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Maximizers of a Trudinger-Moser-type inequality with the critical growth on the whole plane

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In this talk, I reported recent progress in the joint work with Slim Ibrahim, Nader Masmoudi and Feberica Sani. The Trudinger-Moser inequality on $\mathbb{R}^2$:

$$\sup_{u\in H_0^1(\Omega),\ \|\nabla u\|_2^2\leq 2\pi}\int_{\Omega}e^{2u^2}dx\leq C|\Omega| \tag{1}$$

is a celebrated substitute for the forbidden Sobolev embedding $H^1(\mathbb{R}^2)\not\subset L^{\infty}(\mathbb{R}^2)$ for bounded domains. Henceforth $\|\varphi\|_p$ denotes the L_x^p norm. Moser [6] proved the critical case $\|\nabla u\|_2^2=2\pi$, as well as the optimality of the growth e^{2u^2} as $u\rightarrow\infty$, and loss of compactness by concentration for some bounded sequences in $H^1(\mathbb{R}^2)$. Carleson and Chang [2] proved however that the concentration cannot occur for maximizing sequences of the inequality, hence existence of maximizers.

We consider a similar question on the whole plane $\mathbb{R}^2$. Cao [1] proved a variant of (1) on $\mathbb{R}^2$ in the subcritical case:

$$\alpha<2\pi\implies\sup_{\|\nabla u\|_2^2\leq\alpha}\int_{\mathbb{R}^2}(e^{2u^2}-1)dx\leq C_{\alpha}\|u\|_2^2. \tag{2}$$

It is indeed a natural formulation for the nonlinear Schrödinger equation

$$i\dot{u}-\Delta u=f'(u) \tag{3}$$

or similar Klein-Gordon equation on $\mathbb{R}^{1+2}$, where L^2-invariance plays crucial roles in the dynamics, cf. [3]. It is easy to observe by Moser's sequence of functions [6] that the critical exponent $\alpha=2\pi$ is prohibited on $\mathbb{R}^2$. Motivated by the dynamical problems, the optimal version of (2) was obtained in [5]:

Theorem 1. *Let $f:\mathbb{R}\rightarrow[0,\infty)$ be any continuous function. Then* (i)

$$S(f):=\sup_{\|\nabla u\|_2^2\leq 2\pi}\|u\|_2^{-2}\int_{\mathbb{R}^2}f(u)dx \tag{4}$$

is finite if and only if

$$\limsup_{|u|\rightarrow\infty}f(u)\frac{|u|^2}{e^{2|u|^2}}+\limsup_{|u|\rightarrow 0}\frac{f(u)}{|u|^2}<\infty. \tag{5}$$

(ii) *For every sequence of radial functions u_n satisfying $\|\nabla u_n\|_2^2\leq 2\pi$ and weakly convergent in $H^1(\mathbb{R}^2)$, $f(u_n)$ is strongly convergent in $L^1(\mathbb{R}^2)$, if and only if*

$$\lim_{|u|\rightarrow\infty}|f(u)|\frac{|u|^2}{e^{2|u|^2}}+\lim_{|u|\rightarrow 0}\frac{f(u)}{|u|^2}=0. \tag{6}$$

If f is smooth and $S(f)$ is attained, then each maximizer $u\in H^1(\mathbb{R}^2)$ solves the static nonlinear Schrödinger equation:

$$-\Delta u+\omega u=f'(u) \tag{7}$$

with $\omega = 2S(f)$. The above compactness (ii) implies that $S(f)$ is attained if (6) holds. Note that the decay for $|u| \to 0$ can generally be assumed after subtracting the limit mass:

$$f(u) - u^2 \lim_{u \to 0} \frac{f(u)}{u^2}, \tag{8}$$

at least for smooth f. Hence the vanishing (or spreading) phenomenon is not a compactness issue in our setting. On the other hand, if $f(u)$ grows faster than the critical one, namely

$$\forall \varepsilon > 0, \quad f(u) = o(e^{(2+\varepsilon)|u|^2}), \quad \limsup_{|u| \to \infty} f(u) \frac{|u|^2}{e^{2|u|^2}} = \infty, \tag{9}$$

then for every $0 < \omega < \infty$, $\omega = 2\|u\|_2^{-2} \int f(u)dx$ is reached with a subcritical kinetic energy $\|\nabla u\|_2^2 < 2\pi$, hence (7) has a solution by the above compactness result. Then essentially the only remaining case is the exactly critical growth:

$$f(u) \sim \frac{e^{2|u|^2}}{|u|^2} \quad (|u| \to \infty). \tag{10}$$

It turned out that the right hand side is a sharp threshold also for the existence of maximizers, which differs from the original Trudinger-Moser inequality (1). To state it more precisely, let $f_h^* : \mathbb{R} \to [0, \infty)$ be the model function with a cut-off parameter $h > 0$, defined by

$$f_h^*(u) = \begin{cases} |u|^{-2} e^{2u^2} & (|u| > h), \\ 0 & (|u| \le h). \end{cases} \tag{11}$$

Theorem 2. *Let $\delta > 0$ and let $f : \mathbb{R} \to [0, \infty)$ be a continuous function satisfying*

$$\lim_{|u| \to \infty} f(u)/f_1^*(u) = 1, \quad \lim_{|u| \to 0} f(u)/u^2 = m \in [0, \infty). \tag{12}$$

(i) *$S(f)$ is attained if f satisfies, for some $h > 0$,*

$$f(u) \ge f_h^*(u)(1 + \delta u^{-2}). \tag{13}$$

(ii) *$S(f)$ is not attained if f satisfies, for some sufficiently large $h \gg 1$,*

$$f(u) \le f_h^*(u)(1 - \delta u^{-2}) + mu^2. \tag{14}$$

(iii) *$S(f)$ is attained if f satisfies, for some $h > 0$,*

$$f(u) \ge f_h^*(u) + mu^2 + \delta u^4. \tag{15}$$

(iv) *$S(f)$ is not attained if f satisfies, for some sufficiently large $h \gg 1$,*

$$f(u) \le f_h^*(u) + mu^2 - \delta u^4. \tag{16}$$

In other words, the existence is determined by the sign of small order perturbations around the critical nonlinearity $f_h^*(u)$. Note that for the original Trudinger-Moser inequality (1) the existence of maximizer is stable for small perturbations.

Our proof of Theorem 2 is inspired by [2], based on asymptotic expansion of $S(f)$ along concentration. More precisely, we introduce another height parameter $H > 0$ as follows. Restricting to radially symmetric decreasing $u \in H^1(\mathbb{R}^2)$, let

$$\begin{aligned} &X_H := \{u \mid \exists R > 0, \ \|\nabla u\|^2_{L^2(|x|<R)} \le \pi, \ \|\nabla u\|^2_{L^2(|x|>R)} \le \pi, \ u(R) = H\}, \\ &S_H(f) := \sup_{u \in X_H} \|u\|_2^{-2} \int_{\mathbb{R}^2} f(u)dx. \end{aligned} \tag{17}$$

Theorem 2 follows from the asymptotic expansion as $h \to \infty$ of

$$S_h(f_h^*) = 2e^{-\gamma} + O(h^{-4}), \tag{18}$$

where γ denotes Euler's constant. The vanishing of the order $O(h^{-2})$ is the special property of f_h^* (besides the critical growth for Theorem 1), which allows the small order perturbations to determine the existence in Theorem 2.

We also need asymptotic expansion of an exponential radial Sobolev inequality:

$$\begin{aligned} &\inf\{\|u\|^2_{L^2(|x|>1)}; u(1) = h, \ \|\nabla u\|^2_{L^2(|x|>1)} \le 2\pi\} \\ &\qquad = \frac{\pi e^{2h^2+\gamma-1}}{4h^2}\left\{1 + \frac{1}{2h^2} + O(h^{-4})\right\}, \end{aligned} \tag{19}$$

whose growth order was obtained in [5] to prove Theorem 1. It is noteworthy that $S(f) < \infty$ can be derived from the original Trudinger-Moser inequality (1) using the above inequality, while (1) can be derived from Theorem 1 using the Hardy-type inequality:

$$0 < r < R \implies |u(r)| - |u(R)| \le \sqrt{\frac{1}{2\pi}\|\nabla u\|^2_{L^2(|x|<R)} \log(R/r)}. \tag{20}$$

At the time of the conference, we were missing a logic to connect the asymptotics of $S_H(f_H^*)$ with that of $S_H(f_h^*)$ for fixed $h \gg 1$ and $H \to \infty$, which requires us to control the "tail" left behind the concentration. Using carefully the above two inequalities, however, we are now able to prove that maximizers of $S_H(f_h^*)$ should almost optimize the height H in the exterior region $|x| > R$ and the nonlinear energy $f_H^*(u)$ in the interior region $|x| < R$.

A full proof will be published elsewhere for Theorem 2, as well as for (19).

References

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