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Properties characterized by generalized indiscernible

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Abstract

Some model theoretic properties, stability, NIP and n-dependence of formulas are preserved under taking boolean combinations of formulas. We prove those results by using Ramsey classes which characterize such properties.

1 Introduction

The classification theory of first order theories introduced by S. Shelah[7] has been developed by many model theorists with applications to other mathematic fields. Shelah considered several dividing lines of first order theories, for example, stability, simplicity, NIP, NTP_1, NTP_2, n-dependence, which have alternative definitions.

Definition 1.

1. A theory T is said to be stable if there is a cardinal λ such that for any $M \models T$ of cardinality λ the number $|S(M)|$ of types over M is $\leq \lambda$.
2. A formula $\varphi(x, y)$ is said to have IP (under T) if there is $(a_i)_{i\in\omega}$ (in a big model M of T) such that for any $X \subset \omega$ there is b such that $M \models \varphi(b, a_i)$ if and only if $i \in X$. NIP is the negation of IP.
3. A formula $\varphi(x, y)$ is said to have k-TP_1 (under T) if there is $(a_\eta)_{\eta\in\omega^{<\omega}}$ (in a big model M of T) such that $\{\varphi(x, a_{\mu|n}) : n \in \omega\}$ is consistent for any $\mu \in \omega^\omega$ and $\{\varphi(x, a_{\eta_i}) : i < k\}$ is inconsistent for any pairwise incomparable $\eta_i \in \omega^{<\omega}$ $(i < k)$. A formula φ is NTP_1 if it has no k-TP_1 for any finite $k \geq 2$.

On the study of such theories, indiscernible sequences, which is obtained by Ramsey theorem and compactness, have an important role for analyzing structures. Recent development of them shows that some model theoretic properties are essentially related to a kind of generalization of Ramsey theory through generalized indiscernible.

Example 2.
1. T is stable if and only if every indiscernible sequence (in a model of T) is actually an indiscernible set.
2. (L. Scow[6]) T is NIP if and only if every ordered random graph indiscernible (in a model of T) is actually an indiscernible sequence.
3. (A. Chernikov, D. Palacin and K. Takeuchi[1]) T is n-dependence if and only if every ordered $(n+1)$-uniform random hypergraph indiscernible (in a model of T) is actually an indiscernible sequence.
4. If $\varphi(x, y)$ has k-TP_1 then we can find an witness $(a_\eta)_{\eta \in \omega^{<\omega}}$ of k-TP_1, which is an indiscernible tree. This implies that k-$TP_1 \Leftrightarrow 2$-TP_1. (See for example, [3] or [9].)

Many similar results are also found in other contexts. In fact, L. Scow[6] pointed out that Ramsey classes have a correspondence to generalized indiscernible.

Fact 3. (Very roughly speaking. For the precise form see Fact 13.) Under some regular conditions, a class K of finite L_0-structures is a Ramsey class if and only if you can find K-indiscernible in any L-theory T.

Hence it is natural to expect that some of model theoretic results are follow from combinatorial arguments in Ramsey class. In this article we prove:

Theorem 4. Suppose that there is a good expansion $(N; R, ...)$ of $(N; R)$ to a finite relational language L_0 such that $Age((N; R, ...))$ is a Ramsey class of finite L_0-structures. If $\varphi \vee \psi$ partly encodes $(N; R)$ in T, then either φ or ψ partly encodes $(N; R)$ in T.

It is a demonstration of the above idea on the following.

Fact 5. A formula φ is not stable (NIP, n-dependent) under T if it encodes a total order (a random graph, a $(n+1)$-uniform random hypergraph, respectively) in a model of T.

Fact 6. Let $P \in \{\text{stable, NIP}, n\text{-dependent}\}$. Suppose that formulas φ and ψ have the property P (under a theory T). Then so does $\varphi \vee \psi$. In particular if every atomic formula has the property P then so does every quantifier free formula, since P is preserved under taking negation.

2 Preliminaries

Let L_0 be a finite relational language. Finite tuples of variables are denoted by $x, y, z, \ldots$ and finite tuples of elements in structures are denoted by $a, b, c, \ldots$. If we need to clarify that x (or a) is singleton or not, we explicitly denote that $|x| = 1$ or $= n$.

Definition 7. Let A, B and C are L_0-structures.

1. $\binom{B}{A} = \{A' : A' \subset B, A' \cong A\}$.
2. The arrow notation $C \to (B)^A_k$ denotes the condition that for any $f : \binom{C}{A} \to k$ there is $B' \in \binom{C}{B}$ such that $f|\binom{B'}{A}$ is constant.
3. A class K of (isomorphism types of) finite L_0-structure is said to be a Ramsey class if for any $k \in \omega$ and $A, B \in K$ there is $C \in K$ such that $C \to (B)^A_k$.
4. A class K is said to be hereditary if $A \subset B \in K$ implies $A \in K$.
5. A class K is said to have joint embedding property (JEP) if for any $A, B \in K$ there is $C \in K$ such that $A, B \subset C$.
6. Let N be an L_0-structure. $Age(N) = \{A : A \subset_{\text{fin}} N\}$.

It is interesting that Ramsey property implies amalgamation property (AP).

Fact 8 (Nešetřil[5]). Let K be a hereditary Ramsey class with JEP. Then K has AP.

Fact 9 (Fraïssé's Theorem). Let K be a hereditary class of finite L_0-structures. Then the following are equivalent.

1. K has AP.
2. There is a (unique) countable homogeneous L_0-structure N such that $K = Age(N)$. Such a structure N is called the Fraïssé limit of K denoted as $\lim K$.

Example 10. 1. Let $L_0 = \emptyset$ and K be the class of finite sets. The Ramsey property of K is an alternative expression of the classical finite Ramsey theorem. The Fraïssé limit of K is ω.

2. Let $L_0 = \{<\}$ and K be the set of totally ordered finite sets. Then K is a Ramsey class with $\lim K = (\mathbb{Q}; <)$.

3. Let $L_0 = \{<, E(x, y)\}$ and K be the set of totally ordered finite graphs. Then K is a Ramsey class with $\lim K$ an ordered random graph.

4. Let $L_0 = \{<, E(x, y), P_0(x), P_1(x)\}$ and K be the set of totally ordered bipartite finite graphs. Then K is a Ramsey class with $\lim K$ a ordered bipartite random graph.

The next theorem by Nešetřil and Rödl is useful to examine if a class K is Ramsey.

Fact 11 ([4]). Suppose that K is a hereditary class of finite L_0-structures. If K has free amalgamation property, then $K_<$ has Ramsey property, where $K_< = \{(A, <) : A \in K, <$ is a total ordering on $A\}$.

In what follows we assume every class K in this article is hereditary and has AP with Fraïssé limit $\lim K$ unless otherwise noted.

Let K be a class of finite L_0-structures (which is hereditary and has AP with Fraïssé limit $\lim K$).

Definition 12 ([6]). Let T be an L-theory and M a model of T.

1. A sequence $(b_\eta)_{\eta \in \lim K} \subset M$ is said to be based on $(a_\eta)_{\eta \in \lim K} \subset M$ if for every finite $X \subset \lim K$ and a formula $\varphi((x_\eta)_{\eta \in X})$ there is a subset $X' \cong_{L_0} X$ of $\lim K$ such that $\varphi((b_\eta)_{\eta \in X}) \leftrightarrow \varphi((a_\eta)_{\eta \in X'})$.

2. A sequence $(a_\eta)_{\eta \in \lim K} \subset M$ is said to be a K-indiscernible if for every finite $X \cong_{L_0} Y \subset \lim K$, $\mathrm{tp}((a_\eta)_{\eta \in X}) = \mathrm{tp}((a_\eta)_{\eta \in Y})$ (in theory T).

Fact 13 ([6]). Suppose that for any $A \in K$, $\mathrm{Aut}(A)$ is trivial. Then K is a Ramsey class if and only if for any structure M in any language L and for any $(a_\eta)_{\eta \in \lim K} \subset M$, there is K-indiscernible $(b_\eta)_\eta$ based on $(a_\eta)_\eta$ in an elementary extension of M.

3 The Ramsey property implies a preservation theorem

Recall that L_0 is a finite relational language, K a hereditary class of finite L_0-structures with Fraïssé limit $\lim K$.

Definition 14. 1. An n-partitioned structure $(N; R)$ is a structure with n-partitioned universe $N = \sqcup_{i<n} N_i$ and n-ary predicate R.

2. Let $(N; R)$ be an n-partitioned structure and let $(N; R, ...)$ be any expansion of $(N; R)$ by adding finitely many predicates. We say that the expansion is good if for any finite $A \subset N$ there is $A' \cong_R A$ in N such that if $b_i, b'_i \in N_i \cap A'$ $(i < n)$ and $N \models R(b_0, ..., b_{n-1}) \wedge R(b'_0, ..., b'_{n-1})$, then $b_0...b_{n-1} \cong_{L_0} b'_0...b'_{n-1}$

Example 15. 1. Let $N_{ladder} = (N_0 \sqcup N_1; R(x, y))$ be a 2-partitioned structure such that $N_k = \{b^k_i : i < \omega\}$ and $R(b^0_i, b^1_j)$ if and only if $i < j$. Consider an expansion $\bar{N}_{ladder}$ of N_{ladder} by predicates $P_k(x)$ $(k < 2)$ expressing the partition. It is clearly a good extension and moreover, $Age(\bar{N}_{ladder})$ is a Ramsey class (in the language $\{R, P_0, P_1\}$). The Ramsey property of the class is proved by Fact 13. To see this, let $(a_\eta)_{\eta \in N_{ladder}}$ be any sequence in any structure. Put $a^k_i = a_{b^k_i}$. Then we can consider a sequence $(a^0_i a^1_i)_{i \in \omega}$. By the classical Ramsey theorem, we can find an indiscernible sequence $(d^0_i d^1_i)_i$ which is based on $(a^0_i a^1_i)_i$. It is easy to check that $(d_\eta)_{\eta \in N_{ladder}}$ is N_{ladder}-indiscernible where $d_{b^k_i} = d^k_i$.

2. Let $N_{RG} = (N_0 \sqcup N_1; R(x, y))$ be a bipartite random graph. Consider an expansion of N_{RG}, as the same in the case of N_{ladder}, by $P_k(x)$ $(k < 2)$ with any total ordering $<$ on N_{RG}. It is also a good extension and $Age((N_{RG}, P_0, P_1, <))$ is a Ramsey class. The Ramsey property of the class is immediately followed from Nešetřil and Rödl's theorem (Fact 11). To see that this is a good expansion, notice that for any finite subgraph A of N_{RG} there is an isomorphic graph $A' \subset N_{RG}$ such that $A' \cap N_0 < A' \cap N_1$.

3. Let $N_{nRG} = (N_0 \sqcup ... \sqcup N_{n-1}; R(x_0, ..., x_{n-1}))$ be an n-partite random graph. As the same in the above, by adding $P_k(x)$ $(k < n)$ and $<$, $Age((N_{nRG}, P_0, ..., P_{n-1}, <))$ is a Ramsey class and it is a good expansion.

Definition 16. Let $(N;R)$ be an n-partitioned structure and let M be an L-structure. We say an L-formula $\varphi(x_0,...,x_{n-1})$ with $|x_i|=k$ partly encodes $(N;R)$ in M through an injection $\sigma: N \to M^k$ if fore every $b_i \in N_i$ $(i<n)$, we have $N \models R(b_0,...,b_{n-1}) \Leftrightarrow M \models \varphi(\sigma(b_0),...,\sigma(b_{n-1}))$. We say $\varphi(x_0,...,x_{n-1})$ partly encodes $(N;R)$ in a theory T if there is such a model M of T.

Example 17.

1. A formula $\varphi(x,y)$ is unstable if and only if φ partly encodes N_{ladder} (in T).
2. A formula $\varphi(x,y)$ is not NIP if and only if φ partly encodes N_{RG} (in T). (See [6].)
3. A formula $\varphi(x,y_0,...,y_{n-1})$ is not n-dependent if and only if φ partly encodes N_{nRG} (in T). (See [1].)

Let T be an L-theory and $(N;R)$ be an n-partitioned structure.

Theorem 18. Suppose that there is a good expansion $(N;R,...)$ of $(N;R)$ to a finite relational language L_0 such that $Age((N;R,...))$ is a Ramsey class of finite L_0-structures. If $\varphi \vee \psi$ partly encodes $(N;R)$ in T, then either φ or ψ partly encodes $(N;R)$ in T.

Skech of Proof. Suppose that $\varphi \vee \psi$ partly encodes $(N;R)$ through σ in M. Let $N=\bigcup_{i<\omega} B_i$ with finite B_i. Fix $j \in \omega$ and take B'_j which is an witness of good extension with respect to B_j. Let $A=\{b_0,...,b_{n-1}\}$ be such that $b_i \in N_i \cap B'_j$ and $R(b_0,...,b_{n-1})$ holds. Then by the Ramsey property of $Age((N;R,...))$ in the language L_0, we can find $C \subset_{\text{fin}} N$ such that $C \to (B'_j)^A_2$. Let $f: \binom{C}{A} \to 2$ be a coloring such that $f(b'_0...b'_{n-1})=0$ if and only if $M \models \varphi(\sigma(b'_0),...,\sigma(b'_{n-1}))$. Take a monochromatic $B''_j \in \binom{C}{B'_j}$. By the choice of B'_j and B''_j, we have

$$\{b'_0...b'_{n-1} : b'_i \in N_i \cap B''_j \text{ and } R(b'_0,...,b'_{n-1})\} \subset \binom{B''_j}{A}.$$

Since f is constant in B''_j, either φ or ψ partly encodes $(B''_j;R)$ in M through $\sigma|B''_j$. Hence for each $j<\omega$, $(B_j;R)$ is partly encoded by φ or ψ. By considering a subsequence, we may assume either φ or ψ partly encodes every $(B_j;R)$ in M. By the compactness, we got the conclusion. □

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