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# Allen-Cahn equation with strong irreversibility

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**Abstract**

This note mainly presents a review of recent works [3, 4] on a variant of the Allen-Cahn equation with a nondecreasing constraint on flow.

## 1 Introduction

*Gradient flows* appear in various fields to describe dynamics of nonequilibrium systems, e.g., diffusion process, phase transition. They are usually formulated in terms of an evolution equation governed by the gradient $\mathrm{d}\mathcal{F}$ of a free energy functional $\mathcal{F}$,

$$u'(t) = -\mathrm{d}\mathcal{F}(u(t)), \quad 0 < t < +\infty. \tag{1}$$

Then the nonincrease of the free energy, $t \mapsto \mathcal{F}(u(t))$, naturally follows from (1). In particular, the Allen-Cahn equation (2) is a typical example of gradient systems,

$$u_t = \Delta u - W'(u) \quad \text{in } \Omega \times (0, +\infty), \tag{2}$$

where $u = u(x,t)$, $x \in \Omega$, $t > 0$, $\Omega$ is a domain of $\mathbb{R}^N$, $u_t = \partial u/\partial t$ and $W(u)$ is a double-well potential (e.g., $W(u) = u^4/4 - u^2/2$), equipped with the homogeneous Dirichlet or Neumann boundary condition (when $\partial\Omega \neq \emptyset$), and it is rephrased as an $L^2$-gradient system associated with the Ginzburg-Landau free energy,

$$\mathcal{F}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_\Omega W(u(x))\,\mathrm{d}x.$$

Indeed, the derivative $\mathrm{d}\mathcal{F}(u)$ of $\mathcal{F}$ (e.g., in $L^2(\Omega)$) corresponds to the terms $-\Delta u + W'(u)$. Among various features of the Allen-Cahn equation, we particularly recall the following properties. Here let us focus on the bounded domain case (with the homogeneous Dirichlet or Neumann boundary condition).

- *Smoothing effect*: even though initial data are not so smooth, solutions immediately recover certain regularity. For instance, one can construct an $L^2(\Omega)$-*solution* of (2) (i.e., the each term of (2) lies on $L^2(\Omega)$) such that $u(\cdot, t)$ belongs to $H^2(\Omega)$ for any $t > 0$ even for initial data $u_0$ belonging to a wider class, say $u_0 \in L^2(\Omega)$.

- *Energy dissipation and existence of global attractors*: besides the nonincrease of the free energy, one can derive *energy dissipation estimates*, which exhibit the (strict) decrease of the free energy and other quantities along the evolution of solutions which are still far from equilibrium. Moreover, in a proper function space setting, one can construct an *absorbing set*, into which the orbit of any bounded set enters in finite time. Furthermore, these properties may enable us to construct a *global attractor*, which is a compact subset of a phase space and attracts any bounded set in the phase space (see, e.g., [50, 14, 28] for more details).

- *Convergence to an equilibrium and Lyapunov stability*: each orbit $u(x,t)$ converges to an equilibrium $\phi(x)$ as $t \to \infty$ (namely, the equilibrium is uniquely determined by the initial datum) and every equilibrium is characterized as a solution to the stationary problem,

  $$-\Delta\phi + W'(\phi) = 0 \quad \text{in } \Omega$$

  equipped with a corresponding boundary condition. Hence the stationary equation itself is independent of the choice of initial data, and moreover, it may have multiple solutions and cover all the possible equilibria of the system. Therefore from a variational analysis on the (single) stationary problem, one may determine stability and instability (e.g., in Lyapunov's sense) of each equilibrium.

In the context of *Damage Mechanics*, the evolution of *damage*, e.g., brittle fracture [32, 27, 7, 8, 33, 40, 41, 13, 30, 31, 32], crack propagation [49, 38] and damage accumulation [17, 18] (see also [37]), is often described in terms of phase field model, where an order parameter $u(x,t)$ is introduced to represent the degree of damage; more precisely, we mean that the point $x$ of a specimen is completely damaged at time $t$ by $u(x,t) = 1$ and it is not at all damaged by $u(x,t) = 0$. And then, the evolution of $u(x,t)$ is given in such a way as to decrease a certain free energy. For instance, the so-called *Ambrosio-Tortorelli regularization* (see [7, 8]) of the *Francfort-Marigo energy* (see [32]) is used:

$$\mathcal{E}_\varepsilon(z,u) := \underbrace{\int_\Omega (1-u^2)|\nabla z|^2 \, dx}_{\text{bulk energy}} + \underbrace{\int_\Omega \left( \frac{\varepsilon}{2}|\nabla u|^2 + \frac{V(u)}{\varepsilon} \right) dx}_{\text{surface energy (Modica-Mortola funct.)}},$$

where $\varepsilon > 0$ is a small (regularization) parameter, $z$ denotes the deformation of the specimen and $V$ is a potential function (e.g., $V(u) = u^2/2$), equipped with some boundary condition including an external load onto the specimen. However, in contrast with standard phase field models, the evolution of the order parameter $u(x,t)$ is constrained to be nondecreasing in time from the nature of damage phenomena. Indeed, the evolution of damage is always unidirectional. In order to realize such constrained gradient flows, let us modify (1) as follows:

$$u_t = \Big( - \mathrm{d}\mathcal{F}(u) \Big)_+,$$

where $(\,\cdot\,)_+ := \max\{\,\cdot\,, 0\} \geq 0$ is the so-called *positive-part function* (cf. see also [34, 35]).[1] Such nondecreasing constrained problems are also studied in other context (see, e.g., [10, 11, 19, 20, 21, 46, 43, 47]).

In order to understand how the nondecreasing constraint (i.e., the presence of the positive-part function) influences the dynamics of the gradient-like systems, one may start with analyzing concrete and simple examples, which often enable us to investigate more precise properties of the dynamics. As for the diffusion equation with a given function $f(x,t)$, the following variant

$$u_t = \Big( \Delta u + f(x,t) \Big)_+,$$

is studied in [5] (see also [45]), where some regularity result on elliptic variational inequalities is developed (cf. see also [39]) and well-posedness, comparison principle and convergence of solutions for the Cauchy-Dirichlet problem are studied based on the regularity result. Then it is also pointed out that equilibria can be characterized by an elliptic obstacle problem whose obstacle function is given by the initial data. Furthermore, the positive-part modification is also applied to a nonlinear diffusion equation along with a blow-up term (see [18] for $N = 1$ and [1] for general $N$), that is,

$$u_t = u^\alpha \Big( \Delta u + u \Big)_+, \quad \alpha \geq 0,$$

which arises from a damage accumulation model proposed by Barenblatt and Prostokishin [17]. In this study, analysis requires somewhat involved arguments to deal with the fully nonlinear equation. On the other hand, behaviors of solutions do not change drastically (indeed, due to the blow-up term, solutions tend to increase and the nondecreasing constraint does not prevent such increasing behaviors).

[1] This formulation may sacrifice accuracy, since the original flow is posed on an abstract space and the positive function is not defined there. However, it would be more helpful to understand a central idea of this research, which has already been proposed in [5]. A more rigorous formulation can be given by using subdifferential operators of indicator functions over closed convex cones (see, e.g., [48, 51] and also [6]).

This note is concerned with a constrained version of the Allen-Cahn equation (2). More precisely, let us consider the Cauchy-Dirichlet problem (denoted by (irAC)),

$$u_t = \Big(\Delta u - W'(u)\Big)_+ \quad \text{in } \Omega \times (0, \infty),$$
$$u = 0 \quad \text{on } \partial\Omega \times (0, \infty), \quad u|_{t=0} = u_0 \quad \text{in } \Omega,$$

where $\Omega$ is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. For simplicity, we set $W(u) = u^4/4 - u^2/2$. As mentioned above, one of main features of the problem above lies on the nondecrease of $t \mapsto u(x,t)$ for each $x \in \Omega$. On the analogy of the Allen-Cahn equation, solutions may decrease and increase to be stabilized (note that $W(u)$ has two global minimizers, 1 and $-1$). However, the nondecreasing constraint prohibits decreasing behaviors of solutions. So one may expect that solutions may face obstructions to be stabilized. The main purpose of this study is to reveal where and how such obstructions appear and influence the dynamics of the system (irAC).

In Section 2, we shall reformulate (irAC) into an evolution equation of doubly-nonlinear type which restores a gradient structure to be better fitted for energy techniques. Section 3 is devoted to existence and uniqueness of ($L^2$-)solutions for (irAC). Furthermore, Section 4 concerns the convergence of solutions to equilibria as $t \to +\infty$, and then, in Section 5, we shall discuss Lyapunov stability of equilibria. Finally, we shall close this paper with a brief discussion on related topics, mainly on the existence of global attractors, in Section 6. All the results presented below and their proofs are reported in [3, 4]. So we refer the reader to these papers for more details.

# 2 Reformulations of the problem

First of all, let us note that, as will be shown below, the energy $t \mapsto \mathcal{F}(u(t))$ is nonincreasing along the evolution of each solution $t \mapsto u(t) := u(\cdot, t)$ (this property is shared with the classical Allen-Cahn equation (2)). However, this fact may not be straightforward from the equation (irAC), which is classified as a fully nonlinear (parabolic) equation and not presented in divergence form. However, through a certain reformulation of the equation, a gradient structure will (partially) recover and one can readily find out the nonincrease of the energy from the reformulated equation.

By applying the (multi-valued) inverse mapping $(\,\cdot\,)_+^{-1}$ of the positive part function $(\cdot)_+$,

$$(s)_+^{-1} = s + \partial I_{[0,\infty)}(s) \qquad \text{for} \quad s \in \mathbb{R}$$

(see below for more details) to both sides of (irAC), we readily derive an equivalent form (denoted by $(\text{irAC})_{\text{DN}}$), which is better fitted for energy methods:

$$u_t + \eta = \Delta u - u^3 + u, \qquad \eta \in \partial I_{[0,\infty)}(u_t) \quad \text{in } \Omega \times (0, \infty),$$
$$u = 0 \quad \text{on } \partial\Omega \times (0, \infty), \quad u|_{t=0} = u_0 \quad \text{in } \Omega,$$

where $\partial I_{[0,\infty)}$ denotes the subdifferential operator[2] of the indicator function $I_{[0,\infty)}$ over $[0,\infty)$, that is,

$$I_{[0,\infty)}(s) = \begin{cases} 0 & \text{if } s \geq 0 \\ +\infty & \text{otherwise,} \end{cases} \quad \partial I_{[0,\infty)}(s) = \begin{cases} \{0\} & \text{if } s > 0 \\ (-\infty, 0] & \text{if } s = 0 \\ \emptyset & \text{if } s < 0 \end{cases} \quad \text{for } s \in \mathbb{R}.$$

Here, evolution equations such as $(\mathrm{irAC})_{\mathrm{DN}}$ (namely, equations with nonlinear operators acting on $u_t$ as well as $u$) are often called *doubly nonlinear* evolution equations (see, e.g., [16, 9, 26, 25, 52, 29]).

Multiply the equation in $(\mathrm{irAC})_{\mathrm{DN}}$ by $u_t$ and integrate it over $\Omega$. Then applying a chain-rule for the functional derivative (see, e.g., [23]) to the right-hand side, we find that

$$\|u_t\|_{L^2}^2 + (\eta, u_t) = -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(u),$$

where $(\cdot,\cdot)$ stands for the inner product of $L^2(\Omega)$. Moreover, note that $\eta u_t \equiv 0$, since $\partial I_{[0,\infty)}(s) = \{0\}$ if $s > 0$ (otherwise, $s = 0$, and hence, the product is also zero). Therefore we obtain the same energy identity as that of (2),

$$\|u_t\|_{L^2}^2 = -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(u), \tag{3}$$

and the nondecrease of the energy $\mathcal{F}(u(t))$ follows. On the other hand, if we multiply the equation by $u$, then the additional term $(\eta, u)$ cannot be eliminated as above (also note that $\partial I_{[0,\infty)}$ is an unbounded operator), and then, it yields an explicit difference from (2).

**Remark 2.1.** (i) By comparison between (irAC) and $(\mathrm{irAC})_{\mathrm{DN}}$, one can readily find that

$$\eta = -\left(\Delta u - W'(u)\right)_-,$$

where $(\,\cdot\,)_- := \max\{-\,\cdot\,, 0\} \geq 0$.

(ii) Throughout this paper, we shall work in the $L^2(\Omega)$ framework, where both the Laplace operator $\Delta$ and the subdifferential operator $\partial I_{[0,\infty)}$ are unbounded. Another possibility would be an $H_0^1(\Omega)$ framework, where the Laplace operator turns out to be bounded from $H_0^1(\Omega)$ into its dual space. On the other hand, in the $H_0^1(\Omega)$ framework, it is more delicate to obtain the representation of the subdifferential of $I_{[0,\infty)}$ in $H_0^1(\Omega)$ in order to check the equivalence between (irAC) and $(\mathrm{irAC})_{\mathrm{DN}}$.

[2]The *subdifferential operator* $\partial\varphi : H \to 2^H$ of a proper ($\varphi \not\equiv +\infty$) lower semicontinuous convex functional $\varphi : H \to (-\infty, +\infty]$ on a Hilbert space $H$ is defined by $\partial\varphi(u) = \{\xi \in H \colon \varphi(v) - \varphi(u) \geq (\xi, v-u)_H$ for all $v \in H\}$, where $(\cdot,\cdot)_H$ is an inner product of $H$, with domain $D(\partial\varphi) = \{u \in D(\varphi) \colon \partial\varphi(u) \neq \emptyset\}$, where $D(\varphi) := \{u \in H \colon \varphi(u) < +\infty\}$ is called the *effective domain* of $\varphi$. Every subdifferential operator is cyclic monotone, and particularly, it is maximal monotone. We refer the reader to [23, 15] for more details.

We shall give another equivalent form (denoted by $(\mathrm{irAC})_{\mathrm{OP}}$) in terms of a parabolic obstacle problem as the third reformulation (see Theorem 4.4 below).

# 3 Existence and smoothing effect of solutions

In this section, we discuss existence and uniqueness of $L^2$-solutions. Let us start with recalling a definition of $L^2$ solution,

**Definition 3.1.** *A function $u \in C([0,\infty); L^2(\Omega))$ is said to be a* solution *(or an $L^2(\Omega)$-solution) of* (irAC) *(equivalently,* $(\mathrm{irAC})_{\mathrm{DN}}$*), if the following conditions are all satisfied:*

(i) *$u$ belongs to $W^{1,2}(\delta, T; L^2(\Omega))$, $C([\delta, T]; H_0^1(\Omega) \cap L^4(\Omega))$ and $L^2(\delta, T; H^2(\Omega) \cap L^6(\Omega))$ for any $0 < \delta < T < \infty$,*

(ii) *there exists $\eta \in L^\infty(0,\infty; L^2(\Omega))$ such that*

$$u_t + \eta - \Delta u + u^3 - u = 0, \quad \eta \in \partial I_{[0,\infty)}(u_t) \quad \text{for a.e. } (x,t) \in \Omega \times (0,\infty) \tag{4}$$

*and $\eta = -\big(\Delta u - u^3 + u\big)_-$ for a.e. $(x,t) \in \Omega \times (0,\infty)$,*

(iii) *$u(\cdot, 0) = u_0$ a.e. in $\Omega$.*

Now, our main result reads,

**Theorem 3.2** (Existence of $L^2$-solutions [3]). *For any $T > 0$, the following holds true:*

(i) *For $u_0 \in H^2(\Omega) \cap L^6(\Omega)$, there exists an $L^2$-solution $u = u(x,t)$ of* (irAC) *on $[0,T]$ such that $u \in W^{1,2}(0,T; H_0^1(\Omega)) \cap C_w([0,T]; H^2(\Omega) \cap L^6(\Omega))$, where $C_w([0,T]; X)$ denotes the space of weakly continuous functions on $[0,T]$ with values in a Banach space $X$.*

*For $r > 0$, define*

$$D_r := \Big\{ u \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^6(\Omega) \colon \|(\Delta u - u^3 + u)_-\|_{L^2}^2 \le r \Big\}.$$

(ii) *For $u_0 \in \overline{D_r}^{H_0^1 \cap L^4}$,* (irAC) *admits an $L^2$-solution*

$$u \in W^{1,2}_{loc}((0,T]; H_0^1(\Omega)) \cap C_w((0,T]; H^2(\Omega) \cap L^6(\Omega)) \cap C([0,T]; H_0^1(\Omega) \cap L^4(\Omega))$$

*satisfying $u(t) \in D_r$ for any $t > 0$.*

(iii) *For $u_0 \in \overline{D_r}^{L^2}$, there is an $L^2$-solution $u = u(x,t) \in C([0,T]; L^2(\Omega))$ of* (irAC) *such that*

$$u \in W^{1,2}_{loc}((0,T]; H^1_0(\Omega)) \cap C_w((0,T]; H^2(\Omega) \cap L^6(\Omega)) \cap C((0,T]; H^1_0(\Omega) \cap L^4(\Omega))$$

*and $u(t) \in D_r$ for any $t > 0$.*

Equation $(\mathrm{irAC})_{\mathrm{DN}}$ falls within the scope of an abstract theory developed by Arai [9], where existence of solution is proved for initial data lying on the domain of the operator $A : u \mapsto -\Delta u + W'(u)$ in $L^2(\Omega)$, that is, $D(A) = H^2(\Omega) \cap H^1_0(\Omega) \cap L^6(\Omega)$. So (i) follows immediately. On the other hand, (ii) and (iii) on *smoothing effect* come from a specific structure of $(\mathrm{irAC})_{\mathrm{DN}}$ and are not derived from the abstract theory. Here we reconstruct an existence result by inserting a specific structure of the equation $(\mathrm{irAC})_{\mathrm{DN}}$ to the framework of [9]. Moreover, we also derive energy estimates, which will be used to prove the convergence of solutions to equilibria and existence of global attractors, in parallel with constructing a solution. To this end, we approximate $(\mathrm{irAC})_{\mathrm{DN}}$ as an evolution equation on $H := L^2(\Omega)$,

$$\partial_t u_\lambda + \eta_\lambda + \partial\psi_\lambda(u_\lambda) = u_\lambda, \quad \eta_\lambda \in \partial I_{[0,\infty)}(\partial_t u_\lambda), \quad 0 < t < T, \quad u_\lambda(0) = u_0, \qquad (5)$$

where $\psi_\lambda$ is the so-called Moreau-Yosida regularization

$$\psi_\lambda(u) := \min_{v \in H} \left\{ \frac{1}{2\lambda} \|u - v\|_H^2 + \psi(v) \right\}$$

of the functional $\psi : H \to [0, +\infty]$ given by

$$\psi_\lambda(u) := \begin{cases} \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + \frac{1}{4} \int_\Omega |u|^4 \, dx & \text{if } u \in H^1_0(\Omega) \cap L^4(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Then the subdifferential $\partial\psi_\lambda$ of $\psi_\lambda$ coincides with the Yosida approximation of $\partial\psi$, and hence, it is Lipschitz continuous in $H$. Moreover, since the inverse mapping of $u \mapsto u + \partial I_{[0,\infty)}(u)$ (namely, the resolvent of $\partial I_{[0,\infty)}$) is nonexpansive (i.e., Lipschitz continuous with Lipschitz constant 1), for each $\lambda > 0$ the approximate problem above admits a (classical) solution $u_\lambda \in C^{1,1}([0,T]; H)$ such that $\eta_\lambda \in C^{0,1}([0,T]; H)$ and the equation holds on $[0,T]$ (hence, $\eta_\lambda(0)$ is well defined). Besides the energy estimate as in §2, we need to derive an estimate to control the section $\eta_\lambda$ of the unbounded operator $\partial I_{[0,\infty)}$ evaluated at $u_t$. To this end, differentiate (5) and multiply it by $\eta_\lambda$ to get

$$(\partial_t^2 u_\lambda, \eta_\lambda) + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\eta_\lambda\|_{L^2}^2 + \left( \frac{\mathrm{d}}{\mathrm{d}t} \partial\psi_\lambda(u_\lambda), \eta_\lambda \right) = (\partial_t u_\lambda, \eta_\lambda) = 0,$$

where the last equality follows as in §2. Therefore it suffices to show the nonnegativity of the first and third terms of the left-hand side, and then, one obtains

$$\sup_{t \in [0,T]} \|\eta_\lambda(t)\|_{L^2} \leq \|\eta_\lambda(0)\|_{L^2}.$$

Here we note that

$$\|\eta_\lambda(0)\|_{L^2} \approx \|(\Delta u_0 - u_0^3 + u_0)_-\|_{L^2}^2.$$

Hence, even though $u_0$ lies on a closure of $D_r$ (for some $r > 0$ fixed), then $\|\eta_\lambda(0)\|_{L^2}$ turns out to be still bounded for any $\lambda > 0$. Convergence of approximate solutions follows from standard weak compactness technique as well as the Aubin-Lions compactness lemma. Finally, the limits of nonlinear terms are identified by using the so-called Minty's trick (or demiclosedness of maximal monotone operators).

**Remark 3.3.** (i) The closure of $D_r$ in a space $X$ (e.g., $L^2(\Omega)$) may not coincide with $X$ itself (i.e., $\overline{D_r}^X \neq X$). In order to observe how smoothing effect occurs (in Theorem 3.2), let us consider the initial datum $u_0(x) = |x| - 1 \in H_0^1(-1,1)$ (with $N = 1$ and $\Omega = (-1,1)$). Then set $u_{0,\varepsilon} \in W^{2,\infty}(-1,1)$ by

$$u_{0,\varepsilon}(x) = \begin{cases} |x| - 1 & \text{if } |x| > \varepsilon, \\ \frac{1}{\varepsilon}\frac{x^2}{2} + \frac{\varepsilon}{2} - 1 & \text{if } |x| \le \varepsilon \end{cases}$$

for $\varepsilon > 0$. Then one observes that

$$u_{0,\varepsilon}'' - u_{0,\varepsilon}^3 + u_{0,\varepsilon} = \begin{cases} -u_{0,\varepsilon}^3 + u_{0,\varepsilon} & \text{if } |x| > \varepsilon, \\ \frac{1}{\varepsilon} \underbrace{-u_{0,\varepsilon}^3 + u_{0,\varepsilon}}_{\text{close to zero}} > 0 & \text{if } |x| \le \varepsilon \end{cases}$$

for $\varepsilon > 0$ enough small. Therefore

$$\|(u_{0,\varepsilon}'' - u_{0,\varepsilon}^3 + u_{0,\varepsilon})_-\|_2 \le \int_{|x|>\varepsilon} (u_{0,\varepsilon}^3 - u_{0,\varepsilon})^2 \, dx \le \|u_0^3 - u_0\|_{L^2}^2 =: r < +\infty.$$

Moreover, one can check that $u_{0,\varepsilon} \to u_0$ strongly in $H_0^1(-1,1)$. Hence $u_0$ belongs to the closure of $D_r$ in $H_0^1(-1,1)$. On the other hand, $u_0$ does not belong to $H^2(-1,1)$; indeed, $u_0'$ is not continuous at $x = 0$ but $H^2(-1,1)$ is embedded in $C^{1,\alpha}([-1,1])$ by Sobolev embeddings. Due to Theorem 3.2, the solution $u(\cdot,t)$ turns to lie on $H^2(-1,1) \subset C^{1,\alpha}([-1,1])$ for any $t > 0$. However, this observation may not be true if we change the sign of $u_0$.

(ii) Thanks to Theorem 3.2, we assure that the set $D_r$ (and its closures) is invariant under the flow generated by the solutions to (irAC). This fact will play a fundamental role in analysis of long-time behaviors of solutions (see §5 and §6 below).

Concerning uniqueness of solution, we have:

**Theorem 3.4** (Uniqueness of solution [3]). *It holds that*

(i) *Let $T > 0$ be fixed. For $N \leq 3$, solutions $u$ belonging to the class*

$$W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \tag{6}$$

*are uniquely determined for each initial data $u_0 \in H_0^1(\Omega)$.*

(ii) *Furthermore, for general $N$, bounded solutions belonging to $L^\infty(\Omega \times (0,T))$ as well as (6) are uniquely determined for each initial data $u_0 \in H_0^1(\Omega)$.*

In order to handle the nonlinearity of $u_t$, we subtract equations of two solutions and multiply the time-derivative of the difference of the two solutions (instead of the difference itself). Then the dimension restriction, $N \leq 3$ (then $H_0^1(\Omega)$ is continuously embedded in $L^6(\Omega)$), arises from the cubic nonlinearity of $u$. On the other hand, such a restriction can be removed for bounded solutions.

# 4 Convergence of solutions to equilibria

From now on, we assume $N \leq 3$ to guarantee the uniqueness of solution. The following theorem concerns the convergence of each solution $u = u(x,t)$ to an equilibrium $\phi = \phi(x)$ as $t \to \infty$.

**Theorem 4.1** (Convergence to equilibria [3]). *Let $u_0 \in \overline{D_r}^{H_0^1}$ and let $u = u(x,t)$ be the solution of* (irAC). *Then there exists $\phi \in H^2(\Omega) \cap L^6(\Omega) \cap H_0^1(\Omega)$ such that $\phi$ solves the following* $(\mathrm{E})_{\mathrm{OP}}$:

$$\partial I_{[u_0(x),\infty)}(\phi) \ni \Delta\phi - \phi^3 + \phi \quad \textit{in } \Omega.$$

Inclusion $(\mathrm{E})_{\mathrm{OP}}$ can be equivalently rewritten as the following elliptic obstacle problem,

$$\begin{gathered} \phi \geq u_0, \quad 0 \geq \Delta\phi - \phi^3 + \phi \ \text{ in } \Omega \times (0,\infty), \\ (\phi - u_0)\left(-\Delta\phi + \phi^3 - \phi\right) = 0 \ \text{ in } \Omega \times (0,\infty). \end{gathered}$$

**Remark 4.2.** (i) It is noteworthy that $(\mathrm{E})_{\mathrm{OP}}$ includes the initial datum $u_0$ in itself.

(ii) Formally replacing $u$ and $u_t$ by $\phi$ and $0$, respectively, in (irAC),

$$u_t + \partial I_{[0,\infty)}(u_t) - \Delta u + u^3 \ni u,$$

we see that

$$\partial I_{[0,\infty)}(0) - \Delta\phi + \phi^3 \ni \phi.$$

However, it is just a necessary condition for $(\mathrm{E})_{\mathrm{OP}}$ and it is not sufficient (indeed, $\partial I_{[0,\infty)}(0) = (-\infty, 0]$).

(iii) Since $\phi$ solves $(\mathrm{irAC})_{\mathrm{OP}}$, one may not expect classical regularity for $\phi$. Indeed, as for obstacle problems for the Dirichlet integral, minimizers lose $C^2$ regularity on their contact sets where the minimizers touch obstacle functions and $C^{1,1}$ is the optimal regularity (see [24]). On the other hand, a corresponding regularity issue for $(\mathrm{E})_{\mathrm{OP}}$ seems to be open.

Let us give a sketch of proof (for simplicity, we shall consider $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ only).

**Step 1 (Pre-compactness of orbits).** By performing an energy method along with the monotonicity of $t \mapsto u(x,t)$, we shall show that

**Lemma 4.3.** *The $\omega$-limit set of $u(x,t)$ is singleton, that is, there exists a limit $\phi$ such that*

$$u(\cdot,t) \to \phi \quad \textit{strongly in } H_0^1(\Omega) \cap L^4(\Omega), \qquad \textit{weakly in } H^2(\Omega) \cap L^6(\Omega)$$

*as $t \to \infty$.*

*Proof.* Indeed, since $\mathcal{F}$ is bounded from below, from the energy identity (3), we have

$$\int_0^\infty \|u_t\|_{L^2}^2 \,\mathrm{d}t - C \le \mathcal{F}(u_0).$$

Thus for each $n \in \mathbb{N}$ there exists $\tau_n \in [n, n+1]$ such that

$$u_t(\cdot,\tau_n) \to 0 \quad \text{strongly in } L^2(\Omega).$$

Then

$$\eta(\tau_n) - \Delta u(\tau_n) + W'(u(\tau_n)) = -u_t(\tau_n) \to 0 \quad \text{strongly in } L^2(\Omega),$$

where $\eta(\tau_n)$ is a section of $\partial I_{[0,\infty)}(u_t(\tau_n))$ as in (4). Formally, differentiate (irAC) in $t$ and test it by $\eta \in \partial I_{[0,\infty)}(u_t)$ to get

$$(u_{tt},\eta)_{L^2} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|_{L^2}^2 + (-\Delta u_t,\eta)_{L^2} + \int_\Omega W''(u)u_t\eta \,\mathrm{d}x = 0.$$

Here we note that, by using a chain-rule for subdifferential,

$$(u_{tt},\eta)_{L^2} = \frac{\mathrm{d}}{\mathrm{d}t} I_{[0,\infty)}(u_t) = 0$$

and also that

$$(-\Delta u_t,\eta)_{L^2} \ge 0 \quad \text{and} \quad u_t\eta = 0.$$

Therefore we deduce that

$$\|\eta(\cdot,t)\|_{L^2} \le \|\eta(\cdot,0)\|_{L^2} = \|(\Delta u_0 - W'(u_0))_-\|_{L^2}.$$

Furthermore, test (irAC) by $(-\Delta u + W'(u))_t$. Then it follows that

$$\big(u_t, (-\Delta u + W'(u))_t\big)_{L^2} + \big(\eta, (-\Delta u + W'(u))_t\big)_{L^2} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\| -\Delta u + W'(u)\|_{L^2}^2 = 0.$$

Note that

$$\big(\eta, (-\Delta u + W'(u))_t\big)_{L^2} = \big(\eta, -\Delta u_t + W''(u)u_t\big)_{L^2} \geq 0$$

and

$$\big(u_t, (-\Delta u + W'(u))_t\big)_{L^2} = \|\nabla u_t\|_{L^2}^2 + 3\int_\Omega u^2 u_t^2 \,\mathrm{d}x - \|u_t\|_{L^2}^2.$$

Hence

$$\|\nabla u_t\|_{L^2}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\| -\Delta u + W'(u)\|_{L^2}^2 \leq \|u_t\|_{L^2}^2 \overset{(3)}{=} -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(u(t)).$$

Thus we conclude that

$$\begin{aligned}\int_0^t \|\nabla u_t(\tau)\|_{L^2}^2 \,\mathrm{d}\tau + \frac{1}{2}\| -\Delta u(t) + W'(u(t))\|_{L^2}^2 + \mathcal{F}(u(t)) \\ \leq \frac{1}{2}\| -\Delta u_0 + W'(u_0)\|_{L^2}^2 + \mathcal{F}(u_0).\end{aligned}$$

Note that

$$\begin{aligned}\| -\Delta u + u^3 - u\|_{L^2}^2 = \| -\Delta u\|_{L^2}^2 + \|u^3\|_{L^2}^2 + \|u\|_{L^2}^2 \\ + 2(-\Delta u, u^3)_{L^2} - \|\nabla u\|_{L^2}^2 - \|u\|_{L^4}^4.\end{aligned}$$

Therefore $u(\cdot, t)$ is uniformly bounded in $H^2(\Omega) \cap L^6(\Omega)$ as $t \to \infty$. Consequently, we obtain

$$\begin{aligned} u(\cdot, \tau_n) \to \phi \quad & \text{weakly in } H^2(\Omega) \cap L^6(\Omega), \\ & \text{strongly in } H_0^1(\Omega) \cap L^4(\Omega), \\ \eta(\cdot, \tau_n) \to \eta_* \quad & \text{weakly in } L^2(\Omega), \end{aligned}$$

and hence,

$$\eta_* - \Delta\phi + \phi^3 - \phi = 0 \quad \text{in } L^2(\Omega).$$

By $\eta \in \partial I_{[0,\infty)}(u_t)$ along with the demiclosedness of maximal monotone operators, one can identify the limit,

$$\eta_* \in \partial I_{[0,\infty)}(0).$$

Consequently,

$$\partial I_{[0,\infty)}(0) - \Delta\phi + W'(\phi) \ni 0 \quad \text{in } L^2(\Omega), \quad \phi \in H_0^1(\Omega).$$

We emphasize again that the relation above is not sufficient to derive $(\mathrm{E})_{\mathrm{OP}}$, although it corresponds to a formal stationary problem for (irAC) (see Remark 4.2).

Furthermore, due to the uniform boundedness of $u(\cdot,t)$ in $H^2(\Omega)\cap L^6(\Omega)$,

$$u(\cdot,t)\to\phi \quad \begin{array}{l}\text{weakly in } H^2(\Omega)\cap L^6(\Omega),\\ \text{strongly in } H^1_0(\Omega)\cap L^4(\Omega) \quad \text{as } t\to\infty.\end{array}$$

Indeed, as the evolution $t\mapsto u(x,t)$ is nondecreasing by $u_t\geq 0$, the limit $\phi$ must be unique and independent of the choice of subsequence. □

Now, we move on to

**Step 2 (Further identification of the limit).** We claim that

$$\eta_*\in\partial I_{[u_0(x),\infty)}(\phi).$$

Then one can conclude that

$$\partial I_{[u_0(x),\infty)}(\phi)-\Delta\phi+W'(\phi)\ni 0 \ \text{ in } L^2(\Omega), \quad \phi\in H^1_0(\Omega).$$

To this end, we shall reformulate (irAC) as a parabolic obstacle problem, which is the following third reformulation of (irAC).

**Theorem 4.4** (Reformulation by an obstacle problem [3]). *Let $u_0\in\overline{D_r}^{L^2}$, (irAC)$_{\mathrm{DN}}$ admits an $L^2$-solution $u=u(x,t)$ which also solves the following Cauchy-Dirichlet problem (denoted by (irAC)$_{\mathrm{OP}}$):*

$$\begin{gathered} u_t+\eta=\Delta u-u^3+u, \quad \eta\in\partial I_{[u_0(x),\infty)}(u) \ \text{ in } \Omega\times(0,\infty),\\ u=0 \ \text{ on } \partial\Omega\times(0,\infty), \quad u|_{t=0}=u_0 \ \text{ in } \Omega,\end{gathered}$$

*where $\partial I_{[u_0(x),\infty)}$ stands for the subdifferential operator of the indicator function $I_{[u_0(x),\infty)}$ over $[u_0(x),+\infty)$. In particular, if the solution to (irAC) is unique, then (irAC) is equivalent to (irAC)$_{\mathrm{OP}}$.*

Problem (irAC)$_{\mathrm{OP}}$ is equivalent to the following obstacle problem:

$$\begin{gathered} u\geq u_0, \quad u_t\geq \Delta u-u^3+u \ \text{ in } \Omega\times(0,\infty),\\ (u-u_0)\left(u_t-\Delta u+u^3-u\right)=0 \ \text{ in } \Omega\times(0,\infty).\end{gathered}$$

**Remark 4.5.** Parabolic obstacle problems whose obstacle functions coincide with initial data also appear in the study of an optimal stopping time problem related to American options (see, e.g., [42] and references therein).

To give a rigorous proof, we need an involved (measure theoretic) argument to overcome some difficulty arising from the lack of smoothness in the $L^2$ framework. Instead, we shall give a more intuitive argument to formally derive the theorem stated above.

*Idea of proof.* Let $\eta \in \partial I_{[0,\infty)}(u_t)$ and suppose that $u$ and $\eta$ are sufficiently smooth. Then we claim that

$$(\star) \quad \text{the region } \{x \in \Omega \colon \eta(x,t) = 0\} \text{ is expanding in } t.$$

**Lemma 4.6** (Nondecreasing of $\eta \in \partial I_{[0,\infty)}(u_t)$ in time). *Let $u$ be a solution of* $(\mathrm{irAC})_{\mathrm{DN}}$ *and let $\eta \in \partial I_{[0,\infty)}(u_t)$. Then $\eta(x,t) = -(\Delta u - W'(u))_-$ is nondecreasing in $t$ for a.e. $x \in \Omega$.*

To prove the lemma above, differentiate $(\mathrm{irAC})_{\mathrm{DN}}$ in $t$,

$$u_{tt} + \eta_t - \Delta u_t + W'(u)u_t = 0$$

and test it by $\rho\eta \in \partial I_{[0,\infty)}(u_t)$ with $\rho \in C_0^\infty(\Omega)$, $\rho \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} I_{[0,\infty)}(u_t) + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\eta|^2 \rho + (-\Delta u_t, \rho\eta) + \int_\Omega W'(u)u_t\eta\rho = 0,$$

which implies $t \mapsto \eta(x,t)$ is nondecreasing. Thus the assertion follows.

Let $(x_0, t_0) \in \Omega \times (0,T)$. We have the following alternative:

- In case $u(x_0,t_0) = u_0(x)$, we have

  $$\partial I_{[u_0(x),\infty)}(u(x_0,t_0)) = (-\infty, 0].$$

  Hence it is easy to see that $\eta \in \partial I_{[u_0(x),\infty)}(u)$ at $(x_0,t_0)$.
- In case $u(x_0,t_0) > u_0(x)$, it yields that $\partial I_{[u_0(x),\infty)}(u(x_0,t_0)) = \{0\}$.

  On the other hand, it follows that

  $$\exists t_1 \in [0,t_0] \quad u_t(x_0,t_1) > 0\,; \text{ hence } \eta(x_0,t_1) = 0.$$

  By $(\star)$, $\eta(x_0,t) = 0$ for all $t \geq t_1$. In particular, $\eta(x_0,t_0) = 0$. Thus $\eta \in \partial I_{[u_0(x),\infty)}(u)$ at $(x_0,t_0)$.

□

We have already shown that

$$\eta(\cdot,t) \in \partial I_{[u_0(x),\infty)}(u(\cdot,t))$$

and

$$\begin{aligned} \eta(\cdot,\tau_n) &\to \eta_* && \text{weakly in } L^2(\Omega),\\ u(\cdot,\tau_n) &\to \phi && \text{strongly in } H_0^1(\Omega) \cap L^4(\Omega). \end{aligned}$$

Therefore thanks to the demiclosedness of $\partial I_{[u_0(x),\infty)}$, we conclude that

$$\eta_* \in \partial I_{[u_0(x),\infty)}(\phi),$$

and hence, $\phi$ solves $(\mathrm{E})_{\mathrm{OP}}$.

# 5 Lyapunov stability of equilibria

This section is devoted to discussing the Lyapunov stability of equilibria. Here it is noteworthy that each equilibrium $\phi$ of (irAC) may be an accumulation point of the set of equilibria,

$$\mathcal{S} := \bigcup \left\{ \mathcal{S}(u_0) \colon u_0 \in \overline{D_r}^{H_0^1} \right\},$$

where $\mathcal{S}(u_0)$ stands for the set of solutions for $(\mathrm{E})_{\mathrm{OP}}$ associated with the obstacle function $u_0$ (see Corollary 5.3 below). So there is no hope of proving asymptotic stability (in Lyapunov's sense) of such non-isolated equilibria. Indeed, every non-isolated equilibrium $\phi$ has different equilibria in its arbitrarily small neighbourhood and solutions emanating from such neighboring equilibria never move (and hence, they never converge to $\phi$). On the other hand, one may expect Lyapunov stability of non-isolated equilibria. Stability analysis of non-isolated equilibria poses a major difficulty even for gradient flows. In the case of gradient flows associated with smooth energy functionals, one may apply the so-called Łojasiewicz-Simon inequality (cf. see [36]). For instance, a stability issue on non-isolated asymptotic profiles of vanishing solutions to fast diffusion equations is revealed by exploiting a Łojasiewicz-Simon inequality in [2]. On the other hand, in the case of (irAC) and equivalent equations, these gradient-like systems are associated with *nonsmooth* functionals, and therefore, there arises a fatal problem to apply techniques based on Łojasiewicz-Simon type inequalities, which essentially require the smoothness of functionals.

Recall again that $\overline{D_r}^{H_0^1}$ is invariant under the flow generated by the solutions to (irAC) (see Theorem 3.2). Now, our result reads,

**Theorem 5.1** (Lyapunov stability of equilibria [4]). *Let $\phi$ be a solution of* $(\mathrm{E})_{\mathrm{OP}}$ *for some $u_0 \in \overline{D_r}^{H_0^1}$. Suppose that $\phi$ lies on a small (in $H_0^1(\Omega)$) neighbourhood of the positive ground state $\phi_{ac}$ of the classical elliptic Allen-Cahn equation (i.e., $\mathcal{F}(\phi_{ac}) = \min_{H_0^1(\Omega)} \mathcal{F}$ ). Then $\phi$ is (Lyapunov) stable, that is, for any $\varepsilon > 0$ there exists $\delta > 0$ such that any solution $u = u(x,t)$ of* (irAC) *satisfies*

$$\sup_{t\in[0,+\infty)} \|u(t) - \phi\|_{H_0^1(\Omega)} < \varepsilon,$$

*whenever $\|u(0) - \phi\|_{H_0^1(\Omega)} < \delta$ and $u(0) \in \overline{D_r}^{H_0^1}$.*

Instead of Łojasiewicz-Simon type inequalities, we shall employ the monotone evolution of solutions as well as variational convergence of a functional corresponding to $(\mathrm{irAC})_{\mathrm{OP}}$ with respect to limiting a sequence of obstacle functions (see [12]). Let us give an outline of proof below.

*Outline of proof.* We first prove that equilibria $\phi$ of (irAC) in a small neighbourhood $V$ (in $H_0^1(\Omega)$) of $\phi_{\mathrm{ac}}$ entail

$$\phi = \arg\min_{H_0^1(\Omega)} \left( \mathcal{F} + I_{[\,\cdot\, \geq \phi]} \right),$$

which means $\mathcal{F}(\phi) = \min_{w \in H_0^1(\Omega)} \left(\mathcal{F}(w) + I_{[\,\cdot\, \geq \phi]}(w)\right)$. If $\|u_0 - \phi\|_{H_0^1(\Omega)} \ll 1$, then

- $u(\cdot, t) \in V$ for all $t \geq 0$,
- the limit $\hat{\phi} \in V$ of $u(\cdot, t)$ as $t \to \infty$ is also characterized as

$$\hat{\phi} = \arg \min_{H_0^1(\Omega)} \left(\mathcal{F} + I_{[\,\cdot\, \geq u_0]}\right).$$

We next claim that

**Lemma 5.2** (Convergence of minimizers). *If $w_n \to w$ strongly in $H_0^1(\Omega)$, then minimizers $\phi_n$ of $\mathcal{F} + I_{[\,\cdot\, \geq w_n]}$ converge to a minimizer $\phi$ of $\mathcal{F} + I_{[\,\cdot\, \geq w]}$ strongly in $H_0^1(\Omega)$.*

Then one can deduce that $\|\phi - \hat{\phi}\|_{H_0^1(\Omega)} \ll 1$. We find that, by the nondecrease of the evolution $t \mapsto u(x, t)$,

$$u_0(x) - \phi(x) \leq u(x, t) - \phi(x) \leq \hat{\phi}(x) - \phi(x),$$

whence

$$\sup_{t \geq 0} \|u(\cdot, t) - \phi\|_p \ll 1, \quad 1 \leq p \leq 6.$$

By performing an energy argument, one may improve the topology of the neighbourhood and finally conclude that

$$\sup_{t \geq 0} \|u(\cdot, t) - \phi\|_{H_0^1(\Omega)} \leq \varepsilon.$$

Indeed, subtracting equations,

$$\begin{aligned} u_t + \eta - \Delta u + u^3 - u = 0, \quad & \eta \in \partial I_{[0,\infty)}(u_t), \\ \zeta - \Delta \phi + \phi^3 - \phi = 0, \quad & \zeta \in \partial I_{[u_0(x),\infty)}(\phi), \end{aligned}$$

setting $w := u - \phi$ and testing it by $w_t = u_t$, we obtain

$$\|u_t\|_{L^2}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w\|_{L^2}^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{4}\|u\|_{L^4}^4 - (\phi^3, w)_{L^2}\right) \leq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2.$$

Here we used $(\eta, w_t)_{L^2} = 0$ and $(-\zeta, u_t)_{L^2} \geq 0$. Thus

$$\begin{aligned} &\frac{1}{2}\|\nabla w(t)\|_{L^2}^2 + \frac{1}{4}\|u(t)\|_{L^4}^4 - (\psi^3, w(t)) - \frac{1}{2}\|w(t)\|_{L^2}^2 \\ &\leq \frac{1}{2}\|\nabla w(0)\|_{L^2}^2 + \frac{1}{4}\|u(0)\|_{L^4}^4 - (\psi^3, w(0)) - \frac{1}{2}\|w(0)\|_{L^2}^2. \end{aligned}$$

Moreover, note that

$$
\begin{aligned}
&\|u(0)\|_{L^4}^4 - \|u(t)\|_{L^4}^4 \\
&\leq \int_\Omega \left(|u(x,0)|^2 + |u(x,t)|^2\right) \\
&\qquad \times \left(|u(x,0)| + |u(x,t)|\right)|u(x,t) - u(x,0)| \, \mathrm{d}x \\
&\leq \int_\Omega \left(|u(x,0)|^2 + |u(x,t)|^2\right) \\
&\qquad \times \left(|u(x,0)| + |u(x,t)|\right) \left(|w(x,t)| + |w(x,0)|\right) \, \mathrm{d}x \\
&\leq C \left(\|w(t)\|_{L^4} + \|w(0)\|_{L^4}\right).
\end{aligned}
$$

Accordingly, we conclude that

$$\|\nabla w(t)\|_{L^2}^2 \leq \omega \left(\|w(t)\|_{L^4} + \|w(0)\|_{L^4} + \|\nabla w(0)\|_{L^2}\right)$$

with a modulus $\omega(s) \to 0$ as $s \to 0_+$. □

As a by-product of the proof above, one can assure that

**Corollary 5.3.** *Every solution of* $(\mathrm{E})_{\mathrm{OP}}$ *in a small neighbourhood* $V$ *of* $\phi_{ac}$ *is an accumulation point of the set* $\mathcal{S}$.

*Proof.* Let $\phi \in V$ be a solution of $(\mathrm{E})_{\mathrm{OP}}$ (with $u_0$). Then it also minimizes the functional,

$$\mathcal{F} + I_{[\,\cdot\, \geq \phi]}$$

over $H_0^1(\Omega)$. One can take a sequence $(u_{0,n})$ in $H^2(\Omega) \cap H_0^1(\Omega)$ of initial data such that $u_{0,n}(x)$ is strictly decreasing at each $x \in \Omega$ and $u_{0,n}$ converges to $\phi$ strongly in $H_0^1(\Omega)$. Then the positive minimizer $\phi_n$ of $\mathcal{F} + I_{[\,\cdot\, \geq u_{0,n}]}$ belongs to $\mathcal{S}$ and converges to $\phi$ strongly in $H_0^1(\Omega)$ by Lemma 5.2. On the other hand, since $u_{0,n} > \phi$, the minimizer $\phi_n$ is greater than $\phi$, and hence, $\phi \neq \phi_n$. Hence $\phi$ is an accumulation point of $\mathcal{S}$. □

# 6 Other related topics

In contrast with (2), one cannot expect existence of global attractors in any $L^p(\Omega)$ spaces for any $1 \leq p \leq \infty$. Indeed, if one takes nonnegative data $u_0 \in L^p(\Omega)$, then by means of the nondecrease of $u(x,t)$ in $t$, one observes that

$$\|u_0\|_{L^p(\Omega)} \leq \|u(t)\|_{L^p(\Omega)} \quad \text{for} \quad t \geq 0.$$

Hence, one cannot construct any absorbing set in (the whole of) $L^p$ spaces. On the other hand, if the dynamical system (DS for short) generated by (irAC) possesses a global attractor $\mathcal{A}$, then an $\varepsilon$-neighbourhood of $\mathcal{A}$ must be an absorbing set. Therefore there is no global attractor for (irAC) in (the whole of) any $L^p$ spaces. On the other hand, setting $D_r$ given by §3 to the phase set of the DS, one can construct a global attractor. It indicates that due to the nondecreasing constraint, on some *part* of domain, a solution never evolves, and hence, energy dissipation may not emerge in a usual sense. However, on the other part of domain, the solution behaves like those of (2), and hence, the DS exhibits *partial* energy dissipation. The set $D_r$ plays a role of mask to conceal the portion of domain where the solution never evolves. Therefore working on the masked set, one can find out energy dissipation and construct a global attractor.

Set $D = D_r$ for arbitrary $r > 0$ and set a metric $\mathrm{d}(\cdot, \cdot)$ by

$$\mathrm{d}(u, v) := \|u - v\|_{H_0^1(\Omega)} + \|u - v\|_{L^4(\Omega)} \quad \text{for } u, v \in D.$$

Let $(S_t)$ be the semigroup generated by (irAC). Here $\mathcal{A}$ is called a *global attractor* if the following (i)–(iii) hold true:

(i) $\mathcal{A}$ is *compact* in $(D, \mathrm{d})$.

(ii) $\mathcal{A}$ satisfies an *attraction property* in $(D, \mathrm{d})$, that is, let $B \subset D$ be a d-bounded subset of $D$. Then for any neighborhood $\mathcal{O}$ of $\mathcal{A}$ in $(D, \mathrm{d})$, there exists $\tau_{\mathcal{O}} \geq 0$ such that $S_t B \subset \mathcal{O}$ for all $t \geq \tau_{\mathcal{O}}$.

(iii) $\mathcal{A}$ is *strictly invariant*, i.e., for any $t \geq 0$ it holds that $S_t \mathcal{A} = \mathcal{A}$.

**Theorem 6.1** (Existence of global attractor [3]). *Let $N \leq 3$. Then the DS $(S_t, (D_r, \mathrm{d}))$ generated by* (irAC) *admits a global attractor $\mathcal{A}$.*

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