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Positive solutions of Kirchhoff type elliptic equations involving the critical Sobolev exponent

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Abstract

We report our recent studies on Kirchhoff type elliptic equations involving the critical Sobolev exponent. The interaction between the Kirchhoff type nonlocality and the Sobolev criticality leads us to several new phenomena, techniques and results depending on the dimension of the domain. More precisely, if the dimension is equal to 3, we observe the multiplicity of solutions induced by the nonlocal coefficient. If it is 4, we encounter an additional difficulty in proving the existence of solutions because of the lack of the Ambrosetti-Rabinowitz type condition. With the aid of the well known nonexistence result by the Pohozaev identity, we overcome this difficulty and give a positive answer for the solvability. For higher dimension, the Kirchhoff type nonlocality may break the uniqueness of solutions of an associated limiting problem. This crucially affects the behavior of Palais-Smale sequences. Because of this, we need nontrivial modification for the concentration compactness analysis. Introducing a new technique based on the method of the Nehari manifold and the fibering map, we succeed in showing the existence of two solutions. This report is based on our talk entitled "Two positive solutions of the Kirchhoff type elliptic problem with critical nonlinearity in high dimension" on RIMS workshop "Analysis on Shapes of Solutions to Partial Differential Equations" on November 9–11, 2016. This report includes a joint work with Prof. Shibata at Tokyo Institute of Technology.

1 Introduction

1.1 A Kirchhoff type problem

We consider a Kirchhoff type elliptic problem.

$$\begin{cases} -\left(1+\alpha\displaystyle\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda u^q + u^{2^*-1},\ u>0 & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $N \geq 3$. Furthermore, we set $2^* = 2N/(N-2)$, $1 \leq q < 2^* - 1$ and $\alpha, \lambda > 0$. In this report, we give our recent results on the existence of solutions of (P). (P) is usually called a Kirchhoff type equation because the principal term has a coefficient which depends on the Dirichlet energy of the solution. An equation of this type was first proposed by Kirchhoff [12] in 1876. It is a wave equation which describes free vibration of elastic strings. On the other hand, Berstein [4] first studied a similar equation from the mathematical point of view. After a formulation by J.L. Lions [14], now a days many mathematicians investigate the solvability and the asymptotic behavior of solutions of such wave equations. See the survey [3] for more detail. We also point out that a parabolic type equation related to (P) was introduced in [7] by the physical and biological motivation. After that, Chipot et al. [8] studied its solvability and the asymptotic behavior of the solutions. Here we remark that, in the introduction, they indicated two interesting points. The first one is that the nonlocal coefficient may induce multiplicity of stationary solutions. The second one is that the problem admits a Lyapunov functional. Furthermore, the stationary problem permits a variational structure. That is, we can study the existence of solutions via the variational method. After their work, many researchers began to study the existence solutions of the stationary problem involving nonlinear force terms. The first work on this direction seems [1].

1.2 Sobolev critical problems

In view of variational studies on such nonlinear elliptic problems, one of the most interesting problems occurs when we consider the Sobolev critical nonlinearity u^{2^*-1} as in (P). Because of the lack of the compactness of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, a serious difficulty occurs in proving the existence of solutions. Furthermore, it is known that if $\alpha = 0$, $\lambda \le 0$ and Ω is star-shaped, (P) has no solution differently from the subcritical case. Hence to prove the existence of solutions for the critical case becomes a very challenging and interesting problem. By these facts, (P) with $\alpha = 0$ has been extensively studied by many authors. Here let us give the celebrated result by Brezis-Nirenberg [5] which first showed the existence of solutions of (P) for $\lambda > 0$. Define $\lambda_1 = \lambda_1(\Omega) > 0$ as the first eigenvalue of $-\Delta$ on Ω.

Theorem 1.1 (Brezis-Nirenberg '83 [5]). *For the case $\alpha = 0$, we have the following.*

(i) Let $N = 3$ and Ω be a ball. Then, if $q = 1$, (P) has at least one solution if and only if $\lambda \in (\lambda_1/4, \lambda_1)$. On the other hand, if $q \in (1, 3]$, (P) admits at least one solution for sufficiently large $\lambda > 0$ and if $q \in (3, 5)$, (P) permits at least one solution for all $\lambda > 0$.

(ii) Assume $N \ge 4$. Then, if $q = 1$, (P) possesses at least one solution if and only if $\lambda \in (0, \lambda_1)$. On the other hand, if $q \in (1, 2^* - 1)$, (P) has at least one solution for all $\lambda > 0$.

Our question is what happens on these existence and nonexistence results on (P) if it has a Kirchhoff type nonlocal coefficient, i.e., $\alpha > 0$.

1.3 A previous work and our aim

Before our study, we could find an interesting work by G.M. Figueiredo [10]. We remark that he treated more general problem than (P). Especially, he considered a nonlocal coefficient which generalizes that in (P). By a truncation argument, he got an existence result on his problem. A direct consequence is the following.

Theorem 1.2 (G.M.Figueiredo '13 [10]). *If $N \ge 3$, $\alpha > 0$ and $q \in (1, 2^* - 1)$, (P) permits at least one solution if $\lambda > 0$ is sufficiently large.*

When we compare Theorems 1.2 with 1.1, we get some natural questions. The first one is what happens on the case $\alpha > 0$ and $q = 1$ since Figueiredo's assumption on the nolinearity admits only superlinear case $q > 1$. Of course it was treated by Brezis-Nirenberg for the case $\alpha = 0$. The second one is that if we can prove the existence solutions for $\alpha > 0$ and small $\lambda > 0$. It is not clear from Theorem 1.2 if the condition $\lambda > 0$ to be large is essential for the case $\alpha > 0$. Notice that Brezis-Nirenberg showed the existence of solutions for all $\lambda > 0$ if $q > 1$ and $\alpha = 0$. The last one is that if we can get the multiplicity of solutions induced by the nonlocal coefficient as was pointed out by [8] for the nonhomogeneous case. Our aim is to give answers for these questions.

1.4 Variational Setting

Here, to start our argument, we give the variational setting for our problem. Let us define the energy functional associated to (P). For any $u \in H_0^1(\Omega)$, we set

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{\alpha}{4}\|u\|^4 - \frac{\lambda}{q+1}\int_\Omega u_+^{q+1}dx - \frac{1}{2^*}\int_\Omega u_+^{2^*}dx,$$

where $\|u\| := (\int_\Omega |\nabla u|^2 dx)^{1/2}$ and $u_+ := \max\{0, u\}$. We emphasize that the forth order term $\alpha\|u\|^4/4$ appears as a result of taking $\alpha > 0$. By a usual elliptic estimate and the maximum principle, we have that every critical point u of I (i.e., $I'(u) = 0$) is nothing but a solution of (P). Hence in order to prove the existence of solutions of (P), we only have to show the existence of critical points of I. From now on, let us prove that. When we look for critical points of I, we usually first observe the geometry of I. As we will see later, the interaction between the forth order term $\alpha\|u\|^4/4$ and the critical term $\int_\Omega u_+^{2^*}dx/2^*$ crucially affects that. Here, notice that

$$2^* \begin{cases} = 6 > 4 \text{ if } N = 3, \\ = 4 \text{ if } N = 4, \\ < 4 \text{ if } N \geq 5. \end{cases}$$

Then it is natural to divide our study into three cases, i.e., $N = 3, 4$ and $N \geq 5$. In the following sections we give our results on each case. As an introduction, we here summarize interesting points for each case as follows.

(i) If $N = 3$, we observe a multiplicity result which is induced by the nonlocal coefficient. In other words, we see that it can break the uniqueness of solution of (P) for the case $\alpha = 0$.

(ii) If $N = 4$, we encounter an additional difficulty in proving the existence of solutions because of the lack of the Ambrosetti-Rabinowitz type condition if $\alpha > 0$. In particular, it is not clear there exists a bounded Palais-Smale sequence for I. With the help of Pohozaev's nonexistence result, we construct a desired bounded Palais-Smale sequence for I. Consequently, we get a positive answer for the existence of solutions of (P).

(iii) If $N \geq 5$, the nonlocal coefficient may break the uniqueness of the limiting problem associated to (P). This crucially affects the concentration compactness analysis on Palais-Smale sequences. Introducing new techniques utilizing the method of the Nehari manifold and the fibering map, we overcome this difficulty and succeed in proving the multiplicity of solutions of (P).

1.5 Organization of this report

This report is consisted of 4 sections. In Sections 2 and Section 3, we briefly discuss our previous studies on the cases $N = 3$ and 4. In Section 4, we give our recent result and its proof on the higher dimensional case. In the following we define the Sobolev constant $S > 0$ by

$$S = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{2^*} dx\right)^{\frac{2}{2^*}}}$$

2 Dimension 3

Let us first see our result on the case $N = 3$. The case $q = 1$ is treated in [17] and $q > 1$ is in [15]. Here we give a result from [17] where a new multiplicity result induced by the nonlocal coefficient is obtained. We note that the case $q = 1$ is very delicate as is known for the case $\alpha = 0$. Following the argument in [5], we also assume Ω is a ball. The next one is a direct consequence of Theorem 5.1 in [17]. For simplicity we only consider the case $\alpha > 0$ is small.

Theorem 2.1 (N. '15 [17]). *Let $N = 3$, $q = 1$ and Ω be a ball. In addition, we assume $\alpha > 0$ is small enough. Then, there exist constants $c_i = c_i(\alpha) > 0$ for $i = 1, 2$ such that $c_i(\alpha) \to 0$ as $\alpha \to 0$ for $i = 1, 2$ and satisfy the following.*

(i) If $\lambda_1/4 + c_1(\alpha) < \lambda \leq \lambda_1$, (P) has at least one solution.

(ii) If $\lambda_1 < \lambda < \lambda_1 + c_2(\alpha)$, (P) admits at least two solutions.

Comparing this result with Theorem 1.1, we observe the effect of the nonlocal coefficient on the existence of solutions of (P). First notice that even if $\lambda \geq \lambda_1$, (P) can have solutions if $\alpha > 0$ in contrast to the case $\alpha = 0$. Furthermore, we obtain the existence of multiple solutions when $\lambda > \lambda_1$ is not too large. We recall that if $\alpha = 0$ and Ω is a ball, (P) admits the uniqueness of solutions. See, for example, [22]. Hence we may say the nonlocal coefficient can break the uniqueness of solutions of our critical problem.

Now, in order to understand why the nonlocal coefficient can induce the multiplicity of solutions, let us see the geometry of I. To this end, we define the fibering map [9][6]. For all $u \in H_0^1(\Omega) \setminus \{0\}$, set

$$f_u(t) := I(tu) \quad (t > 0).$$

As a test function, we choose $u = \phi_1$, the first eigenfunction of $-\Delta$ on Ω. We may assume $\phi_1 > 0$ in Ω. Then noting $\|\phi_1\|^2 = \lambda_1 \int_\Omega \phi_1^2 dx$, we have

$$f_{\phi_1}(t) = \frac{t^2}{2}\left(1 - \frac{\lambda}{\lambda_1}\right)\|\phi_1\|^2 + \frac{\alpha t^4}{4}\|\phi_1\|^4 - \frac{\lambda t^{q+1}}{q+1}\int_\Omega u_+^{q+1} dx - \frac{t^{2^*}}{2^*}\int_\Omega u_+^{2^*} dx.$$

Clearly, if $\alpha = 0$, f_{ϕ_1} has a non zero critical point if and only if $\lambda < \lambda_1$. But, setting $\alpha > 0$, it can admit that even if $\lambda = \lambda_1$. Moreover, if $\lambda > \lambda_1$ is not too large, f_{ϕ_1} permits both a local minimum point and a maximum one. This observation leads us to expect that I has two critical points. Actually, using standard techniques from the critical point theory and carrying out the concentration compactness analysis of associated Palais-Smale sequences, we obtain the desired result as in Theorem 2.1. For more detail, see [17]. In addition, we remark that a bifurcation diagram for this case is obtained in [18]. See Section 3.5 there.

3 Dimension 4

In this section, we give our result on 4 dimensional case. We refer readers to [16]. As is noted there, if $N = 4$, we encounter an additional difficulty which comes from the fact that the nonlinearity may lack the Ambrosetti-Rabinowitz type condition. The condition is known as a sufficient condition to ensure the boundedness of Palais-Smale sequences. The original one for the semilinear problem is found in [2]. On the other hand, for the case $\alpha > 0$, it is summarized, for example, in Section 1 of [13]. See conditions $(f), (f_0), (f_1)$ and (f_2) there. Because of the lack of such a condition it is difficult to construct a bounded Palais-Smale sequences for the case $N = 4$ if $\alpha > 0$. This seems make the problem very challenging. Here we give our result on the interesting case $q > 1$ in which the nonlinearity actually does not satisfy the Ambrosetti-Rabinowitz type condition.

Theorem 3.1 (N. '14 [16]). *Let $N = 4$ and $1 < q < 3$. Then if $1/(2S^2) < \alpha < 1/S^2$ and Ω is star-shaped,* (P) *admits at least one solution if $\lambda > 0$ is sufficiently small.*

Theorem 3.1 compensates the result in Theorem 1.2 for the case $N = 4$ since we get a solution for small $\lambda > 0$ here. But some additional conditions are assumed in Theorem 3.1. The first one is the condition on $\alpha > 0$ to be small and not too small. The condition $\alpha < 1/S^2$ is natural for the energy functional I to admit the mountain pass geometry [2]. But, the assumption $\alpha > 1/(2S^2)$ seems technical. It is supposed for the concentration compactness analysis. See Lemma 3.2 in [16]. The second one is the assumption on the domain Ω to be star-shaped. Although this seems also technical, it allows us to utilize Pohozaev's nonexistence result in constructing a bounded Palais-Smale sequence. Here let us see the argument. After this, we call $(u_n) \subset H_0^1(\Omega)$ a $(\mathrm{PS})_c$ sequence for I if $I(u_n) \to c \in \mathbb{R}$ and $I'(u_n) \to 0$ in $H^{-1}(\Omega)$ as $n \to \infty$.

Lemma 3.2 (N. '14 [16]). *Let $N = 4$, $q > 1$, Ω be star-shaped. In addition, suppose $\alpha \notin \{1/(kS^2) \mid k \in \mathbb{N}\}$. Then every $(PS)_c$ sequence (u_n) for I is bounded in $H_0^1(\Omega)$.*

Proof. The original proof is given in that for Theorem 1.6 in [16]. We argue by contradiction. Suppose $\|u_n\| \to \infty$ as $n \to \infty$. Then put $w_n := u_n/\|u_n\|$. Since $\|w_n\| = 1$, there exists a function $w_0 \in H_0^1(\Omega)$ such that $w_n \rightharpoonup w_0$ weakly in $H_0^1(\Omega)$ up to subsequences. Notice that w_n satisfies

$$\left(\frac{1}{\|u_n\|^2} + \alpha\right) \int_\Omega \nabla w_n \cdot \nabla h dx = \frac{\lambda}{\|u_n\|^{3-q}} \int_\Omega w_n^q h dx + \int_\Omega w_n^3 h dx + o(1) \quad (1)$$

for all $h \in H_0^1(\Omega)$ where $o(1) \to 0$ as $n \to \infty$. It follows that

$$\alpha \int_\Omega \nabla w_0 \cdot \nabla h dx = \int_\Omega w_0^3 h dx,$$

for all $h \in H_0^1(\Omega)$. Then as Ω is star-shaped, we have $w_0 = 0$ by the Pohozaev identity [20]. Moreover, we note that (1) implies (w_n) is an approximate solutions sequence for a semilinear critical problem. Then the result by Struwe [21] ensures that there exists a number $l \in \mathbb{N}$ and for every $i \in \{1, 2, \cdots, l\}$, sequences of values $(R_n^i) \subset \mathbb{R}^+$, points $(x_n^i) \subset \overline{\Omega}$ with $R_n^i \mathrm{dist}(x_n^i, \partial\Omega) \to \infty$ as $n \to \infty$, and a nonnegative function $v_i \in D^{1,2}(\mathbb{R}^4)$ satisfying

$$-\alpha \Delta v_i = v_i^3 \text{ in } \mathbb{R}^4,$$

such that up to subsequences,

$$1 = \|w_n\|^2 = \sum_{i=1}^{l} \|v_i\|_{1,2}^2 + o(1), \tag{2}$$

where $o(1) \to 0$ as $n \to \infty$. Since $\tilde{v}_i := 1/\alpha^{1/2} v_i \in D^{1,2}(\mathbb{R}^4)$ is a nonnegative solution of

$$-\Delta \tilde{v} = \tilde{v}^3 \text{ in } \mathbb{R}^4, \ \tilde{v}(x) \to 0 \text{ as } |x| \to \infty,$$

the uniqueness result by [11] suggests that for every $i \in \{1, 2, \cdots, l\}$, there exist a constant $\varepsilon_i > 0$ and a point $x_i \in \mathbb{R}^4$ such that

$$\tilde{v}_i = \frac{8^{\frac{1}{2}} \varepsilon_i}{\varepsilon_i^2 - |x - x_i|^2}.$$

Therefore we have

$$\|v_i\|_{D^{1,2}(\mathbb{R}^4)}^2 = \alpha \|\tilde{v}_i\|_{D^{1,2}(\mathbb{R}^4)}^2 = \alpha S^2,$$

for all $i \in \{1, 2, \cdots, l\}$. Finally recalling (2), we get

$$1 = l \alpha S^2,$$

which is impossible by our assumption on α. We finish the proof. □

Thanks to this lemma, we can, as usual, carry out the concentration compactness argument for the bounded Palais-Smale sequence. But, as is noted there, we then add an assumption on $\alpha > 0$ to be not too small. In our opinion, these additional conditions should be avoided. It seems an interesting future problem. For more detail for the proof of Theorem 3.1, see [16].

4 Higher dimension

Finally, we shall consider the higher dimensional case. Let $N \geq 5$. We here only deal with the case $\alpha > 0$ is small, which is, as we will see later, the most interesting and difficult case. To give our result, we set $\lambda_* = \lambda_1$ if $q = 1$ and $\lambda_* = \infty$ if $1 < q < 2^* - 1$. The next theorem is obtained in [19].

Theorem 4.1 (N.-Shibata [19] Submitted). *Let $N \geq 5$ and $1 \leq q < 2^* - 1$. Then there exists a constant $\alpha_0 > 0$ such that for all $\alpha \in (0, \alpha_0)$ and $\lambda \in (0, \lambda_*)$, (P) has at least two solutions.*

Notice that in Theorem 4.1, we prove the existence of solutions for $\lambda \in (0, \lambda_*)$ for which Brezis-Nirenberg showed the existence of at least one solution in Theorem 1.1. A different point is that we get at least two solutions. We can say this multiplicity actually comes from the effect of the nonlocal coefficient since we know that if $\alpha = 0$, $q = 1$ and Ω is a ball, (P) has at most one solution [22]. Because we consider a general bounded domain Ω, of course, we obtain the existence of two solutions even if Ω is a ball. In fact, we will see that we can obtain a global minimizer of I in addition to a mountain pass type critical point. Finally, we shall show Theorem 4.1. In the following, we choose $q = 1$ for simplicity. For the case $q > 1$, the argument is similar.

Let us first observe the geometry of I. As in Section 2, we consider the fibring map.

$$f_{\phi_1}(t) = \frac{t^2}{2}\left(1 - \frac{\lambda}{\lambda_1}\right)\|\phi_1\|^2 + \frac{\alpha t^4}{4}\|\phi_1\|^4 - \frac{\lambda t^2}{2}\int_\Omega \phi_1^2 dx - \frac{t^{2^*}}{2^*}\int_\Omega \phi_1^{2^*} dx.$$

Noting $2 < 2^* < 4$ if $N \geq 5$, we can conclude that if $\alpha > 0$ is small enough, f_{ϕ_1} admits just two critical points for all $\lambda \in (0, \lambda_1)$. In addition, the first one is a unique local maximum and second one is a global minimum. This observation allows us to expect the existence of at least two critical points of I. Actually, the Sobolev inequalities show that I satisfies the mountain pass geometry and is coercive.

4.1 Palais-Smale condition and the limiting problem

Now it suffices to show the Palais-Smale condition for I [2]. But since we are considering the critical case, a crucial difficulty occurs in proving the compactness of Palais-Smale sequences for I. This is caused by the lack of the compactness of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$. Here, applying the blow up analysis by Struwe [21], which was done for the case $\alpha = 0$, we obtain the next description of Palais-smale sequences for I. Set $\|v\|_{1,2} = (\int_{\mathbb{R}^N} |\nabla v|^2 dx)^{1/2}$.

Proposition 4.2 (N. '14 [16]). *Let $(u_n) \subset H_0^1(\Omega) \subset D^{1,2}(\mathbb{R}^N)$ be a $H_0^1(\Omega)$-bounded PS sequence for I. Then (u_n) has a subsequence which converges strongly in $H_0^1(\Omega)$ or otherwise, there exist a nonnegative function $u_0 \in H_0^1(\Omega)$ which is a weak limit of (u_n), a number $k \in \mathbb{N}$ and further, for every $i \in \{1, 2, \ldots, k\}$, sequences of values $(R_n^i) \subset (0, \infty)$, points $(x_n^i) \subset \Omega$ and a nonnegative function $v_i \in D^{1,2}(\mathbb{R}^N)$ which satisfies*

$$-\left\{1 + \alpha \left(\|u_0\|^2 + \sum_{j=1}^{k} \|v_j\|_{1,2}^2\right)\right\} \Delta u_0 = \lambda u_0 + u_0^{2^*-1} \ \text{in } \Omega, \tag{3}$$

$$-\left\{1 + \alpha \left(\|u_0\|^2 + \sum_{j=1}^{k} \|v_j\|_{1,2}^2\right)\right\} \Delta v_i = v_i^{2^*-1} \ \text{in } \mathbb{R}^N, \tag{4}$$

such that up to subsequences, $R_n^i \mathrm{dist}(x_n^i, \partial\Omega) \to \infty$ as $n \to \infty$,

$$\left\| u_n - u_0 - \sum_{i=1}^{k} (R_n^i)^{\frac{N-2}{2}} v_i(R_n^i(\cdot - x_n^i)) \right\|_{1,2} = o(1), \tag{5}$$

$$\|u_n\|^2 = \|u_0\|^2 + \sum_{i=1}^{k} \|v_i\|_{1,2}^2 + o(1), \tag{6}$$

$$\int_\Omega (u_n)_+^{2^*} dx = \int_\Omega u_0^{2^*} dx + \sum_{i=1}^{k} \int_{\mathbb{R}^N} v_i^{2^*} dx + o(1), \tag{7}$$

and

$$I(u_n) = \tilde{I}(u_0) + \sum_{i=1}^{k} \tilde{I}_\infty(v_i) + o(1), \tag{8}$$

where $o(1) \to 0$ as $n \to \infty$ and we put

$$\tilde{I}(u_0) := \frac{1}{2}\|u_0\|^2 + \frac{\alpha}{4}\left(\|u_0\|^2 + \sum_{j=1}^{k} \|v_j\|_{1,2}^2\right)\|u_0\|^2$$
$$- \frac{\lambda}{q+1}\int_\Omega u_0^{q+1} dx - \frac{1}{2^*}\int_\Omega u_0^{2^*} dx,$$

$$\tilde{I}_\infty(v_i) := \frac{1}{2}\|v_i\|_{1,2}^2 + \frac{\alpha}{4}\left(\|u_0\|^2 + \sum_{j=1}^k \|v_j\|_{1,2}^2\right)\|v_i\|_{1,2}^2 - \frac{1}{2^*}\int_{\mathbb{R}^N} v_i^{2^*}\,dx.$$

We interpret this proposition as follows. (5) implies that the lack of compactness of Palais-Smale sequences is caused by the functions v_i for $i \in \{1, 2, \cdots, k\}$ which satisfy the equations in whole space (4). (After this, we call the equation (4) a limiting problem.) Hence we can conclude that the existence of solutions of the limiting problem is crucial for the compactness of Palais-Smale sequences. Here let us check it for the simplest case. If $u_0 = 0$ and $k = 1$, (4) becomes

$$\begin{cases} -\left(1 + \alpha \int_{\mathbb{R}^N} |\nabla V|^2 dx\right) \Delta V = V^{2^*-1}, \ V > 0 \text{ in } \mathbb{R}^N, \\ V \in D^{1,2}(\mathbb{R}^N). \end{cases} \tag{9}$$

First notice that for every solution V we can regard the nonlocal coefficient as just a constant. Then it clearly follows from the uniqueness result [11] that V must be the Talenti function [23] multiplied by suitable constants. Then an easy calculation shows that the existence and nonexistence of such constants. We get the next result.

Proposition 4.3 (N.-Shibata [19]). *There exists a constant $\alpha_* > 0$ such that*

(i) if $\alpha > \alpha_*$, (9) has no solution,

(ii) if $\alpha = \alpha_*$, (9) admits a unique solution (up to dilation and translation),

(iii) if $\alpha \in (0, \alpha_*)$, (9) permits just two solutions (up to dilation and translation).

We remark that if $\alpha = 0$, (P) has a unique solution up to dilation and translation [11]. From this proposition, we conclude the following. If $\alpha > 0$ is very large, (i) implies that Palais-Smaile sequences can not include the concentration part in (5) since there exists no solution of (9). That is, all Palais-Smale sequences must be compact. This suggests that if $\alpha > 0$ is large the problem is rather very easy. If $\alpha = \alpha^*$, since (9) has a unique solution by (ii), the situation is very similar to the case $\alpha = 0$. Actually, we do not encounter any additional difficulty in this case. Now, an interesting phenomenon occurs if $\alpha > 0$ is small. In this case, the nonlocal coefficient breaks the uniqueness of solutions of (9) as in (iii). Furthermore, we can clearly check that one of the solution correspond to a mountain pass type critical point of associated functional and the other does a global minimum one. As a consequence, it is not clear if the energy $\tilde{I}_\infty(v_i)$ in (8) has a positive energy or a negative one. Recall that if $\alpha = 0$, the solution is unique and it always has a positive energy. Hence we can immediately conclude from (8) that if $\alpha = 0$ and a Palais-Smale sequence is not compact, it must have the energy greater than a positive value. This was a crucial fact in the argument for the case $\alpha = 0$. But, if $\alpha > 0$ is small, even if a Palais-Smale sequence is not compact, we can not immediately obtain such a lower bound for the energy because of the reason mentioned above. Now, the usual concentration compactness argument does not seem enough for our proof. Hence we need a new idea. In the following we introduce our idea utilizing the method of the Nehari manifold and the fibering map [9][6]. After this, we always assume $\alpha \in (0, \alpha_*)$ since it is the interesting case. Moreover, we suppose V_1 and V_2 are the solutions of (9) such that $c^* := I^\infty(V_1) > I^\infty(V_2)$.

4.2 Proof for a mountain pass type solution

Lastly we give the outline of the proof of Theorem 4.1. Here we only demonstrate the existence of a mountain pass type solution. For the global minimizer, see [19]. Because of the difficulty mentioned above, the usual mountain pass lemma does not seem work well for our aim. Then, instead of that we utilize the method of the Nehari manifold together with the fibring map. First we define the Nehari manifold,

$$\mathcal{N} := \{u \in H_0^1(\Omega) \setminus \{0\} \mid f_u'(1) = 0\},$$

and its submanifold

$$\mathcal{N}^- := \{u \in \mathcal{N} \mid f_u''(1) < 0\}.$$

We solve a minimization problem on $\mathcal{N}^-$. More precisely, we prove the following.

1. We construct a minimizing $(\mathrm{PS})_c$ sequence on $\mathcal{N}^-$, i.e., a sequence $(u_n) \subset \mathcal{N}^-$ such that $I(u_n) \to c^- := \inf_{u \in \mathcal{N}^-} I(u)$ and $I'(u_n) \to 0$ in $H^{-1}(\Omega)$ as $n \to \infty$.

2. Assuming $c^- < c^*$, we show (u_n) contains a subsequence which weakly converges to a function u_0 and $u_0 \neq 0$.

3. Using the fact $u_0 \neq 0$ in (ii) we show $u_n \to u_0$ strongly in $H_0^1(\Omega)$ up to subsequences.

4. We prove $c^- < c_*$ by the estimate using the Talenti function.

For Step 1, we should be careful of the boundary of $\mathcal{N}^-$ which is defined by

$$\mathcal{N}^0 := \{u \in \mathcal{N} \mid f_u''(1) = 0\}.$$

Notice that if $\alpha = 0$, we have $\mathcal{N}_0 = \emptyset$. But if $\alpha > 0$ it may not be an empty set. This causes a difficulty in constructing a desired minimizing Palais-Smale sequence. Then we need the following lemma to avoid the minimizing sequence to get close to the boundary $\mathcal{N}_0$.

Lemma 4.4. *There exists a constant $\alpha_1 > 0$ such that for all $\alpha \in (0, \alpha_1)$, there exists a constant $C(\alpha) > 0$ such that $C(\alpha) \to \infty$ as $\alpha \to 0$ and*

$$c^- < C(\alpha) \leq \inf_{u \in \mathcal{N}^0} I(u)$$

For the proof, see Proposition 3.2 in [19]. Then, the essential idea to accomplish Step 2 is similar to the case $\alpha = 0$ although the argument becomes more delicate. We refer the reader to Lemma D.1 in [19]. Here we only remark that if $\alpha = 0$, (3) implies that u_0 is a solution of (P). Hence the proof is finished by this step since we can get $u_0 \neq 0$ here. But as is observed in (3), if $\alpha > 0$, u_0 is not a solution of (P) in general because of the nonlocal dependence. Therefore, we must prove the strong convergence of (u_n) as in Step 3. This is the most important argument on our proof. Now, let us give the outline. We assume $u_0 \neq 0$ and (u_n) has no subsequence which strongly converges in $H_0^1(\Omega)$ on the contrary. Then we define a function on $t > 0$ by

$$f^*(t) := \lim_{n \to \infty} f_{u_n}(t).$$

We then decompose $f^*(t)$ by using (6) and (7). That is, noting the formulas and setting

$$\tilde{f}_{u_0}(t) := \frac{\|u_0\|^2}{2} t^2 + \frac{\alpha A \|u_0\|^2}{4} t^4 - \frac{\lambda \int_\Omega u_0^2 dx}{2} t^2 - \frac{\int_\Omega u_0^{2^*} dx}{2^*} t^{2^*},$$

and

$$\tilde{f}_\infty(t) = \sum_{i=1}^{k} \left(\frac{\|v_i\|_{1,2}^2}{2} t^2 + \frac{\alpha A \|v_i\|_{1,2}^2}{4} t^4 - \frac{\int_{\mathbb{R}^N} v_i^{2^*} dx}{2^*} t^{2^*} \right),$$

where $A := \lim_{n\to\infty} \|u_n\|^2$, we get

$$f^*(t) = \tilde{f}_{u_0}(t) + \tilde{f}_\infty(t).$$

Now, using (3), we can first deduce that $f'_{u_0}(1) < 0$. This implies that there exists a constant $t_0 \in (0,1)$ such that $t_0 u_0 \in \mathcal{N}^-$. Moreover notice that since we constructed a Palais-Smale sequence (u_n) on $\mathcal{N}^-$, we have additional information $f''_{u_n}(1) < 0$. Using this, (3) and (4), we can next conclude that $(f^*)'(1) = (\tilde{f}_\infty)'(1) = 0$ and $(f^*)''(1), (\tilde{f}_\infty)''(1) \leq 0$. Then, these facts lead us to conclude that $f^*(t)$ and $\tilde{f}_\infty(t)$ are increasing on $(0,1)$. Lastly we get by the definition that

$$c^- \leq I(t_0 u_0) = f_{u_0}(t_0) < f^*_{u_0}(t_0) + f^*_\infty(t_0) = f^*(t_0) < f^*(1) = c^-,$$

which is a contradiction. This completes Step 3. Finally, Step 4 is proved by carrying out the energy estimate by the Talenti function similarly to the case $\alpha = 0$. See Lemma 4.3 in [19]. This completes the proof for the existence of a mountain pass type critical of I. For more detailed discussion, we refer the reader to [19].

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