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CURVE DIFFUSION AND STRAIGHTENING FLOWS WITH FREE BOUNDARY

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ABSTRACT. In this announcement paper, we discuss upcoming results on families of immersed curves $\gamma : (-1,1) \times [0,T) \to \mathbb{R}^2$ with free boundary supported on parallel lines $\{\eta_1, \eta_2\} : \mathbb{R} \to \mathbb{R}^2$ evolving by the curve diffusion flow and the curve straightening flow. The evolving curves are orthogonal to the boundary and satisfy a no-flux condition.

1. INTRODUCTION

Fourth-order extrinsic curvature flow have recently enjoyed considerable attention in the literature. Two model flows are the surface diffusion flow, where points move with velocity $\Delta^\perp \vec{H}$, and the Willmore flow, where points move with velocity $\Delta^\perp \vec{H} + \vec{H}|A^o|^2$. These curvature flow are one-parameter families of surfaces immersed in $\mathbb{R}^3$ via immersions $f : \Sigma \times [0,T) \to \mathbb{R}^3$, with $\vec{H}$ the mean curvature vector, $\Delta^\perp$ the Laplacian on the normal bundle along f, and A^o the tracefree second fundamental form.

Surface diffusion flow, proposed by Mullins [48] in 1956, arises as a model for several phenomena [10, 61]. As such it has received and continues to receive intense attention from the applied mathematics community. Global analysis for the surface diffusion flow is restricted at the moment to special situations, and although the theory of singularities for the flow has received some attention [67, 68] it is far from well-understood. The surface diffusion flow is variational, being the H^{-1}-gradient flow for the area functional. The Willmore flow is also variational, being the steepest descent L^2-gradient flow for the Willmore functional. The Willmore functional is, up to normalisation, the integral of the mean curvature $\vec{H}$ squared. A prototypical bending energy, it has been argued that the Willmore functional was considered first by Sophie Germain in the early 19th century. The Willmore functional drew significant interest from Blaschke [5, 6, 7] and his school, including Thomsen and Schadow, who first presented the Euler-Lagrange operator. Their interest in the Willmore functional stems from its invariance under the Möbius group of $\mathbb{R}^3$ (so long as inversions are not centred on the surface, see [3, 4, 12, 33] for example for a precise formula). This invariance lies at the heart of many of its applications, both to physics and back to mathematics itself, for example in embedding problems. The appeal of the functional is so universal that the Willmore conjecture [71], asserting that the global minimiser among surfaces in $\mathbb{R}^3$ with genus one is achieved by the Clifford torus (and closed conformal images thereof), generated significant attention (a selection is [11, 39, 54, 57]), before being recently solved in a brealthrough work [44]. The Willmore flow was first studied by Kuwert and Schätzle [34, 35, 36] who set up a general framework that is by now a standard methodology used to understand large varieties of higher-order curvature flow. Applications and modifications of this framework exist for the surface diffusion flow [68, 64], the geometric triharmonic heat flow [46], and polyharmonic flows [52].

Although in some special cases maximum-principle style results hold, more typical is a kind of 'almost' maximum principle, and an 'eventual' positivity, see [17, 23, 26, 27] for the parabolic and [28] and for the elliptic settings respectively). Many of the tools and techniques used in the analysis of second-order curvature flow can not be applied to the study of fourth and higher-order curvature flow. In addition to the development of new techniques, it is a natural focus of research effort to determine the extent to which modifications of known techniques apply to various fourth-order curvature flow in different scenarios. This is where the results announced in this paper fit into the picture. We treat the one-dimensional case for the surface diffusion and Willmore flows with free boundary, called the *curve diffusion flow* and *elastic flow* (or curve lengthening/straightening flow) respectively.

In order to differentiate easily between these three flows, we label them as follows:

(CD) Curve diffusion flow

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(E) Elastic flow

The main announced results, Theorem 1.2 for (CD) and Theorem 1.4 for (E), consider the question of geometric stability, where closeness to an equilibrium is measured explicitly in terms of a geometric quantity. We also present some conjectures and a question on a suitable adaptation of Proposition 1.5 from [64]. This directly addresses for (CD) the question of preservation of positivity raised above by measuring the total amount of time during which a global solution may remain not strictly graphical. The evolving families of curves we study have *free boundary*, supported on parallel lines in the plane (see Figure 2).

Second-order curvature flow with free boundary have been considered since the 90s [53, 58, 59, 60] and continues to receive significant research attention (for a sample of the growing literature, see [9, 18, 32, 37, 42, 43, 47, 62, 63, 65, 69, 70]). Fourth-order curvature flow with various boundary conditions have received some recent attention, with work particularly relevant to this paper in [14, 15, 16, 24, 25, 40, 41, 49, 51]. In [24, 25] stability results are proved for curves evolving by (CD) that are graphical and nearby equilibria (with closeness measured in terms of height and $\|k_s\|_2^2$) evolving in bounded domains with free boundary. Although our setting is fundamentally parametric and therefore distinct, our results here, for the curve diffusion flow, can be thought of as naturally complementing these. The evolving curves considered in this paper are supported on straight lines, so the analogue of 'domain' from [24, 25] is always unbounded. We consider immersed curves, with possibly self-intersecting image. Intersections in the image may result from the curve touching itself, or from the curve intersecting one of the straight supporting lines. This allows global results for perturbations of arcs of multiply-covered circles for instance. Considering curves supported on parallel lines allows for results on unbounded, cocompact initial data as well. As the supporting curves are parallel, repeated reflection produces an entire curve.

Stability for the elastic flow is a classically difficult problem. The flow (E) is the steepest descent L^2-gradient flow for the elastic energy:

$$E(\gamma) = \int_\gamma k^2\, ds\,,$$

where $\gamma : [-1, 1] \to \mathbb{R}^2$ is a smooth immersed plane curve, k its scalar curvature and ds the arclength element. This energy is *not* scale-invariant, and can be decreased by enlarging the curve through homothety. Circles and curves with constant curvature are not equilibria; they are expanders.

There exist infinitely many straight line segments in that are stationary under the flow. Despite this it seems difficult to imagine that the flow (E) without a constraint would be stable, especially without imposing an additional symmetry condition, as glued in arcs of circles would still prefer to expand under the flow. In fact, if the distance between the parallel lines $|e|$ is zero, then circles expand. By slowly separating the two lines (continuously increasing $|e|$ for example) and using a continuous dependence on data result in appropriate spaces, there seems to exist many non-compact trajectories for the flow. With this in mind, stability of the straight line under (E) seems unlikely. Nevertheless we do achieve stability for (E) without needing to resort to a length constraint. This argument requires an initial condition.

Let us formally introduce the evolution equations. Suppose $\gamma : [-1, 1] \to \mathbb{R}^2$, $\eta_i : \mathbb{R} \to \mathbb{R}^2$ $(i = 1, 2)$ are regular smooth immersed plane curves such that γ meets η_i perpendicularly with zero flux at its endpoints; that is,

$$\gamma(-1) \in \eta_1(\mathbb{R})\,; \quad \gamma(1) \in \eta_2(\mathbb{R})\,, \quad \langle \nu, \nu_{\eta_i}\rangle\,(\pm 1) = 0\,, \quad k_s(\pm 1) = 0\,. \tag{1}$$

Above we have used ν to denote a unit normal vector field on γ, s is the Euclidean arc-length parameter, and $k = \langle \kappa, \nu\rangle = \langle \gamma_{ss}, \nu\rangle$. We choose ν by setting $\nu = (\tau_2, -\tau_1)$ where $\tau = \gamma_s$ is the tangent vector with direction induced by the given parametrisation. We call η_i *supporting curves* for the flow.

The length of γ is

$$L(\gamma) = \int_{-1}^{1} |\gamma_u|\, du\,.$$

Another important quantity, in addition to the elastic energy E introduced earlier, is

$$A(\gamma) = -\frac{1}{2}\int_{-1}^{1} \langle \gamma, \nu\rangle\ |\gamma_u|\, du\,, \tag{2}$$

which is the usual notion of area for closed plane curves. Here, A corresponds to the area of the star-shaped domain (with multiplicity) traced out by rays connecting the position vector γ and the origin.

Consider a one-parameter family of immersed curves $\gamma : [-1,1] \times [0,T) \to \mathbb{R}^2$ satisfying the boundary conditions (1) and have normal speed given by F, that is

$$\partial_t\gamma = -F\nu\,.$$

The flows are:

(CD): normal velocity equal to $-\text{grad}_{H^{-1}}(L(\gamma))$, that is,

$$F = k_{ss}\,;$$

(E): normal velocity equal to $-\text{grad}_{L^2}(E(\gamma))$, that is,

$$F = k_{ss} + \frac{1}{2}k^3\,;$$

The (free) boundary value problem that we wish to consider for these flows is the following:

$$\text{(CD/E)}\qquad \begin{cases} (\partial_t\gamma)(u,t) = -(F\nu)(u,t) & \text{for all } (u,t)\in(-1,1)\times(0,T)\\ \gamma(-1,t)\in\eta_1(\mathbb{R});\quad \gamma(1,t)\in\eta_2(\mathbb{R}) & \text{for all } t\in\times[0,T)\\ \langle\nu,\nu_{\eta_1}\rangle(-1,t) = \langle\nu,\nu_{\eta_2}\rangle(1,t) = 0 & \text{for all } t\in\times[0,T)\\ k_s(-1,t) = k_s(1,t) = 0 & \text{for all } t\in\times[0,T)\,. \end{cases}$$

Note that we do not prescribe the tangential movement in (CD/E). In the closed case, tangential movements leave the image invariant and correspond to reparametrisations in the domain. For the boundary case, this is no longer true and tangential movements typically correspond to stretching the image (if not periodic for example). We therefore have no freedom in choosing a tangential movement that will simplify analysis, as it will typically be forced upon us by the existence theory. Nevertheless tangential motion, as in the closed case, plays almost no role in the (global) analysis as all quantities that arise from the commutator relations (see [66, Lemma 2.1]) depend only on the normal component of the velocity.

The curve diffusion flow is the steepest descent gradient flow for length in H^{-1}. Since the velocity is a potential, signed enclosed area A in the case of closed curves is constant along the flow. This shows that the isoperimetric ratio is a scale-invariant monotone quantity for the flow, and this fact can be useful for analysis of solutions to the flow (see [64] for example). In the case of the boundary problems considered here, this is no longer true. Here it is difficult to find a useful notion of enclosed area. Indeed, this is a fundamental obstacle to smooth compactness, and can only be overcome in the case when the flow is already in its preferred topological class, that is, when we assume that $\omega = 0$ (see Remark 4 and Figure 3).

Local existence for (CD/E) can be proved by using the standard procedure of solving the flow in the class of graphs over the initial data, as in [58]. As we consider a Neumann problem, we may use a local adapted coordinate system similar to Stahl [58] which does not require a tangential component in the velocity of the flow. This can be continued until the solution leaves this class, at which point there is either some loss of regularity in $C^{4,\alpha}$, or the solution is simply no longer graphical over its initial state. The latter problem is a technicality, and can be resolved by writing the flow in a new coordinate system, as a graph over the solution at a later time. Now if there are uniform $C^{2,\alpha}$-estimates, it is possible to use a standard contraction map argument to continue the solution. To the best of our knowledge the first to observe that only $C^{2,\alpha}$ is required were Ito-Kohsaka, with the map Φ constructed in [31, Proof of Theorem 3.1]. There they are working with (CD) however the additional term added by (E) does not cause any additional difficulty. Therefore by iterating the above procedure we find that the maximal time of existence is either infinity, or the $C^{2,\alpha}$ norm has blown up. In this paper, the most natural norms to control a-priori are L^2 in arc-length derivatives of curvature. The standard Sobolev inequality allows us to control the $C^{2,\alpha}$ norm by the length of the position vector $|\gamma|$ and the L^2-norm of the first derivative of curvature. Note that it is not (without additional arguments) enough to bound only the length of the evolving curves. The statement below is specialised to our current situation, where the supporting curves are straight lines. We note that it is far from optimal.

Theorem 1.1 (Local existence). *Let $\eta_i : \mathbb{R} \to \mathbb{R}^2$ $(i = 1,\ 2)$ be straight lines. Suppose $\gamma_0 : \mathbb{R} \to \mathbb{R}^2$ is a regular smooth curve satisfying the boundary conditions* (1). *Then there exists a maximal $T \in (0,\infty]$ and a unique one-parameter family of regular immersed curves $\gamma : (-1,1)\times[0,T) \to \mathbb{R}^2$ satisfying $\gamma(u,0) = \gamma_0(u)$ and* (CD/E). *Furthermore, if $T < \infty$, then there does not exist a constant C such that*

$$\|\gamma\|_\infty + \|k_s\|_2 \le C \tag{3}$$

for all $t \in [0,T)$.

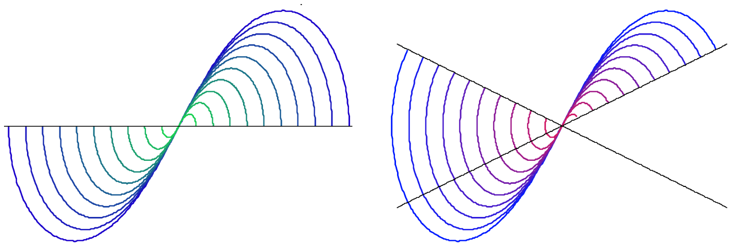


FIGURE 1. The curve diffusion flow with free boundary becoming singular in finite time. The evolution is homothetic.

Remark 1. If the flow is not supported on straight lines, then we require compatibility conditions to produce a solution. If the compatibility conditions are violated by the initial data, then we are still typically able to produce a flow, however convergence as $t \searrow 0$ will be limited by the degree to which the compatibility conditions are satisfied. One interesting investigation into this for the surface diffusion flow is [2], where the degree of incompatibility is finely studied in the context of the original motivation from Mullins [48].

1.1. **Curve diffusion flow.** In light of condition (3), global existence follows if we are able to uniformly bound the length of the position vector and the L^2-norm of the derivative of curvature. The curve diffusion flow is the H^{-1} gradient flow of the length functional, with $L' = -\|k_s\|_2^2$. The length is uniformly controlled a-priori but this does not yield immediately an estimate for $\|\gamma\|_\infty$. It does make $\|k_s\|_2^2$ a natural energy for the flow, with an a-priori uniform estimate in $L^1([0,T))$ depending only on the length of the initial data. Despite this, there are shrinking self-similar solutions to the evolution equation (see Figure 1, which relies upon the lemniscate described in [20]) that are clearly singular in finite-time. Additionally, there is a conjecture due to Giga that implies finite-time singularities can occur from initially embedded data. For the situation with free boundary considered here, we expect that there exist a greater variety of such singularities.

Therefore global existence is not expected to hold generically. It is natural to hope however that in a suitable neighbourhood of minimisers for the energy, global existence and convergence to a minimiser holds. The only global minimisers are straight lines perpendicular to the supporting lines. Our main theorem confirms that these equilibria are stable, with neighbourhood given by the oscillation of curvature.

First let us define:

- Set e to be any vector such that all minimisers of length are translates of e.
- The constant ω and the average curvature are defined by

$$\int_\gamma k\, ds\bigg|_{t=0} = 2\omega\pi\,,$$

$$\overline{k}(\gamma) = \frac{1}{L}\int_\gamma k\, ds\,.$$

  Note that ω is not typically an integer (see [66, Lemma 2.5]).
- The oscillation of curvature and the isoperimetric ratio are defined as

$$K_{osc}(\gamma) = L\int_\gamma \left(k - \overline{k}\right)^2 ds\,,$$

  and

$$I(\gamma) = \frac{L^2(\gamma)}{4\omega\pi A(\gamma)}\,.$$

Theorem 1.2. *Let $\gamma : (-1,1)\times[0,T) \to \mathbb{R}^2$ be a solution to (CD). Suppose γ_0 satisfies*

$$K_{osc}(\gamma_0) = L(\gamma_0)\|k\|_2^2(\gamma_0) < \frac{\pi}{10}\,. \tag{4}$$

Then $\omega = 0$, the flow exists globally $T = \infty$, and $\gamma(\cdot,t)$ converges exponentially fast to a translate of e in the C^∞ topology.

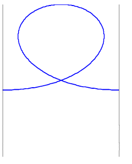


FIGURE 2. Sample initial data. This initial data has winding number 1.

Remark 2. The hypothesis of Theorem 1.2 implies that $\omega = 0$. To see this, we calculate at initial time

$$(2\omega\pi)^2 = \left(\int_\gamma k\,ds\right)^2 \le L\int_\gamma k^2\,ds < \frac{\pi}{10}$$

so

$$\omega^2 < \frac{\pi}{10}\frac{1}{4\pi^2} < \frac{1}{4}\,.$$

The boundary condition implies that ω is an integer multiple of $\frac{1}{2}$, and so must be zero. As ω is constant along the flow (see [66, Lemma 2.5]), it remains zero for all time.

Remark 3. Identifying which translate the solution converges to is a difficult open problem, similar to the problem of identifying the location of the final point singularity that planar curve shortening flow approaches (see [8]).

For closed curves, if the oscillation of curvature is initially small, then the flow exists for all time and converges exponentially fast to a standard circle. This is the main result of [64]. Also in [64] is an estimate of the *waiting time*: as the limit is a circle and convergence is smooth, there exists a T^* such that $k > 0$ for all $t > T^*$, that is, the flow is eventually convex. This is interesting in light of [29], that shows convexity is in general lost under the flow.

This is a symptom of the failure of the maximum principle for fourth-order differential operators. Another such symptom was identified by Elliott and Maier-Paape [21], that graphicality is typically lost in finite time. In our situation here, a natural 'graph direction' exists: the rotation of e by $\frac{\pi}{2}$. Let us denote this rotated vector by f. Indeed, analogously to the situation in [64], there exists a waiting time T^* such that for all $t > T^*$, we have

$$f[\gamma](x,t) := \langle \nu(x), f\rangle > 0\,, \qquad \text{for all } x \in (-1,1)\,.$$

That is, the flow is eventually graphical. This leads us to the natural question:

Question. *Let $\gamma : (-1,1)\times[0,\infty) \to \mathbb{R}^2$ be a solution to (CD) satisfying the assumptions of Theorem 1.2. Does there exist a $C = C(\gamma(\cdot,0))$ depending only on the initial data such that*

$$\mathcal{L}\big\{t \in [0,\infty) : f[\gamma](\cdot,t) \not> 0\big\} \le C(\gamma(\cdot,0))$$

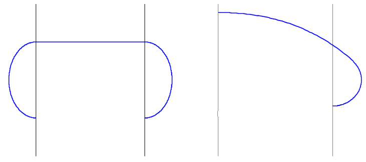
and for every $\varepsilon > 0$ there exists a flow γ_ε such that

$$\mathcal{L}\big\{t \in [0,\infty) : f[\gamma](\cdot,t) \not> 0\big\} > C(\gamma_\varepsilon(\cdot,0)) - \varepsilon\,?$$

In the above we have used $\mathcal{L}$ to denote Lebesgue measure.

Remark 4. Finite-time singularities for the curve diffusion flow with closed data remain difficult to penetrate. Although there are natural Lyapunov functionals for the flow, these do not seem to yield classification results for blowups of singularities. Indeed, it is still unknown if solutions in symmetric perturbation classes near non-trivial shrinkers (such as the figure-8 solution discussed in [20]) converge modulo rescaling to the shrinker. As mentioned, we can also understand this self-similar solution in the free boundary setting (see Figure 1). It seems likely that the free boundary setting will be useful when studying perturbations of the figure-8.

In the free boundary setting, finite-time singularities are more common, and global analysis of the flow can be quite problematic even in a small data regime. For example, the exterior problem, where the flow is supported on parallel lines but with winding number $\omega \ne 0$ (see Figure 3) is in a class of curves whose members all have non-constant curvature. There is no equilibrium in that setting satisfying the boundary



(a) $\omega = 1$ (b) $\omega = \frac{1}{2}$;

FIGURE 3. Sample initial data for the exterior problem.

conditions. Nevertheless, by adjusting the aperture width $|e|$, it is simple to see that one may make the oscillation of curvature arbitrarily small.

There is an interesting technical point here. Some of the estimates used to prove Theorem 1.2 are close to optimal: using initial smallness of the oscillation of curvature, we may use the method of proof from [66, Proposition 3.18] to find that curvature is well-controlled in L^2 if we can control the length difference $L(\gamma_t) - L(\gamma_0)$. If the supporting lines are skew, this follows by using an isoperimetric-type argument. For parallel lines this doesn't work. If $\omega = 0$ then the problematic term is absent, however for $\omega \neq 0$, the term needs to be estimated. An easy condition controlling this term is that $L(\gamma_0) = |e| + \delta$, where $|e|$ is the length of the straight line connecting each of the parallel lines. If it were possible to choose $\delta < K_0$, where K_0 is larger than the initial oscillation of curvature and smaller than K^* from [66, Proposition 3.18], then a stability result would follow. These requirements are in competition with one another: although the oscillation of curvature is scale-invariant, decreasing δ beyond a certain critical level necessitates an increase in the oscillation of curvature. Indeed, the fact that there is no equilibrium in the class of curves satisfying the boundary conditions for $\omega \neq 0$ proves that it is not possible to make this choice. As a corollary of this, we conclude the following lower bound for the oscillation of curvature in the exterior problem.

Corollary 1.3. *Let $\gamma : (-1,1) \to \mathbb{R}^2$ be an immersed curve satisfying the boundary conditions of the exterior problem: $\eta_i : \mathbb{R} \to \mathbb{R}^2$ are parallel straight lines, the origin lies in the interior of η_i[1], γ meets η_i at right angles, with $k_s(\pm 1) = 0$, and at least one of the tangent vectors at the boundary τ_i points away from the interior of η_i.*

Then

$$K_{osc}(\gamma) + 8\pi^2 \log\left(\frac{L(\gamma)}{|e|}\right) \geq \frac{12\pi^2\omega^2 + \pi - 2\omega\pi\sqrt{6\pi(6\pi\omega^2+1)}}{3}.$$

Concerning the global behaviour of this flow, we make the following conjecture.

Conjecture. *Let $\gamma_0 : (-1,1) \to \mathbb{R}^2$ be an immersed curve satisfying the boundary conditions of the exterior problem, as in Corollary 1.3. The curve diffusion flow with free boundary $\gamma : (-1,1) \times [0,T) \to \mathbb{R}^2$ with γ_0 as initial data exists for at most finite time, and $\gamma(\cdot,t)$ approaches a multiply-covered straight line in the C^0 topology and not in C^k for any $k \geq 1$.*

1.2. **Elastic flow.** We finish by announcing a surprising global result on the vanilla elastic flow. As noted earlier, despite $\|k\|_2^2$ being uniformly bounded and non-increasing along an elastic flow, compactness is not expected in general due to the norm $\|\gamma\|_\infty$ typically growing without bound. In order to obtain compactness, a restriction on length is usually imposed.

This makes global results on the vanilla elastic flow quite rare. For the flow supported on parallel lines, we are able to obtain a result of this kind, if the initial oscillation of curvature is not bigger than π. This can be thought of as a stability result for straight lines, as the boundary condition can, via reflection, be understood as imposing a *cocompactness condition* on the flow.

Theorem 1.4. *Let $\gamma : (-1,1) \times [0,T) \to \mathbb{R}^2$ be a solution to (E) given by Theorem 1.1. Assume that*

$$L(\gamma_0)\int_\gamma k^2\,ds\Big|_{t=0} \leq \pi\,. \tag{5}$$

[1]The interior is the region between the two parallel lines.

Then the flow exists globally $T = \infty$ and $\gamma(\cdot, t)$ converges exponentially fast to a translate of e in the C^∞ topology.

Remark 5. As with (CD), it is unknown how to determine, from the initial data, which straight line the flow will converge to.

Sharpness of the given condition is unknown, however, we do not expect it to be sharp. Based on the winding number calculation in [66, Lemma 3.2] and numerical evidence, we make the following conjecture.

Conjecture. *Theorem 1.4 holds with* (5) *replaced by*

$$L(\gamma_0) \int_\gamma k^2 \, ds \bigg|_{t=0} \le \pi^2 . \tag{6}$$

The argument for including equality in (6) above is as follows. It is possible to construct, for any $\delta > 0$, a curve satisfying the boundary conditions with

$$\omega = \frac{1}{2} \quad \text{and} \quad K_{osc} = \pi^2 + \delta .$$

Clearly such curves can not smoothly converge to a straight line; in fact, numerical evidence suggests that (unlike (CD) flow) such curves expand indefinitely and do not display any compactness property. In particular, length is no longer controlled a-priori.

However the limit as $\delta \searrow 0$ has $\omega = 0$ and this does not seem to be avoidable. This is why we conjecture that the sharp energy level that allows compactness and smooth convergence is π^2, with π^2 included.

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