Problem* (1.1) *possesses a local-in-time solution;*
- *Problem* (1.1) *possesses a global-in-time solution.*

Furthermore, if there exists a solution v *of the elliptic problem*

$$-\Delta v = v^p, \quad v \geq 0 \quad \text{in} \quad \Omega, \qquad v = \varphi \quad \text{on} \quad \partial\Omega, \tag{1.14}$$

then problem (1.1) *possesses a global-in-time solution* $U = (u, u^b)$ *such that*

$$\begin{aligned} u(x,t) &\leq v(e^t x) && \text{for almost all } x \in \Omega \text{ and } t \in (0,\infty), \\ u^b(x',t) &\leq v(e^t x') && \text{for almost all } x' \in \partial\Omega \text{ and } t \in (0,\infty). \end{aligned}$$

Next we state our results on the nonexistence of local-in-time solutions of (1.1).

Theorem 1.3 *Assume that* $1 < p \leq p_*$. *Let* φ *be a nonnegative measurable function on* $\partial\Omega$ *such that* $\varphi \not\equiv 0$ *in* Ω. *Then problem* (1.1) *possesses no local-in-time supersolutions.*

Theorem 1.4 *Assume that* $p > p_*$. *Let* ϕ *be a nonnegative measurable function on* $\partial\Omega$ *such that* $\phi \not\equiv 0$ *in* Ω. *Then there exists a constant* $\mu_* > 0$ *such that, if* $\mu \geq \mu_*$ *and* $\varphi = \mu\phi$ *on* $\partial\Omega$, *then problem* (1.1) *possesses no local-in-time supersolutions.*

As a corollary of our theorems, we have:

Corollary 1.1 *Assume that $p > 1$. Let φ be a nonnegative measurable function on $\partial\Omega$ such that $\varphi \not\equiv 0$ in Ω.*

(i) *If there exists a classical supersolution of* (1.14), *then problem* (1.14) *possesses a solution.*

(ii) *If $1 < p \le p_*$, then problem* (1.14) *possesses no supersolutions and no classical supersolutions.*

For similar results as in Corollary 1.1 (ii), see [2, 3], for example. In particular, for $N = 2$, there are no solutions of (1.14) for any φ and $p > 1$ (see [2]).

The rest of this paper is organized as follows. In Section 2 we give some estimates of integrals related to the integral kernels $\mathcal{K}$ and G. Furthermore, we show some lemmas on minimal solutions. In Section 3 we prove Theorem 1.1 by using the results in Section 2. In Section 4 we study the solvability of problem (1.11), and prove Theorem 1.2. In Section 5 we study the nonexistence of solutions of (1.1), and prove Theorem 1.3, Theorem 1.4 and Corollary 1.1.

2 Preliminaries

In this section we obtain some estimates of the integrals related to the kernels $\mathcal{K}$ and G. Furthermore, we prove some fundamental properties of minimal solutions. In what follows, for any $r \in [1, \infty]$, we write $|\cdot|_r = \|\cdot\|_{L^r(\partial\Omega)}$ and $\|\cdot\|_r = \|\cdot\|_{L^r(\Omega)}$ for simplicity.

2.1 Integral kernels $\mathcal{K}$ and G

By using some properties of the Poisson kernel P, we first obtain the following.

Lemma 2.1 *Let $N > 2$ and $\mathcal{K}$ be as in* (1.6). *Then*

$$\begin{aligned} &\int_{\partial\Omega} \mathcal{K}(x,y,t)\, d\sigma_y = (e^t|x|)^{-(N-2)}, \\ &\int_{\partial\Omega} \mathcal{K}(x,y,s)\, \mathcal{K}(y,z,t)\, d\sigma_y = \mathcal{K}(x,z,t+s), \end{aligned} \tag{2.1}$$

for $x \in \overline{\Omega}$, $z \in \partial\Omega$ and s, $t > 0$.

By Lemma 2.1 and the regularity theorems for elliptic equations we have:

Lemma 2.2 *Let $N > 2$. Let φ be a nonnegative measurable function on $\partial\Omega$ such that $\varphi \in L^\infty(\partial\Omega)$. Then*

$$\begin{aligned} &S(\cdot)\varphi \in C(\overline{\Omega} \times (0,\infty)) \cap C^\infty(\Omega \times [0,\infty)), \\ &-\Delta_x S(t)\varphi = 0 \quad in \quad \Omega \quad for\ any\ t \ge 0, \end{aligned} \tag{2.2}$$

$$S(t)\,[S(s)\varphi]^b = S(t+s)\varphi \quad for\ s,\ t \ge 0, \tag{2.3}$$

$$|[S(t)\varphi](x)| \le e^{-(N-2)t}|x|^{-(N-2)}|\varphi|_\infty \quad in \quad \overline{\Omega} \times [0,\infty). \tag{2.4}$$

Here $[S(t)\varphi]^b$ is the restriction of $S(t)\varphi$ to $\partial\Omega$. Furthermore, for any $\theta \in (0,1)$, there exists a constant C such that

$$\|S(t)\varphi\|_{C^{2,\theta}(\Omega)} \le Ct^{-2-\theta}|\varphi|_\infty, \quad t>0,$$
$$\|S(t)\varphi\|_{C^{1,\theta}(\overline{\Omega})} \le C\|\varphi\|_{C^{1,\theta}(\partial\Omega)}, \quad t \ge 0.$$

Next we define two integral operators,

$$W[\psi](x) := \int_\Omega K(y,x)\psi(y)\,dy, \qquad x \in \partial\Omega,$$
$$[(-\Delta_D)^{-1}\psi](x) := \int_\Omega G(x,y)\psi(y)dy, \qquad x \in \Omega,$$

where ψ is a nonnegative measurable function in Ω. Then we have the following lemma.

Lemma 2.3 *Let $N>2$. Let ψ be a nonnegative measurable function in Ω such that*

$$\psi(x) \le c_\psi |x|^{-N-\alpha}, \qquad x \in \Omega,$$

for some $c_\psi > 0$ and $\alpha > 0$. Then there exists a constant C_1 such that

$$|W[\psi]|_\infty \le C_1 c_\psi.$$

Furthermore, there exists a constant C_2 such that

$$[(-\Delta_D)^{-1}\psi](x) \le C_2 c_\psi |x|^{-(N-2)}, \qquad x \in \Omega.$$

2.2 Minimal solutions

In this section we assume that

(2.5) $\quad F = F(t,u)$ is continuous on $(0,\infty)\times[0,\infty)$ and increasing with respect to u,

and construct minimal solutions of (1.4). Let $u_1(x,t) \equiv 0$ in $\Omega \times (0,\infty)$ and $u_1^b(x,t) \equiv 0$ on $\partial\Omega \times (0,\infty)$. For $n = 1,2,\dots$, by induction we define

$$u_{n+1}(x,t) := [S(t)\varphi](x) + f_n(x,t) + w_n(x,t) \tag{2.6}$$

for almost all $x \in \overline{\Omega}$ and $t>0$ and

$$u_{n+1}^b(x',t) := [S(t)\varphi](x') + w_n(x',t) \tag{2.7}$$

for almost all $x' \in \partial\Omega$ and $t>0$, where

$$F_n(x,t) := F(t,u_n(x,t)), \qquad f_n(x,t) := \left[(-\Delta_D)^{-1}F_n(\cdot,t)\right](x),$$
$$W_n(x,t) := W[F_n(\cdot,t)](x), \qquad w_n(x,t) := \int_0^t [S(t-s)W_n(\cdot,s)](x)\,ds.$$

Since $\mathcal{K} = \mathcal{K}(x,y,t)$ and $G = G(x,y)$ are nonnegative, we can prove inductively that

$$\begin{aligned} 0 \le u_{n-1}(x,t) \le u_n(x,t) \quad & \text{for almost all } x \in \Omega \text{ and } t > 0, \\ 0 \le u^b_{n-1}(x',t) \le u^b_n(x',t) \quad & \text{for almost all } x' \in \partial\Omega \text{ and } t > 0, \end{aligned}$$

where $n = 2, 3, \dots$. Then we can define

$$\begin{aligned} u_*(x,t) &:= \lim_{n\to\infty} u_n(x,t) \in [0,\infty] \quad \text{for almost all } x \in \Omega \text{ and } t > 0, \\ u^b_*(x',t) &:= \lim_{n\to\infty} u^b_n(x',t) \in [0,\infty] \quad \text{for almost all } x' \in \partial\Omega \text{ and } t > 0. \end{aligned} \tag{2.8}$$

We first obtain the following.

Lemma 2.4 *Assume* (2.5). *If there exists a supersolution* $U = (u, u^b)$ *of* (1.4) *in* $\Omega\times(0,T)$ *for some* $T > 0$, *then* $U_* = (u_*, u^b_*)$ *is the minimal solution of* (1.4) *in* $\Omega \times (0,T)$.

Next we assume that

$$f \text{ is continuous and increasing on } [0,\infty), \tag{2.9}$$

and construct a minimal solution of (1.11). Let $v_1(x) \equiv 0$ in Ω. For $n = 1, 2, \dots,$ by induction we define

$$v_{n+1}(x,t) := [S(t)\varphi](x) + \int_\Omega G(x,y) f(v_n(y))\, dy$$

for almost all $x \in \Omega$. Then it follows that

$$0 \le v_{n-1}(x) \le v_n(x) \quad \text{for almost all } x \in \Omega,$$

where $n = 2, 3, \dots,$ and we can define

$$v_*(x) = \lim_{n\to\infty} v_n(x) \in [0,\infty] \quad \text{for almost all } x \in \Omega.$$

Similarly to Lemma 2.4, we have:

Lemma 2.5 *Assume* (2.9). *If there exists a supersolution* v *of* (1.11) *in* Ω, *then* v_* *is a minimal solution of* (1.1) *in* Ω.

3 Proof of Theorem 1.1

We prove Theorem 1.1. In this section we use the same notation as in Section 2.2.

Applying Lemmas 2.2 and 2.3 to approximate solutions (2.6), we have the following.

Lemma 3.1 *Assume the same conditions as in Theorem* 1.1. *Furthermore, assume that*

$$D_n := \operatorname*{ess\,sup}_{t>0}\left[e^{(N-2)t}|u_n^b(\cdot,t)|_\infty + e^{(N-2)t}\operatorname*{ess\,sup}_{x\in\Omega}|x|^{N-2}|u_n(x,t)|\right] < \infty$$

for some $n \in \{1,2,\dots\}$. *Then there exists a constant* C_*, *independent of* n, *such that*

$$D_{n+1} \le 2\{|\varphi|_\infty + C_*(\kappa D_n^p + \lambda D_n^q)\}.$$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let δ be a sufficiently small positive constant such that

$$\kappa(C_*+1)^p\delta^{p-1} + \lambda(C_*+1)^q\delta^{q-1} \le 1/2, \tag{3.1}$$

where C_* is the constant as in Lemma 3.1. Assume $|\varphi|_\infty \le \delta/2$. Since $u_2(\cdot,t) = S(t)\varphi$ and $u_2^b(\cdot,t) = [S(t)\varphi]^b$, by (2.4) we see that

$$\operatorname*{ess\,sup}_{t>0}\left[e^{(N-2)t}|u_2^b(\cdot,t)|_\infty + e^{(N-2)t}\operatorname*{ess\,sup}_{x\in\Omega}|x|^{N-2}|u_2(x,t)|\right] \le 2|\varphi|_\infty \le \delta. \tag{3.2}$$

Taking a sufficiently small δ if necessary, by Lemma 3.1, (3.1) and (3.2) we have

$$\begin{aligned}&\operatorname*{ess\,sup}_{t>0}\left[e^{(N-2)t}|u_3^b(\cdot,t)|_\infty + e^{(N-2)t}\operatorname*{ess\,sup}_{x\in\Omega}|x|^{N-2}|u_3(x,t)|\right]\\ &\quad\le 2\{|\varphi|_\infty + C_*(\kappa\delta^p + \lambda\delta^q)\} \le \delta + C_*\delta.\end{aligned}$$

Applying Lemma 3.1 again, by (3.1) and (3.2) we obtain

$$\begin{aligned}&\operatorname*{ess\,sup}_{t>0}\left[e^{(N-2)t}|u_4^b(\cdot,t)|_\infty + e^{(N-2)t}\operatorname*{ess\,sup}_{x\in\Omega}|x|^{N-2}|u_4(x,t)|\right]\\ &\quad\le 2\{|\varphi|_\infty + C_*[\kappa((C_*+1)\delta)^p + \lambda((C_*+1)\delta)^q]\} \le \delta + C_*\delta.\end{aligned}$$

Repeating this argument, we deduce that

$$\operatorname*{ess\,sup}_{t>0}\left[e^{(N-2)t}|u_n^b(\cdot,t)|_\infty + e^{(N-2)t}\operatorname*{ess\,sup}_{x\in\Omega}|x|^{N-2}|u_n(x,t)|\right] \le \delta + C_*\delta$$

for all $n = 2,3,\dots$. This together with (2.8) implies that

$$\operatorname*{ess\,sup}_{t>0}\left[e^{(N-2)t}|u_*^b(\cdot,t)|_\infty + e^{(N-2)t}\operatorname*{ess\,sup}_{x\in\Omega}|x|^{N-2}|u_*(x,t)|\right] \le \delta + C_*\delta.$$

Then, by (2.7) we see that $U_* = (u_*, u_*^b)$ is a solution of (1.1) in $\Omega\times(0,\infty)$. Furthermore, we deduce from Lemma 2.4 that $U_* = (u_*, u_*^b)$ is a minimal solution of (1.1). Thus Theorem 1.1 follows. □

4 Nonlinear elliptic equations

In this section we consider problem (1.11) and prove the following theorem.

Theorem 4.1 *Let φ be a nonnegative measurable function on $\partial\Omega$. Assume that*

$$f \text{ is Hölder continuous and increasing on } [0,\infty). \tag{4.1}$$

Then the following statements are equivalent:

(a) *Problem (1.11) possesses a solution;*

(b) *Problem*

$$\begin{cases} -\Delta u = e^{2t} f(u), \quad u \geq 0 & \text{in} \quad \Omega \times (0,\infty), \\ \partial_t u + \partial_\nu u = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = \varphi(x) & \text{on} \quad \partial\Omega, \end{cases} \tag{4.2}$$

possesses a local-in-time solution;

(c) *Problem (4.2) possesses a global-in-time solution.*

Furthermore, if $v = v(x)$ and $U = (u, u^b)$ are minimal solutions of (1.11) and (4.2), respectively, then

$$v(e^t x) = u(x,t) \quad \text{for almost all } x \in \Omega \text{ and } t > 0,$$
$$v(e^t x) = u^b(x,t) \quad \text{for almost all } x \in \partial\Omega \text{ and } t > 0.$$

We prepare the following Phragmén-Lindelöf theorem for the Laplace equation in Ω. The proof is a modification of the proof of [7, Theorem 3.1].

Lemma 4.1 *Let $\sigma > 0$ and let $u = u(x,t)$ satisfy*

$$u(\cdot,t) \in C^2(\Omega) \cap C^1(\overline{\Omega}) \quad \text{for any} \quad t \in (0,\sigma],$$
$$u \in C(\overline{\Omega} \times (0,\sigma]), \quad \partial_t u \in C(\partial\Omega \times (0,\sigma]),$$

and

$$-\Delta u \geq 0 \quad \text{in} \quad \Omega \times (0,\sigma], \qquad \partial_t u + \partial_\nu u \geq 0 \quad \text{on} \quad \partial\Omega \times (0,\sigma].$$

Assume that

$$\liminf_{t\to+0} \inf_{x\in\partial\Omega} u(x,t) \geq 0,$$
$$\limsup_{R\to\infty} \inf_{|x|=R, t\in(0,\sigma]} u(x,t) \geq 0.$$

Then $u \geq 0$ in $\Omega \times (0,\sigma]$.

By Lemma 4.1 and the regularity theorems for elliptic equations we have:

Lemma 4.2 *Let ϑ be a nonnegative continuous function on $\partial\Omega\times(0,\infty)$ such that $\vartheta(\cdot,s)\in C^{1,\theta}(\partial\Omega)$ for all $s>0$ with $0<\theta<1$ and*

$$\sup_{s\in(0,T)}|\vartheta(\cdot,s)|_\infty<\infty,\qquad \sup_{s\in[\tau,T]}\|\vartheta(\cdot,s)\|_{C^{1,\theta}(\partial\Omega)}<\infty,$$

for any $0<\tau<T<\infty$. Then

$$w(x,t):=\int_0^t\int_{\partial\Omega}\mathcal{K}(x,y,t-s)\vartheta(y,s)\,d\sigma_y\equiv\int_0^t\int_{\partial\Omega}K(e^{t-s}x,y)\vartheta(y,s)\,d\sigma_y$$

is the unique function on $\overline{\Omega}\times(0,\infty)$ with the following properties:

(a) *$w\in C^2(\Omega\times(0,\infty))\cap C^1(\overline{\Omega}\times(0,\infty))$;*

(b) *w satisfies*

$$-\Delta w=0\quad in\quad \Omega\times(0,\infty),\qquad \partial_t w+\partial_\nu w=\vartheta\quad on\quad \partial\Omega\times(0,\infty);$$

(c) *$\lim_{t\to+0}\sup_{x\in\overline{\Omega}}|w(x,t)|=0$;*

(d) *$\lim_{R\to\infty}\sup_{0<t\le\sigma}\|w(\cdot,t)\|_{L^\infty(\partial B(0,R))}=0$ for any $\sigma>0$.*

We construct approximate solutions of (1.11) and (4.2). Let φ be a nonnegative measurable function on $\partial\Omega$. Let ζ be a smooth function in $\mathbb{R}^N$ such that

$$0\le\zeta\le1\quad\text{in}\quad\mathbb{R}^N,\quad \zeta=1\quad\text{in}\quad B(0,1),\quad \zeta=0\quad\text{outside}\quad B(0,2).$$

For any $k=1,2,\dots$, we set

$$\varphi_k(x):=\min\{\varphi(x),k\}\quad\text{on}\quad\partial\Omega,\qquad \zeta_k(x):=\zeta(k^{-1}x)\quad\text{on}\quad\overline{\Omega}.$$

Define a sequence $\{v_{k,n}\}$ inductively by

$$\begin{aligned}v_{k,1}(x)&:=\int_{\partial\Omega}K(x,y)\varphi_k(y)\,d\sigma_y,\\ v_{k,n+1}(x)&:=\int_{\partial\Omega}K(x,y)\varphi_k(y)\,d\sigma_y+\int_\Omega G(x,y)f(v_{k,n}(y))\zeta_k(y)\,dy,\end{aligned}\tag{4.3}$$

where $n=1,2,\dots$. By (2.2) and (2.4) we see that

$$v_{k,1}\in C^2(\Omega)\cap L^\infty(\Omega).$$

This together with (1.7) and (4.1) implies that

$$v_{k,2} \in C^2(\Omega) \cap L^\infty(\Omega), \qquad \lim_{R\to\infty} \sup_{|x|=R} |v_{k,2}(x)| = 0,$$
$$\lim_{R\to+1} \sup_{|x|=R} \left| v_{k,2}(x) - \int_{\partial\Omega} K(x,y)\varphi_k(y)\, d\sigma_y \right| = 0.$$

Repeating this argument, we have

$$\begin{aligned} &v_{k,n} \in C^2(\Omega) \cap L^\infty(\Omega), \qquad \lim_{R\to\infty} \sup_{|x|=R} |v_{k,n}(x)| = 0,\\ &\lim_{R\to+1} \sup_{|x|=R} \left| v_{k,n}(x) - \int_{\partial\Omega} K(x,y)\varphi_k(y)\, d\sigma_y \right| = 0, \end{aligned} \tag{4.4}$$

for $n = 1, 2, \ldots$. Furthermore, it follows that

$$-\Delta v_{k,n} = f(v_{k,n-1})\zeta_k \quad \text{in} \quad \Omega. \tag{4.5}$$

In addition, by the definition of $v_{k,n}$ and the monotonicity of f we see that

$$v_{k,n}(x) \le v_{k,n+1}(x), \qquad v_{k,n}(x) \le v_{k+1,n}(x), \tag{4.6}$$

for all $x \in \Omega$ and k, $n = 1, 2, \ldots$.

Set

$$u_{k,n}(x,t) := v_{k,n}(e^t x), \qquad x \in \overline{\Omega}, \quad t > 0. \tag{4.7}$$

Then we deduce from (4.4) that

$$\begin{aligned} &u_{k,n} \in C^2(\overline{\Omega} \times (0,\infty)) \cap L^\infty(\Omega \times (0,\infty)),\\ &\lim_{R\to\infty} \sup_{|x|=R, t\in(0,\sigma]} |v_{k,n}(x)| = 0 \quad \text{for any } \sigma > 0,\\ &\lim_{t\to+0} \sup_{x\in\partial\Omega} \left| u_{k,n}(x,t) - \int_{\partial\Omega} K(e^t x, y)\varphi_k(y)\, d\sigma_y \right| = 0. \end{aligned} \tag{4.8}$$

Furthermore, by (4.5) and (4.6) we see that

$$\begin{cases} -\Delta u_{k,n} = e^{2t} f(u_{k,n-1})\zeta_k & \text{in} \quad \Omega \times (0,\infty),\\ \partial_t u_{k,n} + \partial_\nu u_{k,n} = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \end{cases} \tag{4.9}$$

and

$$u_{k,n}(x,t) \le u_{k,n+1}(x,t), \qquad u_{k,n}(x,t) \le u_{k+1,n}(x,t), \tag{4.10}$$

for all $(x,t) \in \overline{\Omega} \times (0,\infty)$ and k, $n = 1, 2, \ldots$.

On the other hand, we set

$$
(4.11)\qquad \begin{aligned} g_{k,n}(x,t) &:= e^{2t}\int_\Omega G(x,y)f(u_{k,n-1}(y,t))\zeta_k(e^t y)\,dy,\\ h_{k,n}(x,t) &:= u_{k,n}(x,t) - \int_{\partial\Omega} K(e^t x,y)\varphi(y)\,d\sigma_y - g_{k,n}(x,t). \end{aligned}
$$

By (2.2), (4.8) and (4.9) we see that

$$
g_{k,n},\ h_{k,n} \in C^{2,\theta}(\overline{\Omega}\times(0,\infty))
$$

for some $\theta\in(0,1)$ and

$$
\begin{cases} -\Delta h_{k,n} = 0 & \text{in}\quad \Omega\times(0,\infty),\\ \partial_t h_{k,n} + \partial_\nu h_{k,n} = -\partial_\nu g_{k,n} & \text{on}\quad \partial\Omega\times(0,\infty),\\ \lim\limits_{t\to+0}\sup\limits_{x\in\partial\Omega}|h_{k,n}(x)| = 0. \end{cases}
$$

Therefore, it follows from Lemma 4.2 that

$$
\begin{aligned} h_{k,n}(x,t) &= -\int_0^t\int_{\partial\Omega} K(e^{t-s}x,y)\partial_\nu g_{k,n}\,d\sigma_y\,ds\\ &= -\int_0^t e^{2s}\int_{\partial\Omega} K(e^{t-s}x,y)\left(\int_\Omega(\partial_\nu G)(y,z)f(u_{k,n-1}(z,s))\zeta_k(e^s z)\,dz\right)d\sigma_y\,ds\\ &= \int_0^t e^{2s}\int_{\partial\Omega} K(e^{t-s}x,y)\left(\int_\Omega K(z,y)f(u_{k,n-1}(z,s))\zeta_k(e^s z)\,dz\right)d\sigma_y\,ds. \end{aligned}
$$

This together with (4.11) implies that

$$
(4.12)\qquad \begin{aligned} u_{k,n}(x,t) &= \int_{\partial\Omega} K(e^t x,y)\varphi_k(y)\,d\sigma_y + e^{2t}\int_\Omega G(x,y)f(u_{k,n-1}(y,t))\zeta_k(e^t y)\,dy\\ &\quad + \int_0^t e^{2s}\int_{\partial\Omega} K(e^{t-s}x,y)\left(\int_\Omega K(z,y)f(u_{k,n-1}(z,s))\zeta_k s(e^s z)\,dz\right)d\sigma_y\,ds \end{aligned}
$$

for all $x\in\overline{\Omega}$ and $t>0$.

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Assume that problem (1.11) has a solution v. Since G is positive in $\Omega\times\Omega$, by (1.12) we see that $v\in L^1_{\rm loc}(\Omega)$. Then it follows from the Fubini theorem that

$$
(4.13)\qquad v(x)<\infty\quad\text{for almost all } x\in\partial B(0,R) \text{ and } R>1.
$$

On the other hand, similarly to the proof of Lemma 2.5, it follows that

$$
(4.14)\qquad v_{k,n}(x)\le v(x)<\infty\quad\text{for almost all } x\in\Omega,
$$

where k, $n = 1, 2, \ldots$. Furthermore, similarly to (4.13), we have

$$v_{k,n}(x) \le v(x) < \infty \quad \text{for almost all } x \in \partial B(0,R) \text{ and } R > 1. \tag{4.15}$$

By (4.7), (4.14) and (4.15) we see that

$$u_{k,n}(x,t) = v_{k,n}(e^t x) \le v(e^t x) < \infty \text{ for almost all } x \in \Omega \text{ and } t > 0,$$
$$u_{k,n}(x',t) = v_{k,n}(e^t x') \le v(e^t x') < \infty \text{ for almost all } x' \in \partial\Omega \text{ and } t > 0,$$

where k, $n = 1, 2, \ldots$. Then, by (4.10) we see that

$$u_*(x,t) := \lim_{k\to\infty} \lim_{n\to\infty} u_{k,n}(x,t) \le v(e^t x) < \infty \text{ for almost all } x \in \Omega \text{ and } t > 0,$$
$$u_*^b(x',t) := \lim_{k\to\infty} \lim_{n\to\infty} u_{k,n}(x',t) \le v(e^t x') < \infty \text{ for almost all } x' \in \partial\Omega \text{ and } t > 0.$$

Furthermore, we deduce from (4.12) that $U_* := (u_*, u_*^b)$ is a solution of (4.2) in $\Omega \times (0,\infty)$.

Next we assume that problem (4.2) has a solution $U = (u, u^b)$ in $\Omega \times (0,T)$ for some $T > 0$. Similarly to the proof of Lemma 2.4, by (4.12) we see that

$$u_{k,n}(x,t) \le u(x,t)$$

for almost all $x \in \Omega$ and $t \in (0,T)$, where k, $n = 1, 2, \ldots$. Then, for almost all $t \in (0,T)$, by (4.7) we obtain

$$0 \le v_{k,n}(x) = u_{k,n}(e^{-t}x, t) \le u(e^{-t}x, t) < \infty$$

for almost all $x \in \Omega$ with $e^{-t}x > 1$. Therefore, for almost all $t \in (0,T)$, by (4.6) we see that

$$v_*(x) := \lim_{k\to\infty} \lim_{n\to\infty} v_{k,n}(x) \le u(e^{-t}x, t) < \infty \tag{4.16}$$

for almost all $x \in \Omega$ with $e^{-t}x > 1$. Then we deduce from (4.3) that v_* is a solution of (1.11). Thus Theorem 4.1 follows. □

We prove Theorem 1.2 by using Theorem 4.1

Proof of Theorem 1.2. Assume that problem (1.1) has a solution $U = (u, u^b)$ in $\Omega \times (0,T)$ for some $T > 0$. Then we can find $T_* \in (0,T)$ such that (1.8) and (1.9) with $F = u^p$ hold for almost all $x \in \Omega$ and $x' \in \partial\Omega$ at $t = T_*$, respectively. It follows that

$$\begin{aligned} u^b(x',T_*) &= \int_{\partial\Omega} \mathcal{K}(x',y,T_*)\varphi(y)\,d\sigma_y \\ &\quad + \int_0^{T_*} \int_{\partial\Omega} \mathcal{K}(x',y,T_*-s)\left\{ \int_\Omega K(z,y)u(z,s)^p\,dz \right\} d\sigma_y\,ds \\ &= [S(T_*)\varphi](x') + \int_0^{T_*} \left[S(T_*-s)\left\{ \int_\Omega K(z,\cdot)u(z,s)^p\,dz \right\} \right](x')\,ds < \infty \end{aligned} \tag{4.17}$$

for almost all $x' \in \partial\Omega$. This together with (1.8) and (2.3) implies that

$$\begin{aligned}
u(x,T_*) &= \int_{\partial\Omega} \mathcal{K}(x,y,T_*)\varphi(y)\,d\sigma_y + \int_\Omega G(x,y)u(y,T_*)^p\,dy \\
&\quad + \int_0^{T_*}\int_{\partial\Omega} \mathcal{K}(x,y,T_*-s)\biggl\{\int_\Omega K(z,y)u(z,s)^p\,dz\biggr\}\,d\sigma_y\,ds \\
&= [S(T_*)\varphi](x) + \int_\Omega G(x,y)u(y,T_*)^p\,dy \\
&\quad + \int_0^{T_*}\biggl[S(T_*-s)\biggl\{\int_\Omega K(z,\cdot)u(z,s)^p\,dz\biggr\}\biggr](x)\,ds \\
&= \int_{\partial\Omega} K(x,y)u^b(y',T_*)\,d\sigma_y + \int_\Omega G(x,y)u(y,T_*)^p\,dy < \infty
\end{aligned}$$

for almost all $x \in \Omega$. This means that $u(\cdot,T_*)$ is a solution of (1.14) with $\varphi = u^b(T_*)$. Then, by Lemma 2.4 and Theorem 4.1 we see that problem (1.1) possesses a global-in-time solution $\tilde{U} = (\tilde{u},\tilde{u}^b)$ with $\varphi = u^b(T_*)$. Set

$$w(x,t) = \begin{cases} u(x,t) & \text{for almost all } x \in \Omega \text{ and } t \in (0,T_*), \\ \tilde{u}(x,t-T_*) & \text{for almost all } x \in \Omega \text{ and } t \in (T_*,\infty). \end{cases}$$

Similarly, we set

$$w^b(x',t) = \begin{cases} u^b(x',t) & \text{for almost all } x' \in \partial\Omega \text{ and } t \in (0,T_*), \\ \tilde{u}^b(x',t-T_*) & \text{for almost all } x' \in \partial\Omega \text{ and } t \in (T_*,\infty). \end{cases}$$

Then we see that $W = (w,w^b)$ is a solution of (1.1) with $\lambda = 0$ in $\Omega\times(0,T_*)$. Furthermore, by (1.9) we have

$$\begin{aligned}
w(x,t) &= \tilde{u}(x,t-T_*) \\
&= \int_{\partial\Omega} \mathcal{K}(x,y,t-T_*)u^b(y,T_*)\,d\sigma_y + \int_\Omega G(x,y)\tilde{u}(y,t-T_*)^p\,dy \\
&\quad + \int_0^{t-T_*}\int_{\partial\Omega} \mathcal{K}(x,y,t-T_*-s)\biggl\{\int_\Omega K(z,y)\tilde{u}(z,s)^p\,dz\biggr\}\,ds\,d\sigma_y \\
&= [S(t-T_*)u^b(T_*)](x) + \int_\Omega G(x,y)w(y,t)^p\,dy \\
&\quad + \int_{T_*}^{t}\biggl[S(t-s)\biggl\{\int_\Omega K(z,\cdot)w(z,s)^p\,dz\biggr\}\biggr](x)\,ds
\end{aligned}$$

for almost all $x \in \Omega$ and $t \in (T_*, \infty)$. This together with (2.3) and (4.17) implies that

$$
\begin{aligned}
w(x,t) &= \tilde{u}(x, t-T_*) \\
&= [S(t)\varphi](x) + \int_0^{T_*} \left[S(t-s) \left\{ \int_\Omega K(z,\cdot) u(z,T_*)^p \, dz \right\} \right](x) \, ds \\
&\qquad + \int_\Omega G(x,y) w(y,t)^p \, dy + \int_{T_*}^t \left[S(t-s) \left\{ \int_\Omega K(z,\cdot) w(z,s)^p \, dz \right\} \right](x) \, ds \\
&= \int_{\partial\Omega} \mathcal{K}(x,y,t)\varphi(y) \, d\sigma_y + \int_\Omega G(x,y) w(y,t)^p \, dy \\
&\qquad + \int_0^t \int_{\partial\Omega} \mathcal{K}(x,y,t-s) \left\{ \int_\Omega K(z,y) w(z,s)^p \, dz \right\} ds \, d\sigma_y
\end{aligned}
$$

for almost all $x \in \Omega$ and $t \in (T_*, \infty)$. Similarly, we have

$$
\begin{aligned}
w^b(x',t) = & \int_{\partial\Omega} \mathcal{K}(x',y,t)\varphi(y) \, d\sigma_y \\
& + \kappa \int_0^t \int_{\partial\Omega} \mathcal{K}(x',y,t-s) \left\{ \int_\Omega K(z,y) w(z,s)^p \, dz \right\} ds \, d\sigma_y
\end{aligned}
$$

for almost all $x' \in \partial\Omega$ and $t \in (T_*, \infty)$. Therefore, we see that $W = (w, w^b)$ is a solution of (1.1) with $\lambda = 0$ in $\Omega \times (0, \infty)$. Thus problem (1.1) possesses a global-in-time solution, and Theorem 1.2 follows. □

Furthermore, as a corollary of Theorem 4.1, we have the following result.

Corollary 4.1 *Assume that $p > 1$. Let ϕ be a nonnegative measurable function on $\partial\Omega$. Assume that problem* (1.1) *possesses a local-in-time solution u with the initial data ϕ. Then, for any $\mu \in (0,1)$, problem* (1.14) *possesses a solution with $\varphi = \mu\phi$.*

Proof. Let u be a solution of (1.1) with the initial data ϕ in $\Omega \times (0,T)$ for some $T > 0$. Then $\tilde{u}(x,t) := \mu u(x,t)$ satsfies

$$
\begin{cases}
-\Delta \tilde{u} = \mu^{-(p-1)} \tilde{u}^p, \quad \tilde{u} \ge 0, & x \in \Omega, \ t \in (0,T), \\
\partial_t \tilde{u} + \partial_\nu \tilde{u} = 0, & x \in \partial\Omega, \ t \in (0,T), \\
\tilde{u}(x,0) = \mu\phi(x) \ge 0, & x \in \partial\Omega.
\end{cases}
$$

Since $\mu^{-(p-1)} > 1$, we see that

$$
-\Delta \tilde{u} = \mu^{-(p-1)} \tilde{u}^p \ge e^{2t} \tilde{u}^p, \qquad x \in \Omega, \ t > 0,
$$

provided that $\mu^{-(p-1)} \ge e^{2t}$ and $0 < t < T$. Therefore, the problem

$$
\begin{cases}
-\Delta w = e^{2t} w^p, \quad w \ge 0, & x \in \Omega, \ t > 0, \\
\partial_t w + \partial_\nu w = 0, & x \in \partial\Omega, \ t > 0, \\
w(x,0) = \mu\phi(x) \ge 0, & x \in \partial\Omega,
\end{cases}
$$

possesses a local-in-time supersolution. Then, by Lemma 2.4 and Theorem 4.1 we deduce that problem (1.14) possesses a solution with $\varphi = \mu\phi$. Thus Corollary 4.1 follows. □

On the other hand, by the definition of $v_{k,n}$, we obtain the following.

Theorem 4.2 *Let φ be a nonnegative measurable function on $\partial\Omega$ such that $\varphi \not\equiv 0$ in Ω. Assume* (4.1) *and that there exists a classical supersolution v of* (4.2). *Then problem* (4.2) *possesses a solution.*

Proof. Let v be a classical supersolution of (4.2). Let $k,\ n = 1, 2, \dots$ and let $v_{k,n}$ be as in (4.3). By Definition 1.2 (iii), Lemma 2.2 and (4.3), we apply Lemma 4.1 to v and $v_{k,1}$ and we see that $v(x) \ge v_{k,1}(x)$ in Ω. Then, by (4.4) and (4.5) we apply Lemma 4.1 to v and $v_{k,2}$, and obtain $v(x) \ge v_{k,2}(x)$ in Ω. Repeating this argument, for any $k,\ n = 1, 2, \dots,$ we deduce that $v(x) \ge v_{k,n}(x)$ for all $x \in \Omega$. Similarly to (4.16), by (4.6) we have

$$v_*(x) := \lim_{k\to\infty} \lim_{n\to\infty} v_{k,n}(x,t) \le v(x)$$

for all $x \in \Omega$. Furthermore, we see that v_* is a solution of (4.2). Thus Theorem 4.2 follows. □

5 Proof of Theorems 1.3 and 1.4

For the proof of Theorems 1.3 and 1.4, applying the estimate

$$u(x,t) \ge [(-\Delta_D)^{-1} u(t)^p](x)$$

for almost all $x \in \Omega$ and $t \in (0,T)$, we prepare the following lemma.

Lemma 5.1 *Assume that $p > 1$. Let u be a solution of* (1.1) *in $\Omega \times (0,T)$ for some $T > 0$. Let $R \ge 5$ and $A > 0$. Assume that u satisfies* (1.8) *at $t = T_* \in (0,T)$ and*

$$u(x,T_*) \ge A|x|^{-(N-2)}$$

for almost all $x \in \Omega \setminus B(0,R)$. Then there exists a positive constant K, independent of R, such that, if $A \ge KR^{(\gamma-N)/(p-1)}$, then

$$u(x,T_*) \ge e^{p^{n-1}} |x|^{-(N-2)}$$

for almost all $x \in \Omega \setminus B(0,R_n)$ and all $n = 1, 2, \dots$. Here $\gamma := \max\{p(N-2), N\}$ and $R_n := 3^{n-1}R$.

Now we are ready to prove Theorem 1.3.

Proof Theorem 1.3. Let $1 < p \le p_*$. Then $\gamma := \max\{p(N-2), N\} = N$. Let φ be a nonnegative measurable function on $\partial\Omega$ such that $\varphi \not\equiv 0$ in Ω. Assume that there exists a nonnegative solution u of (1.1) in $\Omega\times(0,T)$ for some $T>0$. Then we can find $T_*\in(0,T)$ that u satisfies (1.8) at $t=T_*$.

On the other hand, since $\varphi\not\equiv 0$ on $\partial\Omega$, we see that $S(T_*/2)\varphi$ is positive on $\overline{\Omega}$. Then, by (2.2) we can find a positive constant m such that

$$S(T_*/2)\varphi\ge m \quad \text{on} \quad \partial\Omega. \tag{5.1}$$

This together with (1.8), (2.1) and (2.3) implies that

$$\begin{aligned} u(x,T_*) &\ge [S(T_*)\varphi](x)=\left[S(T_*/2)[S(T_*/2)\varphi]^b\right](x)\\ &\ge [S(T_*/2)m](x)\ge m[e^{T_*/2}|x|]^{-(N-2)}\ge C_*|x|^{-(N-2)} \end{aligned} \tag{5.2}$$

for almost all $x\in\Omega$, where C_* is a positive constant.

Let $R\ge 5$. Since $1<p\le p_*$, it follows from (1.8) and (5.2) that

$$\begin{aligned} u(x,T_*) &\ge \int_\Omega G(x,y)u(y,T_*)^p\,dy \ge \kappa C_*^p\int_{B(0,R/2)\setminus B(0,1)}|y|^{-\gamma}G(x,y)\,dy\\ &= \frac{\kappa c_N C_*^p}{N-2}\int_{B(0,R/2)\setminus B(0,1)}\frac{|y|^{-\gamma}}{|x-y|^{N-2}}\,dy\\ &\quad -\frac{\kappa c_N C_*^p}{N-2}|x|^{-(N-2)}\int_{B(0,R/2)\setminus B(0,1)}\frac{|y|^{-\gamma}}{|y-x_*|^{N-2}}\,dy \end{aligned} \tag{5.3}$$

for almost all $x\in\Omega$. Furthermore, since $N>2$, there exist constants C_1 and C_2, independent of R, such that

$$\begin{aligned} &\int_{B(0,R/2)\setminus B(0,1)}\frac{|y|^{-\gamma}}{|x-y|^{N-2}}\,dy\ge C_1^{-1}|x|^{-(N-2)}\int_{B(0,R/2)\setminus B(0,1)}|y|^{-\gamma}\,dy,\\ &\int_{B(0,R/2)\setminus B(0,1)}\frac{|y|^{-\gamma}}{|y-x_*|^{N-2}}\,dy\le C_1\int_{B(0,R/2)\setminus B(0,1)}|y|^{-(N-2)-\gamma}\,dy\le C_2, \end{aligned} \tag{5.4}$$

for $x\in\Omega\setminus B(0,R)$. By (5.3) and (5.4) we see that

$$u(x,T_*)\ge\frac{\kappa c_N C_*^p}{N-2}|x|^{-(N-2)}\left[C_1^{-1}\int_{B(0,R/2)\setminus B(0,1)}|y|^{-\gamma}\,dy-C_2\right] \tag{5.5}$$

for almost all $x\in\Omega\setminus B(0,R)$. Let K be the constant given in Lemma 5.1. Since $\gamma=N$, taking a sufficiently large R, by (5.5) we obtain

$$u(x,T_*)\ge K|x|^{-(N-2)}$$

for all $x\in\Omega\setminus B(0,R)$. This together with Lemma 5.1 implies that

$$u(x,T_*)\ge e^{p^{n-1}}|x|^{-(N-2)} \tag{5.6}$$

for almost all $x \in \Omega \setminus B(0, R_n)$ and all $n = 1, 2, \dots$, where $R_n = 3^{n-1}R$.

By (1.7), (1.8) and (5.6), we obtain

$$
\begin{aligned}
u(x, T_*) &\geq \int_\Omega G(x,y)u(y,T_*)^p\,dy \geq e^{p^n} \int_{B(0,2R_n)\setminus B(0,R_n)} G(x,y)|y|^{-N}\,dy \\
&= \frac{c_N e^{p^n}}{N-2} \int_{B(0,2R_n)\setminus B(0,R_n)} \frac{|y|^{-N}}{|x-y|^{N-2}}\,dy \\
&\quad - \frac{c_N e^{p^n}}{N-2}|x|^{-(N-2)} \int_{B(0,2R_n)\setminus B(0,R_n)} \frac{|y|^{-N}}{|y-x_*|^{N-2}}\,dy
\end{aligned}
\tag{5.7}
$$

for almost all $x \in \Omega$ and all $n = 1, 2, \dots$. Let L be a sufficiently large positive constant. Since $R_n \to \infty$ as $n \to \infty$, we can find positive constants C_3, C_4, C_5 and C_6, independent of L and n, such that

$$
\begin{aligned}
\int_{B(0,2R_n)\setminus B(0,R_n)} \frac{|y|^{-N}}{|x-y|^{N-2}}\,dy &\geq C_3 \int_{B(0,2R_n)\setminus B(0,R_n)} |y|^{-(N-2)-N}\,dy \\
&= C_3 R_n^{-(N-2)} \int_{B(0,2)\setminus B(0,1)} |y|^{-(N-2)-N}\,dy \geq C_4 R_n^{-(N-2)}
\end{aligned}
\tag{5.8}
$$

and

$$
\begin{aligned}
\int_{B(0,2R_n)\setminus B(0,R_n)} \frac{|y|^{-N}}{|y-x_*|^{N-2}}\,dy &\leq C_5 \int_{B(0,2R_n)\setminus B(0,R_n)} |y|^{-(N-2)-N}\,dy \\
&\leq C_5 R_n^{-(N-2)} \int_{B(0,2)\setminus B(0,1)} |y|^{-(N-2)-N}\,dy \leq C_6 R_n^{-(N-2)}
\end{aligned}
\tag{5.9}
$$

for all $L \leq |x| \leq R_n$ and sufficiently large n. By (5.7), (5.8) and (5.9) we obtain

$$
u(x, T_*) \geq \frac{c_N e^{p^n}}{N-2} R_n^{-(N-2)} \left[C_4 - C_6|x|^{-(N-2)}\right] \tag{5.10}
$$

for almost all $x \in \Omega$ with $L \leq |x| \leq R_n$ and all $n = 1, 2, \dots$. Taking a sufficiently large L if necessary, we see that $2C_4 \geq C_6 L^{-(N-2)}$. Then, by (5.10) we have

$$
u(x, T_*) \geq \frac{c_N e^{p^n}}{N-2} \frac{C_4}{2} R_n^{-(N-2)}
$$

for almost all $x \in \Omega$ with $L \leq |x| \leq R_n$ and all sufficiently large n. This implies that $u(x, T_*) = \infty$ for almost all $x \in \Omega$ with $|x| \geq L$. This is a contradiction. Therefore we see that problem (1.1) possesses no local-in-time solutions, and Theorem 1.3 follows. □

Proof of Theorem 1.4. Let $p > p_*$. Let ϕ be a nonnegative measurable function on $\partial\Omega$ such that $\varphi \not\equiv 0$ in Ω. Similarly to (5.1), we can find a positive constant m such that

$$
[S(t_*)\phi](x) \geq m \quad \text{on} \quad \partial\Omega \tag{5.11}
$$

for all $1/2 \le t_* \le 1$.

Let μ be a sufficiently large constant. Assume that problem (1.1) possesses a local-in-time solution with $\varphi = \mu\phi$. Then, by Lemma 2.4 and Theorem 1.2 we can find a global-in-time solution u of (1.1) with $\lambda = 0$. Furthermore, by (1.8) and (5.11) we see that,

$$u(x,t) \ge \mu[S(t)\phi](x) = \mu\left[S(t-t_*)[S(t_*)\phi]^b\right](x) \ge m\mu|x|^{-(N-2)}$$

for almost all $x \in \Omega$ and $t \in (1,2)$. Let $R \ge 5$. Then, taking a sufficiently large μ if necessary, by Lemma 5.1 we obtain

$$u(x,t) \ge e^{p^{n-1}}|x|^{-(N-2)} \tag{5.12}$$

for almost all $x \in \Omega \setminus B(0,R_n)$, $t \in (1,2)$ and all $n = 1,2,\dots$, where $R_n = 3^{n-1}R$. Similarly to (5.7), by (1.7), (1.8) and (5.12) we obtain

$$\begin{aligned} u(x,t) \ge\ & \frac{c_N e^{p^n}}{N-2}\int_{B(0,2R_n)\setminus B(0,R_n)} \frac{|y|^{-p(N-2)}}{|x-y|^{N-2}}\,dy \\ & - \frac{c_N e^{p^n}}{N-2}|x|^{-(N-2)}\int_{B(0,2R_n)\setminus B(0,R_n)} \frac{|y|^{-p(N-2)}}{|y-x_*|^{N-2}}\,dy \end{aligned} \tag{5.13}$$

for almost all $x \in \Omega$, $t \in (1,2)$ and all $n = 1,2,\dots$. Let L be a sufficiently large constant. Since $R_n \to \infty$ as $n \to \infty$, similarly to (5.8) and (5.9), we see that

$$\begin{aligned} \int_{B(0,2R_n)\setminus B(0,R_n)} \frac{|y|^{-p(N-2)}}{|x-y|^{N-2}}\,dy &\ge C_1\int_{B(0,2R_n)\setminus B(0,R_n)} |y|^{-(N-2)-p(N-2)}\,dy \\ &= C_1 R_n^{-p(N-2)+2}\int_{B(0,2)\setminus B(0,1)} |z|^{-(N-2)-N}\,dz \ge C_2 R_n^{-p(N-2)+2} \end{aligned} \tag{5.14}$$

and

$$\begin{aligned} \int_{B(0,2R_n)\setminus B(0,R_n)} \frac{|y|^{-p(N-2)}}{|y-x_*|^{N-2}}\,dy &\le C_3\int_{B(0,2R_n)\setminus B(0,R_n)} |y|^{-(N-2)-p(N-2)}\,dy \\ &= C_3 R_n^{-p(N-2)+2}\int_{B(0,2)\setminus B(0,1)} |z|^{-(N-2)-N}\,dz \le C_4 R_n^{-p(N-2)+2} \end{aligned} \tag{5.15}$$

for all $L \le |x| \le R_n$ and all sufficiently large n, where C_i $(i = 1,2,3,4)$ are positive constants independent of L and n. By (5.13), (5.14) and (5.15) we have

$$u(x,t) \ge \frac{c_N e^{p^n}}{N-2}R_n^{-p(N-2)+2}\left[C_2 - C_4|x|^{-(N-2)}\right] \tag{5.16}$$

for almost all $x \in \Omega$ with $L \le |x| \le R_n$, $t \in (1,2)$ and all sufficiently large n. Taking a sufficiently large L if necessary, we have $2C_2 \ge C_4 L^{-(N-2)}$. Then, by (5.16) we have

$$u(x,t) \ge \frac{c_N e^{p^n}}{N-2} \frac{C_2}{2} R_n^{-p(N-2)+2}$$

for almost all $x \in \Omega$ with $L \le |x| \le R_n$, $t \in (1,2)$ and all sufficiently large n. This implies that $u(x,t) = \infty$ for almost all $x \in \Omega$ with $|x| \ge L$ and $t \in (1,2)$. This is a contradiction. Therefore we see that problem (1.1) possesses no local-in-time solution with $\varphi = \mu\phi$. Thus the proof of Theorem 1.4 is complete. □

Proof of Corollary 1.1 Assertion (i) follows from Theorem 4.2. Furthermore, assertion (ii) follows from Theorem 1.2, Theorem 1.3 and assertion (i) of Corollary 1.1. Thus Corollary 1.1 follows. □

References

[1] H. Amann and M. Fila, A Fujita-type theorem for the Laplace equation with a dynamical boundary condition, Acta Math. Univ. Comenianae **66** (1997), 321–328.

[2] M-F. Bidaut-Véron, Local and global behavior of solutions of quasilinear equations of Emden-Fowler type, Arch. Rational Mech. Anal. **107** (1989), 293–324.

[3] M-F. Bidaut-Véron, Local behaviour of solutions of a class of nonlinear elliptic systems, Adv. Differential Equations **5** (2000), 147–192.

[4] M. Fila, K. Ishige and T. Kawakami, Convergence to the Poisson kernel for the Laplace equation with a nonlinear dynamical boundary condition, Commun. Pure Appl. Anal., **11** (2012), 1285–1301.

[5] M. Fila, K. Ishige and T. Kawakami, Large-time behavior of solutions of a semilinear elliptic equation with a dynamical boundary condition, Adv. Differential Equations **18** (2013), 69–100.

[6] M. Fila, K. Ishige and T. Kawakami, Existence of positive solutions of a semilinear elliptic equation with a dynamical boundary condition, Calc. Var. Partial Differential Equations **54** (2015), 2059–2078.

[7] M. Fila, K. Ishige and T. Kawakami, Minimal solutions of a semilinear elliptic equation with a dynamical boundary condition, J. Math. Pures Appl. **105** (2016), 788–809.

[8] M. Fila, K. Ishige and T. Kawakami, An exterior nonlinear elliptic problem with a dynamical boundary condition, to appear in Rev. Mat. Complut.

[9] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1983.

[10] M. Kirane, E. Nabana and S. I. Pokhozhaev, The absence of solutions of elliptic systems with dynamic boundary conditions, Differ. Equ. **38** (2002), 808-815; translation from Differ. Uravn. **38** (2002), 768–774.

[11] M. Kirane, E. Nabana and S. I. Pokhozhaev, Nonexistance of global solutions to an elliptic equation with nonlinear dynamical boundary condition, Bol. Soc. Paran. Mat. **22** (2004), 9–16.