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Extinction profile of solutions of the logarithmic diffusion equation on $\mathbb{R}$

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1 Introduction

This article is based on a joint work with Masahiko Shimojo (Oakayama Science University) and Peter Takáč (Rostock University). Our aim is to investigate the behavior of positive solutions for the logarithmic diffusion equation

$$u_t = (\log u)_{xx}, \quad x \in \mathbb{R}. \tag{1.1}$$

In particular, we are interested in the behavior of solutions which vanish in finite time. Before discussing the above equation, we describe our motivation to study the above equation.

Equation (1.1) is a special case of a more general nonlinear diffusion equation of the form

$$u_t = \nabla \cdot (u^{m-1}\nabla u), \qquad x \in \mathbb{R}^N, \tag{1.2}$$

which has been studied extensively in the last few decades. When $m = 1$, (1.2) is a linear heat equation.

$$u_t = \Delta u, \qquad x \in \mathbb{R}^N.$$

It is not hard to prove by using the representation formula of solutions that any positive L^1-solution behaves like a fundamental solution (Gaussian kernel) for large $t > 0$:

$$u(x,t) \sim \frac{C}{t^{N/2}} \exp(-\frac{|x|^2}{4t}) \qquad (t \simeq \infty).$$

where $C > 0$ is a constant depending on initial data. More precisely, the solution satisfies

$$t^{N/2}u(t^{1/2}x,t) \to C\exp(-|x|^2/4) \qquad (t \to \infty),$$

which gives an asymptotic rescaled profile of the solution.

When $m > 1$, (1.2) is called the porous media equation. Since the diffusion rate u^{m-1} is small if $u > 0$ is small, there may exist a solution with a natural free boundary of its support. An example of such a solution is the Barenblatt solutions (forward self-similar solutions) given by

$$u(x,t) = t^{-\alpha}\varphi(t^{-\beta}x),$$

where

$$\alpha = \frac{N}{2+N(m-1)}, \qquad \beta = \frac{1}{2+N(m-1)},$$
$$\varphi(y) = \left\{C - \frac{(m-1)\beta}{2}|y|^2\right\}_+^{1/(m-1)}.$$

It is known that for any compactly supported initial data, the solution of (1.2) converges to one of the Barenblatt solutions with the same total mass, that is,

$$t^{\alpha}u(t^{\beta}x,t) \to \varphi(x) \qquad (t \to \infty).$$

In other words, the Barenblatt solution plays the same role as the Gaussian solution for the linear heat equation in describing the asymptotic rescaled profile.

When $m < 1$, (1.2) is called a singular diffusion equation or a fast diffusion equation, because the diffusion rate u^{m-1} tends to ∞ as $u \to 0$. The behavior of solutions crucially depends on m and N, and the exponent

$$m_c := \frac{N-2}{N} \qquad (N \geq 3)$$

turns out to be critical. If $N \geq 3$ and $m > m_c$, diffusion is not fast enough so that for any positive initial data, the solution exists globally in time and remains positive for all $t > 0$. It is known that there exist forward self-similar solutions (Barenblatt-like solutions) given by

$$u(x,t) = t^{-\alpha}\varphi(t^{-\beta}|x|) \qquad (t > 0)$$

where

$$\alpha = \frac{N}{Nm-(N-2)} > 0, \qquad \beta = \frac{1}{Nm-(N-2)},$$
$$\varphi(y) := \left\{C + \frac{(1-m)\beta}{2}|y|^2\right\}^{-1/(1-m)}.$$

It was shown by Vázquez [29], Carrillo et al. [7, 8, 9], Doskalopoulos-Sesum [10] that any positive L^1 solution converges to one of the Barenblatt-like solution with the same total mass:

$$t^{\alpha}u(t^{\beta}x,t) \to \varphi(x) \qquad (t \to \infty).$$

On the other hand, if $N < 3$ or if $N \geq 3$ and $m < m_c$, then the diffusion is very fast so that a finite time extinction of solutions may occur [17, 18, 29]. More precisely, if the initial value is positive and decay to 0 as $|x| \to \infty$, then there exists $T < \infty$ such that the solution satisfies

$$\lim_{t \uparrow T} u(x,t) = 0$$

uniformly in $x \in \mathbb{R}$. An example of such a solution is the backward self-similar solutions (backward Barenblatt solutions) gien by

$$u(x,t) = (T-t)^{\alpha} \varphi((T-t)^{\beta} x), \qquad t \in (-\infty, T),$$

where

$$\alpha = \frac{N}{N(1-m)-2} > 0, \qquad \beta = \frac{1}{N(1-m)-2} > 0,$$

and

$$\varphi(y) = \left\{ C + \frac{\beta(1-m)}{2} |y|^2 \right\}^{-1/(1-m)}.$$

It was shown by Blanchet et al. [2], Bonforte et al. [3], Fila et al. [5, 6] that for a wide class of initial data, the solution converges to one of the backward Barenblatt solutions:

$$(T-t)^{-\alpha} u((T-t)^{-\beta} x, t) \to \varphi(x) \qquad (t \uparrow T).$$

Now let us consider the case where $m = 1$:

$$u_t = \Delta(\log u) \tag{1.3}$$

This equation has been studied in various contexts. It appears as the central limit approximation of Carleman's model of the Boltzman equation [23], and the expansion into a vacuum of a thermalized electron cloud [21]. It also arises as a model for long Van-der-Waals interactions in thin films of a liquid spreading on a solid surface, if certain nonlinear fourth order terms are neglected. There is a relation with the Ricci flow on $\mathbb{R}^2$ in differential geometry [13, 16], namely the evolution of a Riemannian metric $g_{ij}(\tau)$ given by $\partial_\tau g_{ij} = -2R_{ij}$. If the metric is conformal, there is a function w such that $g_{ij} = w\delta_{ij}$, where δ_{ij} denotes the standard Euclidean metric, and the problem is reduced to (1.3). For mathematical analysis of (1.3), see [10, 11, 19, 20, 24, 26, 27, 28].

For (1.3), the dimension $N \geq 3$ is the very fast diffusion case. This case is covered by the general results described as above, and the backward self-similar solution is written as

$$u(x,t) = (T-t)^{N/(N-2)} \varphi((T-t)^{1/(N-2)} x) \qquad t \in (-\infty, T),$$

where $T > 0$ is an extinction time and

$$\varphi(y) = \left\{ C + \frac{1}{2(N-2)}|y|^2 \right\}^{-1},$$

and any positive solution converges to one of the self-similar solutions:

$$(T-t)^{-N/(N-2)}u((T-t)^{-1/(N-2)}x,t) \to \varphi(x) \qquad (t \uparrow T).$$

If $N = 2$, by setting

$$u(x,t) := (T-t)\varphi(x),$$

we have

$$u_t = -\varphi, \qquad \Delta(\log u) = \Delta(\log \varphi).$$

Hence u satisfies the logarithmic diffusion equation, if

$$\Delta(\log \varphi) + \varphi = 0.$$

This equation has a positive radial entire solution for $N = 2$. Note that $w = \log \varphi$ satisfies the Gelfand equation $\Delta w + e^w = 0$. It was shown by Hsu [14] that if the initial value $u(x,0) = u_0(x)$ is positive, radially symmetric and $u_0(x) \to 0$ as $t \to \infty$, then

$$(T-t)^{-1}u(x,t) \to \varphi(x) \qquad (t \uparrow T).$$

For $N = 1$, Hsu [15] studied the problem

$$\begin{cases} u_t = (\log u)_{xx}, & & x \in \mathbb{R},\ t > 0, \\ \lim\limits_{x \to -\infty} (\log u)_x = \alpha, & \lim\limits_{x \to +\infty} (\log u)_x = -\beta, & t > 0, \\ u(x,0) = u_0(x), & & x \in \mathbb{R}, \end{cases} \tag{1.4}$$

where α, β are given positive constants corresponding to the decay rate of u as $x \to \pm\infty$. The initial value $u_0 : \mathbb{R} \to (0,\infty)$ at $t = 0$ is assumed to be in the function space $L^1(\mathbb{R})$ incorporating prescribed asymptotic conditions at $x = \pm\infty$. Then it was shown in [14] that the initial value problem (1.4) possesses a unique positive solution which satisfies (1.4) in the classical sense at least for small $t > 0$. Here and in what follows, the interval $[0,T)$ will denote the maximal time interval for the existence of the positive classical solution to (1.4).

Concerning the behavior of solutions of (1.4) near the extinction time, Hsu [15] proved that if $\alpha = \beta$ and $u_0(x)$ is even symmetric, then

$$(T-t)^{-1}u(x,t) \to \varphi(x) \qquad (t \uparrow T),$$

where φ is a positive solution of

$$\begin{cases} (\log\varphi)_{xx} + \varphi = 0, & x \in \mathbb{R}, \\ \dfrac{\varphi_x}{\varphi} \to \alpha > 0 & (x \to \pm\infty). \end{cases} \tag{1.5}$$

The aim of this article is to study the case where $\alpha \neq \beta$ or $u_0(x)$ not even symmetric. Our main result states that the rescaled solution $v(x,s) := e^s u(x, T - e^{-s})$ converges to a traveling pulse solution of

$$v_s = (\log v)_{xx} + v, \qquad x \in \mathbb{R}.$$

Theorem 1 *Let u be a solution of* (1.4). *There exists $\gamma \in \mathbb{R}$ such that*

$$v(x,t) = e^s u(x, T - e^{-s}) \to \varphi(x - cs - \gamma) \qquad (s \to \infty)$$

uniformly in $x \in \mathbb{R}$, where φ satisfies

$$-c\varphi' = (\log\varphi)'' + \varphi, \qquad z \in \mathbb{R},$$

with the asymptotic conditions

$$\lim_{z \to -\infty} \frac{\varphi'(z)}{\varphi(z)} = \alpha, \quad \lim_{z \to +\infty} \frac{\varphi'(z)}{\varphi(z)} = -\beta.$$

The rest of this paper is organized as follows. In Section 2 we describe some fundamental properties of solutions of the logarithmic diffusion equation, and introduce a transformation which is useful in studying the extinction behavior. In Section 3 we consider the existence of traveling solutions of the transformed equation. In Section 4 we sketch the proof of Theorem 1.

2 Fundamental properties

In this section we describe fundamental properties of solutions to (1.4). Integrating the equation in (1.4) over $\mathbb{R}$ and applying the asymptotic conditions at $x = \pm\infty$, we see that the total mass

$$m(t) := \int_{\mathbb{R}} u(x,t)dx$$

satisfies

$$\frac{d}{dt}m(t) = -(\alpha + \beta).$$

Hence the solution vanishes at

$$t = T := \frac{1}{\alpha+\beta}\int_{\mathbb{R}} u_0(x)dx. \tag{2.1}$$

Introducing the transformation of variables

$$u(x,t) = (T-t)v(x,s), \qquad t = T - e^{-s},$$

we have

$$\begin{cases} v_s = (\log v)_{xx} + v, & x \in \mathbb{R}, \quad s \in (-\log T, \infty) \\ (\log v)_x \to +\alpha > 0 & (x \to -\infty) \\ (\log v)_x \to -\beta < 0 & (x \to +\infty) \\ v(x, -\log T) = v_0(x) := \dfrac{1}{T}u_0(x) \end{cases} \tag{2.2}$$

Thus, in order to study the behavior of u as $t \to T$, we need to study the behavior of v as $s \to \infty$.

Integrating the equation in (2.2) and using the asymptotic conditions, we have

$$\frac{d}{ds}\int_{\mathbb{R}} v(x,s)dx = \int_{\mathbb{R}} v(x,s)dx - (\alpha+\beta).$$

Hence

$$\int_{\mathbb{R}} v_0(x)dx \begin{cases} > \alpha+\beta & \Rightarrow \text{total mass grows exponentially} \\ = \alpha+\beta & \Rightarrow \text{total mass is conserved} \\ < \alpha+\beta & \Rightarrow \text{total mass vanishes in finite time} \end{cases}$$

On the other hand, by (2.1) and

$$v(x, -\log T) = v_0(x) = \frac{1}{T}u_0(x),$$

we have

$$\int_{\mathbb{R}} v_0(x)dx = \alpha + \beta.$$

Hence the total mass of v is conserved.

3 Traveling solutions

When the mass is conserved in (2.2), one may expect that the solution converges to a stationary solution. Indeed, when $\alpha = \beta$, problem (1.4) admits a stationary solution and the solution converges to the stationary solution as is discussed in [10, 14]. However, when $\alpha \neq \beta$, there exist no positive stationary solutions, because any stationary solution must be even symmetric with respect to its critical point.

When $\alpha \neq \beta$, one possibility is the convergence to a traveling wave solution of the form $v(x,s) = \varphi(z+\gamma)$, $z = x - cs$, where c denotes the propagation speed, $\varphi > 0$ is the profile of the traveling wave and $\gamma \in \mathbb{R}$ is a phase shift. Substituting $v = \varphi(z)$ in (1.4), we see that φ must satisfy

$$\begin{cases} (\log\varphi)_{zz} + c\varphi_z + \varphi = 0, \quad z \in R \\ (\log\varphi)_z(-\infty) = \alpha, \quad (\log\varphi)_z(-\infty) = -\beta. \end{cases} \tag{3.1}$$

In order to fix the phase, we may require

$$\varphi_z(0) = 0.$$

The unique existence of such a traveling wave will be shown later, and it turns out that $c \gtreqqless 0$ if and only if $\alpha \gtreqqless \beta$. Therefore, for $\alpha = \beta$, the stationary solution can be regarded as a traveling wave solution with $c = 0$.

If we introduce an auxiliary variable $\psi := (\log\varphi)_z$, then (3.1) is equivalent to the system

$$\begin{cases} \varphi_z = \varphi\psi, \\ \psi_z = -c\varphi\psi - \varphi. \end{cases} \tag{3.2}$$

By the phase plane analysis, we can show that for each $\alpha, \beta > 0$, there exists a unique $c = c(\alpha, \beta)$ such that (3.2) has an orbit connecting $(0, \alpha)$ Moreover, the propagation speed of the traveling wave is monotone decreasing in α and monotone increasing in β, and satisfies satisfies

$$c \gtreqless 0 \Longleftrightarrow \alpha \gtreqless \beta$$

Stability of the traveling wave solution can be studied by considering the following linearized eigenvalue problem:

$$\begin{aligned} &\lambda U = (U/\varphi)_{zz} + cU_z + U, \quad z \in \mathbb{R}, \\ &(U/\varphi)_z \to 0 \quad \text{as } z \to \pm\infty. \end{aligned} \tag{3.3}$$

It is easy to see that $\lambda_0 = 1$ is an eigenvalue with the associated eigenfunction given by $U_0 = \varphi + c\varphi_z$. Similarly, $\lambda_1 = 0$ is an eigenvalue with the associated eigenfunction $U_0 = \varphi_z$, which corresponds to the spatial phase shift. Thus the traveling wave solution is unstable in the linearized sense. However, if the disturbance is in a class of mass conservation, then the traveling solution may be stable.

4 Outline of the proof of Theorem 1

We study the case of critical mass, and prove the convergence to a traveling solution.

STEP 1: We first establish the intersection number principle, namely, the number of intersection points of two distinct solutions is non-increasing in s.

The intersection number principle is not trivial for the fast diffusion equation, because we must exclude the possibility of appearance of intersection points from $\pm\infty$. The comparison principle follows immediately. In fact, we need a more precise result about "parabolic words".

STEP 2: We show the boundedness of v and v_x by the comparison argument. We also need lower estimates by the intersection number argument.

It should be noted that this property holds truw only in the case of mass conservation. In fact, if the mass is smaller than $\alpha+\beta$, then the solution vanishes in finite time, whereas if the mass is larger than $\alpha + \beta$, then the solution grows exponentially.

STEP 3: We show that the solution becomes unimodal in finite time.

To show this, we use the fact that t $v = e^s$ satisfies the equation in (2.2). By the boundedness of v and the intersection number principle, the solution must be unimodal eventually.

STEP 4: We show that the solution becomes log-concave in finite time.

To show this, we use the fact that

$$v(x,s) = \exp(Px + Q + s) \qquad (P,\ Q : \text{consttans})$$

satisfies the equation in (2.2). Then the intersection number principle implies that there exists $s_0 < \infty$ such that

$$(\log v)_{xx} < 0 \text{ for } x \in \mathbb{R},\ s > s_0.$$

Once the solution becomes log-concave, then the maximum principle implies that it preserves the log-concavity.

STEP 5: We derive an evolution equation of the curvature of v in terms v_x.

If the solution is log-concave, the flux $\theta := -(\log v)_x$ is increasing in $x \in \mathbb{R}$, which allows us to parametrize the solution by the flux density θ instead of the space variable x. If we can take $\theta \in (-\alpha, \beta)$ as a new independent space variable, the curvature

$$k(x,s) = -\frac{(\log v)_{xx}}{v} > 0$$

as a function of (θ, s) is described as

$$\begin{cases} k_s = k^2(vk_\theta)_\theta + k(k-1), \\ k_\theta = \dfrac{1-k}{\theta}, \quad \theta = -\alpha,\ \beta, \end{cases} \tag{4.1}$$

where $v = -\int_{-\alpha}^{\theta} \frac{\theta}{k} d\theta$.

STEP 6: We analyze the evolution equation of the curvature of solutions to show the convergence of the profile of v to φ.

Using a suitable Lyapunov function

$$\frac{d}{ds}\int_{-\alpha}^{\beta}\left(\frac{kv}{2}k_\theta^2 - \frac{1}{2}k^2 + k\right)d\theta = -\int_{-\psi_+}^{\psi_-}\frac{k_s^2}{k}d\theta < 0,$$

we can apply the standard argument for dynamical systems to show that k converges to to a stationary solution of (4.1) as $s \to \infty$. This stationary solution corresponds to a traveling wave in the original equation, which means that the asymptotic profile of the solution is given by the traveling wave.

STEP 7: We show the convergence to a traveling pulse.

The convergence of the solution of (2.2) to one of the traveling wave is shown by analyzing the crossing property.

We squeeze the solution by two traveling pulses $\varphi(x - cs - \gamma^1)$ and $\varphi(x - cs - \gamma^+)$. This proves the convergence of v to a traveling pulse $\varphi(x - cs - \gamma)$ by observing the crossing property between the two traveling solutions

For more details of the proof, we refer the reader to [25].

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