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Optimal Poincaré type trace inequalities on the Euclidean ball

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Abstract

The shape of extremal functions in Poincaré type trace inequalities for functions of bounded variation in the unit ball $\mathbb{B}^n$ of the n-dimensional Euclidean space $\mathbb{R}^n$ is discussed. Both customary and less standard normalization conditions are considered. The extremals in question turn out to take a different form, depending on the condition imposed. A key step in our analysis is a characterization of the sharp constants in the relevant trace inequalities in any admissible domain $\Omega \subset \mathbb{R}^n$, in terms of isoperimetric inequalities for subsets of Ω.

1 Introduction

The purpose of this note is to survey some results on Poincaré inequalities for the boundary trace of functions in the Eulidean ball from [Ma3] and [Ci3], as well to announce some recent developments on the same topic to appear in [CFNT2].

Assume that Ω is a domain, namely a bounded connected open set in $\mathbb{R}^n$, $n \geq 2$. It is well known that if the boundary $\partial\Omega$ of Ω is sufficiently regular, then a linear operator if defined on the space $BV(\Omega)$ of functions of bounded variation in Ω, which associates with any function $u \in BV(\Omega)$ its (suitably defined) boundary trace $\widetilde{u} \in L^1(\partial\Omega)$. Here, $L^1(\partial\Omega)$ denotes the Lebesgue

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space of integrable functions on $\partial\Omega$ with respect to the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$. Moreover, there exists a constant C, depending on Ω, such that

$$\inf_{c\in\mathbb{R}} \|\widetilde{u}-c\|_{L^1(\partial\Omega)} \leq C\|Du\|(\Omega) \tag{1.1}$$

for every $u \in BV(\Omega)$, where $\|Du\|(\Omega)$ stands for the total variation over Ω of distributional gradient Du of u [Ma3, Theorem 9.6.4].

A property of L^1 norms ensures that the infimum in (1.1) is attained when c agrees with a median of $\widetilde{u}$ on $\partial\Omega$, given by

$$\mathrm{med}_{\partial\Omega}\widetilde{u} = \sup\{t \in \mathbb{R} : \mathcal{H}^{n-1}(\{\widetilde{u} > t\}) > \mathcal{H}^{n-1}(\partial\Omega)/2\}$$

(see e.g. [CP1, Lemma 3.1]) Thus, inequality (1.1) is equivalent to

$$\|\widetilde{u} - \mathrm{med}_{\partial\Omega}\widetilde{u}\|_{L^1(\partial\Omega)} \leq C\|Du\|(\Omega)$$

for every $u \in BV(\Omega)$, with the same constant C.

Other normalizing operators than $\mathrm{med}_{\partial\Omega}\widetilde{u}$ are admissible in inequalities of the form (3.2). General assumptions on an operator T such that

$$BV(\Omega) \ni u \mapsto T(u) \in \mathbb{R}$$

are known for an inequality of the form

$$\|\widetilde{u} - T(u)\|_{L^1(\partial\Omega)} \leq C\|Du\|(\Omega) \tag{1.2}$$

to hold for some constant C, and for every $u \in BV(\Omega)$. These assumptions can be derived, for instance, by specializing an abstract result from [Zi, Lemma 4.1.3].
Besides the median of $\widetilde{u}$ on $\partial\Omega$, another classical choice for $T(u)$ is the mean value $\mathrm{mv}_{\partial\Omega}(\widetilde{u})$ of $\widetilde{u}$ over $\partial\Omega$, given by

$$\mathrm{mv}_{\partial\Omega}(\widetilde{u}) = \frac{1}{\mathcal{H}^{n-1}(\partial\Omega)}\int_{\partial\Omega} \widetilde{u}(x)\, d\mathcal{H}^{n-1}(x).$$

Less conventional admissible operators $T(u)$ amount to

$$T(u) = \mathrm{med}_{\Omega}(u), \tag{1.3}$$

where

$$\mathrm{med}_{\Omega}(u) = \sup\{t \in \mathbb{R} : |\{u > t\}| > |\Omega|/2\},$$

the median of u in the whole of Ω, and

$$T(u) = \mathrm{mv}_{\Omega}(u), \tag{1.4}$$

where

$$\mathrm{mv}_{\Omega}(u) = \frac{1}{|\Omega|}\int_{\Omega} u(x)\, dx,$$

the mean value of u in the whole of Ω Here, $|\cdot|$ denotes Lebesgue measure. The choices (1.3) and (1.4) make inequality (1.2) nonstandard, in that its left-hand side combines quantities depending both on $\widetilde{u}$ and on u.

We are concerned with the problem of the optimal constant C in (1.2) when $T(u)$ is either $\mathrm{med}_{\partial\Omega}\widetilde{u}$, or $\mathrm{mv}_{\partial\Omega}(\widetilde{u})$, or $\mathrm{med}_{\Omega}(u)$, or $\mathrm{mv}_{\Omega}(u)$. For any admissible domain Ω, these optimal constants equal certain geometric constants of isoperimetric type. In the special case when Ω is an Euclidean ball, an explicit description of the extremal functions is possible. In fact, due to the scaling invariance of the relevant inequalities, we shall deal, without loss of generality, with the unit ball $\mathbb{B}^n$, centered at 0, in $\mathbb{R}^n$.

Interestingly, the Poincaré inequalities in question share the same extremals under the constraint on $\mathrm{med}_{\partial\Omega}\widetilde{u}$, $\mathrm{mv}_{\partial\Omega}(\widetilde{u})$ and $\mathrm{mv}_\Omega(u)$, but take a different, non-standard form , for $\mathrm{med}_\Omega(u)$.

The geometric characterizations of the sharp constants in the Poincaré inequalities are stated in Section 2. Section 3 is devoted to the description of the extremals in the Poincaré inequalities in $\mathbb{B}^n$.

Let us mention that trace inequalities in Sobolev type spaces, involving optimal constants, have been extensively investigated in the literature. Contributions along this line of research include [AFV, AMR, BGP, Bro, BrF, Ci2, CFNT2, DDM, Es1, MV1, MV2, Ma1, Ma2, Ma3, Na, Ro, W]. Sharp forms of Poincaré type inequalities for Sobolev functions and functions of bounded variation, involving norms of u in the whole of Ω, are the object of [BK, BoV, BrV, Ci1, DG, DN, EFKNT, ENT, FNT, GW, Le, NR].

2 Geometric constants

Let E be a measurable set in $\mathbb{R}^n$. The essential boundary $\partial^M E$ of E is defined as the complement in $\mathbb{R}^n$ of the sets of points of densities 0 and 1 with respect to E. One has that $\partial^M E$ is a Borel set, and $\partial^M E \subset \partial E$, the topological boundary of E.
The set E is said to be of finite perimeter relative to an open set $\Omega \subset \mathbb{R}^n$ if $D\chi_E$, the distributional derivative of the characteristic function χ_E of E, is a vector-valued Radon measure in Ω with finite total variation in Ω. The perimeter of E relative to Ω is defined as

$$P(E;\Omega) = \|D\chi_E\|(\Omega). \tag{2.1}$$

A result from geometric measure theory tells us that E is of finite perimeter in Ω if and only if $\mathcal{H}^{n-1}(\partial^M E \cap \Omega) < \infty$; moreover,

$$P(E;\Omega) = \mathcal{H}^{n-1}(\partial^M E \cap \Omega) \tag{2.2}$$

[Fe, Theorem 4.5.11]. A domain Ω in $\mathbb{R}^n$ will be called admissible if $\mathcal{H}^{n-1}(\partial\Omega) < \infty$, $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^M\Omega) = 0$, and

$$\min\{\mathcal{H}^{n-1}(\partial^M E \cap \partial\Omega)\,, \mathcal{H}^{n-1}(\partial\Omega \setminus \partial^M E)\} \leq C\mathcal{H}^{n-1}(\partial^M E \cap \Omega) \tag{2.3}$$

for some positive constant C and every measurable set $E \subset \Omega$ [Zi, Definition 5.10.1]. In particular, any Lipschitz domain is an admissible domain.
If Ω is an admissible domain, the boundary trace $\widetilde{u}$ of a function $u \in BV(\Omega)$ is well defined for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$ as

$$\widetilde{u}(x) = \lim_{r\to 0^+} \frac{1}{|B_r(x) \cap \Omega|} \int_{B_r(x)\cap\Omega} u(y)\, dy\,, \tag{2.4}$$

where $B_r(x)$ denotes the ball centered at x, with radius r [Ma3, Corollary 9.6.5]. The assumption that Ω be an admissible domain is necessary and sufficient for $\widetilde{u}$ to belong to $L^1(\partial\Omega)$ for every function $u \in BV(\Omega)$ – see [AG] and [Ma3, Theorem 9.5.2]. Moreover, $L^1(\partial\Omega)$ cannot be replaced with any smaller Lebesgue space independent of u.
Alternate notions of the boundary trace of a function of bounded variation can be found in the literature. One definition relies upon the notion of upper and lower approximate limits of the

extension of u by 0 outside Ω [Zi, Definition 5.10.5]. Another possible definition is that of rough trace in the sense of [Ma3, Section 9.5.1]. Both of them coincide with $\widetilde{u}$, up to subsets of $\partial\Omega$ of $\mathcal{H}^{n-1}$-measure zero.
If Ω is a Lipschitz domain, and the function u enjoys some additional regularity property, such as membership to the Sobolev space $W^{1,1}(\Omega)$, then the trace of u on $\partial\Omega$ defined as the limit of the restrictions to $\partial\Omega$ of approximating sequences of smooth functions on $\overline{\Omega}$ also agrees with $\widetilde{u}$ for $\mathcal{H}^{n-1}$-a.e. point on $\partial\Omega$.

We assume through this section that Ω is an admissible domain in $\mathbb{R}^n$, with $n \geq 2$. Let us denote by $C(\mathrm{med}_{\partial\Omega})$ the optimal constant in the inequality

$$\|\widetilde{u} - \mathrm{med}_{\partial\Omega}(\widetilde{u})\|_{L^1(\partial\Omega)} \leq C(\mathrm{med}_{\partial\Omega})\|Du\|(\Omega) \tag{2.5}$$

for $u \in BV(\Omega)$. A pioneering result by Burago and Maz'ya [Ma3, Theorem 9.5.2] tells us that $C(\mathrm{med}_{\partial\Omega})$ equals the geometric constant $K(\mathrm{med}_{\partial\Omega})$ of Ω defined as

$$K(\mathrm{med}_{\partial\Omega}) = \sup_{E\subset\Omega} \frac{\min\{\mathcal{H}^{n-1}(\partial^M E \cap \partial\Omega), \mathcal{H}^{n-1}(\partial\Omega \setminus \partial^M E)\}}{\mathcal{H}^{n-1}(\partial^M E \cap \Omega)}. \tag{2.6}$$

Here, and in similar occurrences in what follows, we tacitly assume that the supremum is extended over non-negligible subsets E of Ω.

Theorem 2.1 [Ma3, Theorem 9.5.2] *Let Ω be an admissible domain in $\mathbb{R}^n$, with $n \geq 2$. Then*

$$C(\mathrm{med}_{\partial\Omega}) = K(\mathrm{med}_{\partial\Omega}). \tag{2.7}$$

Equality holds in (2.5) for some nonconstant function u if and only if the supremum is attained in (2.6) for some set E. In particular, if E is an extremal set in (2.6), then the function $a\chi_E + b$ is an extremal function in (2.5) for every $a \in \mathbb{R}\setminus\{0\}$ and $b \in \mathbb{R}$.
More generally, if $\{E_k\}$ is an optimizing sequence of sets in (2.6), then the sequence $\{u_k\} = \{a_k\chi_{E_k} + b_k\}$ is an optimizing sequence of functions in (2.5) for every $a_k, b_k \in \mathbb{R}$.

Let us next consider the optimal constant $C(\mathrm{mv}_{\partial\Omega})$ in the inequality

$$\|\widetilde{u} - \mathrm{mv}_{\partial\Omega}(\widetilde{u})\|_{L^1(\partial\Omega)} \leq C(\mathrm{mv}_{\partial\Omega})\|Du\|(\Omega) \tag{2.8}$$

for $u \in BV(\Omega)$. It has been shown in [Ci3] that $C(\mathrm{mv}_{\partial\Omega})$ agrees with another geometric constant $K(\mathrm{mv}_{\partial\Omega})$, given by

$$K(\mathrm{mv}_{\partial\Omega}) = \sup_{E\subset\Omega} \frac{2\min\{\mathcal{H}^{n-1}(\partial^M E \cap \partial\Omega), \mathcal{H}^{n-1}(\partial\Omega \setminus \partial^M E)\}}{\mathcal{H}^{n-1}(\partial\Omega)\mathcal{H}^{n-1}(\partial^M E \cap \Omega)}. \tag{2.9}$$

Theorem 2.2 [Ci3, Theorem 1.1] *Let Ω be an admissible domain in $\mathbb{R}^n$, with $n \geq 2$. Then*

$$C(\mathrm{mv}_{\partial\Omega}) = K(\mathrm{mv}_{\partial\Omega}). \tag{2.10}$$

Equality holds in (2.8) for some nonconstant function u if and only if the supremum is attained in (2.9) for some set E. In particular, if E is an extremal set in (2.9), then the function $a\chi_E + b$ is an extremal function in (2.8) for every $a \in \mathbb{R}\setminus\{0\}$ and $b \in \mathbb{R}$.
More generally, if $\{E_k\}$ is an optimizing sequence of sets in (2.9), then the sequence $\{u_k\} = \{a_k\chi_{E_k} + b_k\}$ is an optimizing sequence of functions in (2.8) for every $a_k, b_k \in \mathbb{R}$.

The geometric constants associated with the Poincaré inequalities with normalization depending on the whole funtion u, instead of just its boundary trace $\widetilde{u}$, are exhibited in [CFNT2]. Specifically, let us denote by $C(\mathrm{mv}_\Omega)$ the optimal constant in the inequality

$$\|\widetilde{u} - \mathrm{mv}_\Omega(u)\|_{L^1(\partial\Omega)} \le C(\mathrm{mv}_\Omega)\|Du\|(\Omega) \tag{2.11}$$

for $u \in BV(\Omega)$. Then $C(\mathrm{mv}_\Omega)$ is related to the isoperimetric constant $K(\mathrm{mv}_\Omega)$ associated with Ω by

$$K(\mathrm{mv}_\Omega) = \sup_{E\subset\Omega} \frac{|E|\,\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^M E) + |\Omega \setminus E|\,\mathcal{H}^{n-1}(\partial^M E \cap \partial\Omega)}{|\Omega|\,\mathcal{H}^{n-1}(\partial^M E \cap \Omega)}. \tag{2.12}$$

Theorem 2.3 [CFNT2, Theorem 2.1] *Let Ω be an admissible domain in $\mathbb{R}^n$, with $n \ge 2$. Then*

$$C(\mathrm{mv}_\Omega) = K(\mathrm{mv}_\Omega). \tag{2.13}$$

Equality holds in (2.11) *for some nonconstant function u if and only if the supremum is attained in* (2.12) *for some set E. In particular, if E is an extremal set in* (2.12)*, then the function $a\chi_E+b$ is an extremal function in* (2.11) *for every $a \in \mathbb{R}\setminus\{0\}$ and $b \in \mathbb{R}$.*
More generally, if $\{E_k\}$ is an optimizing sequence of sets in (2.12)*, then the sequence $\{u_k\} = \{a_k\chi_{E_k} + b_k\}$ is an optimizing sequence of functions in* (2.11) *for every $a_k, b_k \in \mathbb{R}$.*

We conclude this section by a geometric characterization of the optimal constant $C(\mathrm{med}_\Omega)$ in the inequality

$$\|\widetilde{u} - \mathrm{med}_\Omega(u)\|_{L^1(\partial\Omega)} \le C(\mathrm{med}_\Omega)\|Du\|(\Omega) \tag{2.14}$$

for $u \in BV(\Omega)$. The isoperimetric constant which now comes into play is defined as

$$K(\mathrm{med}_\Omega) = \sup_{\substack{E\subset\Omega\\ |E|\le|\Omega|/2}} \frac{\mathcal{H}^{n-1}(\partial^M E \cap \partial\Omega)}{\mathcal{H}^{n-1}(\partial^M E \cap \Omega)}. \tag{2.15}$$

Theorem 2.4 [CFNT2, Theorem 2.3] *Let Ω be an admissible domain in $\mathbb{R}^n$, with $n \ge 2$. Then*

$$C(\mathrm{med}_\Omega) = K(\mathrm{med}_\Omega). \tag{2.16}$$

Equality holds in (2.14) *for some nonconstant function u if and only if the supremum is attained in* (2.15) *for some set E. In particular, if E is an extremal in* (2.15)*, then the function $a\chi_E + b$ is an extremal in* (2.14) *for every $a \in \mathbb{R}\setminus\{0\}$ and $b \in \mathbb{R}$.*
More generally, if $\{E_k\}$ is an optimizing sequence of sets in (2.15)*, then the sequence $\{u_k\} = \{a_k\chi_{E_k} + b_k\}$ is an optimizing sequence of functions in* (2.14) *for every $a_k, b_k \in \mathbb{R}$.*

Remark 2.5 The Poincaré type trace inequalities considered in the present section hold, in particular, with the same constants, for every function u in the Sobolev space $W^{1,1}(\Omega)$. Indeed, the latter is a subspace of $BV(\Omega)$. For any such function u, the total variation $\|Du\|(\Omega)$ agrees with $\|\nabla u\|_{L^1(\Omega)}$, where ∇u denotes the weak gradient of u. The constants in the relevant Poincaré inequalities continue to be optimal in $W^{1,1}(\Omega)$, since any function $u \in BV(\Omega)$ can be approximated by a sequence of functions $u_k \in W^{1,1}(\Omega)$ in such a way that

$$\widetilde{u_k} = \widetilde{u} \quad \text{and} \quad \lim_{k\to\infty}\|\nabla u_k\|_{L^1(\Omega)} = \|Du\|(\Omega).$$

The existence of the sequence $\{u_k\}$ follows, for instance, from [Gi, Theorem 1.17 and Remark 1.18]. Of course, the last part of the statements of Theorems 2.1–2.4 does not apply when dealing with Sobolev functions, since characteristic functions of subsets of Ω are not weakly differentiable.

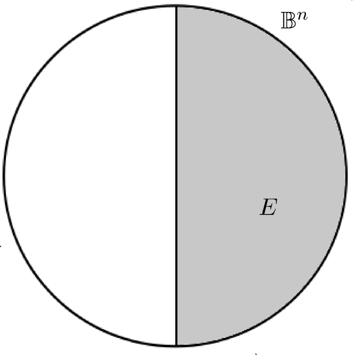

Figure 1

3 Extremal functions in Poincaré inequalities on $\mathbb{B}^n$

In the case when the domain Ω is the ball $\mathbb{B}^n$, the extremal subsets in geometric functionals introduced in section 2 can be exhibited. As a consequence of Theorems 2.1–2.4, the extremal functions in the associated Poincaré trace inequalities can be characterized.

The computation of the optimal constant $C(\mathrm{med}_{\partial\mathbb{B}^n})$ in the Poincaré trace inequality

$$\|\widetilde{u} - \mathrm{med}_{\mathbb{B}^n}(\widetilde{u})\|_{L^1(\partial\mathbb{B}^n)} \leq C(\mathrm{med}_{\partial\mathbb{B}^n})\|Du\|(\mathbb{B}^n) \tag{3.1}$$

for $u \in BV(\mathbb{B}^n)$ goes back to Burago and Maz'ya [Ma3, Corollary 9.4.4/3].
In what follows, $\omega_n = \pi^{n/2}/\Gamma(1+n/2)$, the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

Theorem 3.1 [Ma3, Corollary 9.4.4/3] *Let $n \geq 2$. Then*

$$C(\mathrm{med}_{\partial\mathbb{B}^n}) = \frac{n\omega_n}{2\omega_{n-1}}.$$

Equality holds in (3.1) *if u agrees with the characteristic function of a half-ball (see Figure 1).*

The best constant $C(\mathrm{mv}_{\partial\mathbb{B}^n})$ in the inequality

$$\|\widetilde{u} - \mathrm{mv}_{\partial\mathbb{B}^n}(\widetilde{u})\|_{L^1(\partial B)} \leq C(\mathrm{mv}_{\partial\mathbb{B}^n})\|Du\|(B) \tag{3.2}$$

for $u \in BV(B)$ is provided by a result from [Ci3], which is stated in Theorem 3.2 below. Interestingly, the existence and the form of extremals in inequality (3.2) turns out to depend on the dimension n. In particular, Theorem 3.2 shows that extremals in the trace inequality (2.8) need not exist, even for domains with such a simple geometry as the disk in $\mathbb{R}^2$.

In what follows, we call spherical segment in B the (non empty) intersection of B with a half-space.

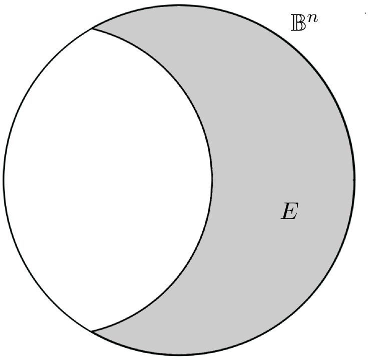

Figure 2

Theorem 3.2 [Ci3, Theorem 1.2] *Let B be a ball in $\mathbb{R}^n$, $n \geq 2$. Then*

$$C(\mathrm{mv}_{\partial\mathbb{B}^n}) = \begin{cases} \dfrac{n\omega_n}{2\omega_{n-1}} & \text{if } n \geq 3, \\ 2 & \text{if } n = 2. \end{cases} \tag{3.3}$$

If $n \geq 4$, equality holds in (3.2) *when u agrees with the characteristic function of a half-ball.*
If $n = 3$, equality holds in (3.2) *when u agrees with the characteristic function of any spherical segment.*
If $n = 2$, equality never holds in (3.2), *unless u is constant. Any sequence of characteristic functions of spherical segments whose measure converges to* 0 *is optimizing in* (3.2).

Remark 3.3 Let us incidentally mention that $\mathbb{B}^n$ enjoys a minimizing property, among all admissible domains in $\mathbb{R}^n$, as far as the constants $C(\mathrm{med}_{\partial\mathbb{B}^n})$ and $C(\mathrm{mv}_{\partial\mathbb{B}^n})$ are concerned. Indeed, as shown in [CFNT1],

$$C(\mathrm{med}_{\partial\Omega}) \geq C(\mathrm{med}_{\partial\mathbb{B}^n}), \tag{3.4}$$

and

$$C(\mathrm{mv}_{\partial\Omega}) \geq C(\mathrm{mv}_{\partial\mathbb{B}^n}). \tag{3.5}$$

Moreover, equality holds in (3.4) if and only if $\Omega = \mathbb{B}^n$, and, when $n \geq 3$, equality holds in (3.5) if and only if $\Omega = \mathbb{B}^n$. On the other hand, if $n = 2$, there also exist domains $\Omega \neq \mathbb{B}^2$ attaining equality in (3.5).

Let us next focus on the estremal functions in Poincaré type inequalities on $\mathbb{B}^n$ under mean value and median constraint over the entire $\mathbb{B}^n$. In the former case, namely in inequalities of the form

$$\|\widetilde{u} - \mathrm{mv}_{\mathbb{B}^n}(u)\|_{L^1(\partial\mathbb{B}^n)} \leq C(\mathrm{mv}_{\mathbb{B}^n})\|Du\|(\mathbb{B}^n) \tag{3.6}$$

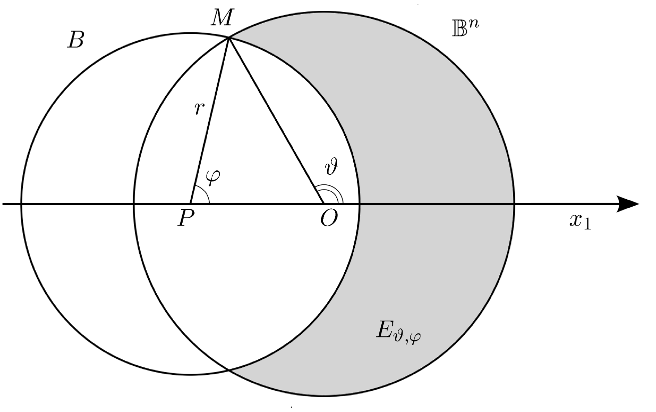

Figure 3

for $u \in BV(\mathbb{B}^n)$ characteristic functions of half-balls are again extremals. Here, $C(\mathrm{mv}_{\mathbb{B}^n})$ stands for the sharp constant in (3.6).

Theorem 3.4 *Let $n \geq 2$. Then*

$$C(\mathrm{mv}_{\mathbb{B}^n}) = \frac{n\omega_n}{2\omega_{n-1}}.$$

Equality holds in (3.6) *if u agrees with the characteristic function of a half-ball.*

In contrast with the previous results of this section, the extremals in the inequality

$$\|\widetilde{u} - \mathrm{med}_{\mathbb{B}^n}(u)\|_{L^1(\partial\mathbb{B}^n)} \leq C(\mathrm{med}_{\mathbb{B}^n})\|Du\|(\mathbb{B}^n) \tag{3.7}$$

for $u \in BV(\mathbb{B}^n)$, with optimal constant $C(\mathrm{med}_{\mathbb{B}^n})$, are characteristic functions of a new kind of subsets of $\mathbb{B}^n$. These subsets are half-moon shaped (Figure 2), and hence, in particular, they are not even convex. This is the content of the next theorem.

In the statement, $E_{\vartheta,\varphi}$ denotes the set depicted in Figure 3, where the couple (ϑ, φ) belongs to the set

$$\Upsilon = \{(\vartheta, \varphi) : 0 < \vartheta < \pi, 0 \leq \varphi < \vartheta\}. \tag{3.8}$$

The isoperimetric nature of the optimal constant in inequality (3.7), as described in Theorem 2.4, helps in accounting for this seemingly striking conclusion.

Theorem 3.5 *Let $n \geq 2$. Then equality holds in* (3.7) *if u is the characteristic function of the half-moon shaped set $E_{\vartheta,\varphi}$ as in Figure 3, where (ϑ, φ) is the unique solution in the set Υ*

(defined by (3.8)*) to the system*

$$
(3.9)\qquad \begin{cases} \dfrac{\Psi_{n-2}(\varphi)\sin^n\vartheta}{\Psi_{n-2}(\vartheta)\sin^n\varphi} = 1 - \dfrac{(n-1)\Psi_{n-2}(\pi)\cos\vartheta}{2[(n-1)\cos\vartheta\,\Psi_{n-2}(\vartheta) - \sin^{n-1}\vartheta]} \\[2ex] \dfrac{\cos\varphi}{\sin\varphi} = \dfrac{\cos\vartheta}{\sin\vartheta}\left(1 - \dfrac{(n-1)\Psi_{n-2}(\pi)}{2[(n-1)\cos^2\vartheta\,\Psi_{n-2}(\vartheta) - \sin^{n-1}\vartheta\cos\vartheta]}\right). \end{cases}
$$

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