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The Dynamics of Duality:
A Fresh Look at the Philosophy of Duality

Yoshihiro Maruyama

The Hakubi Centre and Graduate School of Letters, Kyoto University
Mathematical, Physical, and Life Sciences Division, University of Oxford

"[T]he development of philosophy since the Renaissance has by and large gone from right to left [...] Particularly in physics, this development has reached a peak in our own time, in that, to a large extent, the possibility of knowledge of the objectivisable states of affairs is denied, and it is asserted that we must be content to predict results of observations. This is really the end of all theoretical science in the usual sense" (Kurt Gödel, *The modern development of the foundations of mathematics in the light of philosophy*, lecture never delivered)[1]

1 Unity, Disunity, and Pluralistic Unity

Since the modernist killing of natural philosophy seeking a universal conception of the cosmos as a united whole, our system of knowledge has been optimised for the sake of each particular domain, and has accordingly been massively fragmented and disenchanted (in terms of Weber's theory of modernity). Today we lack a unified view of the world, living in the age of disunity surrounded by myriads of uncertainties and contingencies, which invade both science and society as exemplified by different sciences of chance, including quantum theory and artificial intelligence on the basis of statistical machine learning, and by the distinctively after-modern features of risk society (in terms of the theories of late/reflexive/liquid modernity), respectively (this is not necessarily negative like quite some people respect diversity more than uniformity; contemporary art, for example, gets a lot of inspirations from uncertainties and contingencies out there in nature and society). This modern and late-modern process of disunification and contextualisation in favour of scientific and other sorts of pluralism rather than monism has gone hand-in-hand with:

[1]According to Gödel, scepticism, materialism and positivism stand on the left side, and metaphysics, idealism and theology on the right. Gödel's philosophy of physics is rarely addressed, the only exception being his work on general relativity, and yet this manuscript suggests his position is never naïve realism/platonism. He indeed pays prudent attention to what he calls the left spirit of the time, some sort of antirealism or agnosticism, and seriously attempts to make foundations of mathematics cohere with it through Kantian epistemology and Husserlian phenomenology. It is an endeavour to reconcile the right and the left, and in the present article we shall address exactly the same issue with the idea of duality (between the right and the left). Apart from that, this quote also sounds like Chomsky's criticism of statistical data science, and the leftward propensity Gödel was anxious about seems to have survived until the present, whether unfortunately or fortunately.

- the late-modern shift of analytic philosophy from the Vienna circle to the Stanford school, the former advocating the unity of science, as seen in their series of publications entitled *International Encyclopedia of Unified Science*, and the latter the disunity of science, as seen in their publications such as *The Plurality of Science* by Suppes [52] and *The Disunity of Science* by Galison et al. [23];
- and at the same time with the post-modern shift of continental philosophy as represented by the so-called "end of grand narratives" thesis by (way notorious) Lyotard [35], which was, arguably, a global phenomenon as may be observed even in the analytic philosophy of science move from the global general philosophy of science to the local philosophy of special sciences (from singular "science" to plural "sciences").
- Note that this change further went hand-in-hand with the demise of foundationalism, the unity of science's nearest kin and a particular kind of grand narrative as had been endorsed by quite a majority of past philosophers longing for the so-called Archimedean vantage point according to Rorty in his *Philosophy and the Mirror of Nature* [49], who also emphasised the priority of the contingent over the absolute, and thereby took a step towards post-analytic philosophy (note further that the mirror of nature was thrown away in art as well; Klee says art does not reproduce the visible).

From another angle, the rise of analytic philosophy itself was, in a way, a symptom of disunity or the end of grand narratives in philosophy as a whole, mostly annihilating the tradition of systematic philosophy in the Anglophone world that had still remained even after the death of natural philosophy. Today science (including humanities) is fundamentally pluralist in various respects, and the unity of science is almost empirically refuted as it were, not just in natural sciences but in other sciences as well. To name but a few, the atmosphere of logic has shifted from the absolute logic of the single world to relative logics of different domains, or from the Russell-Wittgenstein's logic as representing the structure of the world to the Carnap's logic as endorsing the so-called principle of tolerance, and now logical pluralism (see, e.g., [2]) is mostly considered something obvious in practice, if not in theory. Pure mathematics has shifted from the Bourbaki-style conceptual modernist mathematics juggling with generic structures to the more down-to-earth computational post-modernist mathematics with nuanced perspectives on particular structures as exemplified by, say, quantum groups and low-dimensional topology (which also utilise category theory in substantial manners). The post-modern mathematics is similar to the 19th century mathematics, and yet it still builds upon the conceptual methods of modern mathematics, which were never available at that time and allowed for solutions to several long-standing problems as even the general public have heard of in popular science readings. The back-to-premodern movement is called the "re-enchantment" of the world by

Berman [7] and others, though it is basically a sociological concept and never applied to mathematics and other formal sciences before. The recent propensity for "retrieving realism" [17] in both analytic and continental traditions, as exemplified by structural realism [30] and speculative realism [41], would count as a case of re-enchantment in philosophy as well; note that philosophy before the realist turn had been disenchanted, from things to ideas (the Copernican turn), and to words (the linguistic turn), according to Rorty [50]. The move in philosophy from things, to ideas, to words, and back to things may be compared with the move in mathematics from things, to infinities, to symbols, and back to things. This is a finer analysis of disenchantment and re-enchantment in philosophy and mathematics; the same, more nuanced analysis could be elaborated for other fields as well.

All this tells us that the notorious idea of the end of grand narratives was indeed generally true in science as well. Yet if the post-modernist terminology is inappropriate I would instead rely upon the concept of "unitism" as opposed to that of "disunitism", which actually better illustrates the plot of the story here, that is, the transition from unity, to disunity, and to something doing justice to both. Here I would characterise unitism and disunitism by the presence of global meaning and absence of it, respectively; remember that disunitism, or course, may still keep some local meanings. The end of grand narrative, then, may be understood as the move from unitism to disunitism about knowledge: the global unity of the kingdom of knowledge collapses (through the "unexit" affair perhaps) whilst globally meaningful knowledge (resp. Truth) cuts down to separate pieces of more local knowledge (resp. truths, which are not yet "post-truth"), and the multiply divided kingdom of disunified knowledge starts to thrive thereafter, followed by the repeated, unending formation of more and more localised knowledge communities in which agents are loosely interlaced with each other by their epistemic family resemblance and in which knowledge production gets more and more localised, and only makes sense more and more locally. Put another way, there is a fundamental discrepancy between the following two views on the terminus of reason/science/civilisation, i.e., the final endpoint or utmost limit to which the dynamics of reason/knowledge/society eventually leads us:

- the modernist view that the world gradually converges into one and the same ideal limit as in the Hegelian and Kantian or neo-Kantian thoughts [20, 21, 22] (for instance because the convergence or patching condition of reason/knowledge/society is satisfied; to the physicist this means physics finally leads to the so-called theory of everything, yet it is still unclear whether it happens at the end of the day or not);
- the post-modernist view that the world diverges into diverse local limits (for instance because reason/knowledge/society is so contextual that there cannot be no unique

limit or global element patching local elements together; to be precise there is yet another case, i.e., there is no limit whatsoever; a toy example of this case would be that philosophical debates or real-life quarrels usually lead to no converging point, people eternally arguing against each other; it is hopeless, but might be reality);

- the mathematician would image what each case would look like in terms of so-called filter convergence or sheaf condition (the latter can actually model quantum contextuality as done by Isham-Butterfield and Abramsky et al.); although every option is mathematically possible, nonetheless, there is usually a unique limit, i.e., if space involved is compact Hausdorff (see any textbook on ultrafilter convergence; note however that ultrafilters may not exist without indeterministic principles; to put it in logical terms, complete or maximally consistent theories may not exist).

The "kingdom of knowledge" is not merely a fancy metaphor: division in society and division in knowledge (and also division in reason) are not separate phenomena, but they are consequences of the same disposition in human history, which I characterise by disunitism here. Whilst Lyotard focused his attention to knowledge in his end of grand narratives thesis, it is part of the even wider tendency of meaning loss or meaning bleach from the unitism/disunitism perspective. And this completes the first half of the shift.

There are of course countermovements against disunitism, and even proponents of disunity are not totally content with the disunity of science in some sense. The concept of re-enchantment by Berman [7] is actually an ideal for the future as well as an account of the after-modern intellectual tendency. Although we have come to naturally separate objects and subjects, fact and value, and so fourth, in the mechanistically divided world after the scientific revolution in the Renaissance, and although holistic meaning has thereby been lost in our conception of the universe, nevertheless, Berman [7] argues that they ought to be reunited in order to overcome the problems of contemporary society and to recover its holistic coherency in the process of the re-enchantment of the world. Yet I would rather argue that the mechanistic view is just dual to the holistic view of the world, and so they are structurally equivalent to each other in a certain sense. It is even mathematically true that the Newtonian mechanistic and Leibnizian holistic views of space are indeed dually equivalent to each other. The same holds for the theory of meaning in the analytic tradition, and the truth-conditional and verification-conditional semantics may be seen as equivalent as I have argued elsewhere. This instantiates the basic idea of duality discussed through the following sections. What is at issue here, however, is the unity versus disunity debate. And my major point about it is that duality allows us to reinstate unity without breaking disunity. Let us unpack the meaning of this in the case of mechanistic and holistic views of nature. The mechanistic view is obviously different from the holistic one, and what duality says, of course, is not that they are equal. Although Hegelian dialectics would urge us to find yet another view to reconcile the two views in conflict, however, duality refuses to do so, and proposes to see them as representing the same structure. Duality keeps pluralism in that any of the two views is not denied, and that there is no global, third view encompassing the two views, and yet it attains unity in that there is one and the same structure shared by them. What this pluralistic unity means further shall be articulated in the following sections.[2]

2 Dualism

The major concern of the present article lies in elucidating the dynamics or "logic" of duality via both philosophical and mathematical analyses. Duality has been the fundamental way of human thinking as broadly seen in human intellectual history; for example, Cartesian dualism and Hegelian dialectics instantiate (of course different) forms of duality in a broader sense. In the following I shall illustrate philosophy as a conceptual enterprise pursing the idea of duality, arguing that a number of fundamental problems in philosophy, including the long-standing realism versus antirealism issue, may be resolved by virtue of duality. The unity of realism and antirealism has been discussed in my preceding works [36, 38] as well. This sort of idea constitutes the philosophical strand of the present article, and yet there is another, mathematical strand to shed light on the mechanism of duality. In light of duality theories developed so far through the duality theorists' dedicated collaborative endeavour, I would say, we are now ready to unpack the general mechanism of the way how duality emerges, changes, and breaks; indeed, a succinct answer to the question shall be given later. In the following I am going to give a conceptual account of dualism, duality, and disduality in the abstract; disduality is basically meant to be the absence of duality, or duality-breaking. Dualism and duality are related to each other, and yet surely different. How they are different is not that obvious, though.

Developments of philosophy have centred around a tension between realism and antirealism (or in Gödel's tems between the "right" and the "left"; see the quotation and footnote above). And the dualistic tension may be illustrated by asking the nature of a variety of fundamental concepts. What is Meaning? The realist asserts it consists in the correspondence of language to reality, whilst the antirealist contends it lies in the autonomous system or internal structure of language or linguistic practice (cf. the early Wittgenstein [57] vs. the later Wittgenstein [58]; Davidson [14] vs. Dummett [18]). What is Truth? To the realist, it is the correspondence of assertions to facts or states of affairs;

[2]In my 2016 *Synthese* paper I have actually formulated the idea of pluralistic unity in a little different, more abstract manner. The conception of pluralistic unity here may be seen as an instance of that.

to the antirealist, it has no outward reference, constituted by the internal coherency of assertions or by some sort of instrumental pragmatics (cf. Russell vs. Bradley [8]). What is Being? To the realist, it is persistent substance; to the antirealist, it emerges within an evolving process, cognition, structure, network, environment, or context (cf. Aristotle [1] vs. Cassirer/Heidegger/Whitehead [10, 26, 56]). What is Intelligence? To the realist, it is more than behavioural simulation, characterised by the intentionality of mind; to the antirealist, it is fully conferred by copycatting as in the Turing test or the Chinese Room (cf. Searle [51] vs. Turing [53]). What is Space? To the realist, it is a collection of points with no extension; to the antirealist, it is a structure of regions, relations, properties, or information (cf. Newton vs. Leibniz [54]; Cantor/Russell vs. Husserl/Whitehead [3]). To cast these instances of dualism in more general terms, dualism may be conceived of as arising between the epistemic and the ontic, or between the formal and the conceptual in Lawvere's terms [32], as in the figure below:

	Ontic	***Epistemic***	
Descartes	*Matter*	*Mind*	Cartesian Dualism
Kant	*Thing-in-itself*	*Appearance*	Idealism
Cassirer	*Substance*	*Function*	Logical Idealism
Heidegger	*Essence*	*Existence*	Analysis of *Dasein*
Whitehead	*Reality*	*Process*	Holism/Organicism
Wittgenstein	*World*	*Language*	Logical Philosophy
Searle	*Intentionality*	*Simulatability*	Philosophy of Mind
Dummett	*Truth*	*Verification*	Theory of Meaning

The best known dualism, presumably, would be the Cartesian dualism between mind and matter (or mind and body), in which the ontic realm of matter and the epistemic realm of mind are separated. The Kantian dualism between thing-in-itself and appearance can readily be seen as a case of the ontic-epistemic dualism. Cassirer, the logical Neo-Kantian of the Marburg School, asserted the priority of the functional over the substantival [10], having built a purely functional, genetic view of knowledge, which was mainly concerned with modern science at an early stage of his thought as in *Substance and Function* [10], and yet eventually evolved to encompass everything including humanities in his mature *Philosophy of Symbolic Form* [11]. It is an all-encompassing magnificent Philosophy of Culture [34], indeed subsuming myth, art, language, humanities, and both empirical and exact sciences. Cassirer now counts as a precursor of what is called Structural Realism [19, 30, 31]. Yet his functionalist philosophy would better be characterised as Higher-Order Structural Realism, allowing for structures of structures (or abstraction of abstraction in a certain sense), just as in category theory. Cassirer, however, does not support ordinary abstractionism from the concrete at all, and that he rather puts strong emphasis on the generation of the abstract via conceptual symbolic formation with no prior reference to reality, which itself comes out of the symbolic construction. The symbolic generation of reality had been a central theme in his philosophy as a whole. His genetic view even payed due attention to the process of how structures are generated, just as in type theory. In light of this, Cassirer's philosophy may be regarded as a conceptual underpinning of the enterprise of category theory, and his dichotomy between substance and function as that of categorical duality. Cassirer actually started his career with work on Leibniz and his relationalism [9], at which we shall have a glance in the following.

Mathematially, the ontic-epistemic duality is best recognised in the nature of space aforementioned. There were two conceptions of space at the dawn of mathematical science: the Newtonian realist conception of absolute space and the Leibnizian antirealist conception of relational space [54]. Hundreds of years later, Whitehead [56] recognised a similar tension between point-free space and point-set space (as part of his inquiry into *Process and Reality*), advocating the latter point-free conception, which may also be found in the phenomenology of Brentano and Husserl as well (see [3]). On the one hand, points are recognition-transcendent entities (just like prime ideals/filters, maximally consistent theories, or what Hilbert called ideal elements; recall that the algebraic geometer indeed identify points with prime ideals, which, in general, only exist with the help of the axiom of choice or some indeterministic principle like that). On the other, regions (or any other aforementioned entities) are more recognition-friendly and epistemically better grounded (just as point-free topology can be developed constructively or even predicatively in the formal topology style). In general, realism and antirealism are grounded upon the ontic and the epistemic, respectively.

As shown in the figure, quite some major philosophers, whether in the analytic or continental tradition, have wondered about versions of the ontic-epistemic dualism. The fundamental problem of such a dualism is to account for how the two different realms can interact with each other; in the particular case of the Cartesian dualism, it boils down to explicating how the mind can know about the material world when they are totally (and so causally) separated. How can they causally interact at all when they are causally separated? It appears impossible; this is the typical way the philosopher gets troubled by dualism in accounting for the ontic-epistemic interaction.

Philosophy of mathematics faces an instance of the interaction problem as well. If the realm of mathematical objects and the realm of human existence are totally separated (in particular causally separated), how can human beings get epistemic access to mathematical entities? If mathematical objects exist in a Platonic universe, as in Gödel's realist philosophy for example, it seems quite hard to account for how it is possible to have a causal connection between humans in the ordinary universe and mathematical entities in the Platonic universe when the two universes are causally disconnected (this sort of problem is known as the Benacerraf's dilemma [5]). Yet if mathematical objects exist in the mind, as in Brouwer's antirealist philosophy (he counts as an antirealist at least in the sense of Dummett [18]), the account of interaction between the ontic and the epistemic is much easier, since the ontic is, just by assumption, reduced to the epistemic in this case; by contrast, then, it gets harder to account for the existence and objectivity of mathematical entities, especially transfinite ones. For instance, how can humans mentally construct far-reaching transfinite entities and does everyone's mental construction really yield the same results for sure? (To remedy the existence problem, Brouwer actually endorsed arguably non-constructive principles. Otherwise a continuum can be countable as in recursive mathematics in the Russian tradition; recall that the number of computable reals are only countably many.) Summing up, realist ontology makes it difficult to account for the possibility of epistemic access to mathematical objects; conversely, antirealist epistemology yields ontological difficulties in constructively justifying their existence and objectivity.

A moral drawn from the above discussion is that there is a trade-off between realism and antirealism: straightforward realist ontology leads to involved epistemology with the urgent issue of epistemic access to entities unable to exist in our ordinary, tangible universe; and straightforward antirealist epistemology to involved ontology with the compelling problem of securing their existence and objectivity. In general, realism gets troubled by our accessibility to abstract entities; antirealism faces a challenge of warranting their existence and objectivity. Put simply, an easier ontology of something often makes its epistemology more difficult, and vice versa. Something seems reversed between realist/ontological and antirealist/epistemological worldviews. And this is where the idea of duality between realism and antirealism comes into the play.

3 Duality

Everything, from Truth and Meaning to Being and Mind, has dual facets as aforementioned. Conceptually, duality theory, in turn, is an attempt to unite them together with the ultimate aim of showing that they are the two sides of one and the same coin. Put another way, duality allows two things opposed in dualism to be reconciled and united as just two different appearances of one and the same fundamental reality; in this sense, duality is a sort of monism established on the top of dualism (cf. Hegelian dialectics as in Lawvere's philosophy of mathematics). In Dummett's philosophy on the theory of meaning, for example, he makes a binary opposition between realism and antirealism (cf. Platonism and Intuitionism/Constructivism); what is at stake there is basically the legitimacy of recognition-transcendent truth conditions, which is allowed in realism, but not in antirealism. As in my recent works [36, 38]), however, the realist and antirealist conceptions of meaning may be reconciled and united as sharing the same sort of structure, even if they are literally opposed. In view of Dummett's constitution thesis, according to which the content of metaphysical (anti)realism is constituted by semantic (anti)realism [18, 42], this would arguably count as a unification of metaphysics as well as the theory of meaning. Duality thus conceived is a constructive canon to deconstruct dualism as it were.

Whilst having posed the Cartesian dualism as aforementioned, Descartes also developed analytic geometry, which is in a sense a precursor of duality between algebra and geometry, even though he might not have been aware that systems of equations are dual to spaces of their zero loci (logically paraphrasing, this amounts to the fact that systems of axioms are dual to spaces of their models; and such correspondence between logic and algebraic geometry can be made precise in duality theory). Notwithstanding that Galois theory may be seen as an instance of duality, Galois himself would not have been aware of the essentially categorical duality underpinning his theory, either. It would, then, be Riemann who first discovered duality between geometry and algebra in a mathematically substantial form; indeed he showed how to reconstruct (what are now called) Riemann surfaces from function fields, and vice versa, thereby establishing the (dual) equivalence between them. Even earlier than Riemann, however, Dedekind-Weber and Kronecker (mathematically) gave a purely algebraic conception of space, a sort of precursor of what is now called point-free geometry (philosophically, it would date back to Leibniz as aforementioned).

The history of duality in mathematical form, thus, goes back to the late 19th century (it ought to be noted here that duality in mathematical form basically means categorically representable duality, and so duality in projective geometry, for example, does count as an origin of duality in this sense; it would not be categorically representable, at least to my knowledge). Duality then flourished in the early 20th century, as exemplified by Hilbert's, Stone's, Gelfand's, and Pontryagin's dualities (Hilbert's Nullstellensatz is essentially a dual equivalence between finitely generated reduced k-algebras and varieties over k for an algebraically closed field k). The discovery of dualities was thereafter followed by applications to functional analysis, general topology, and universal algebra on the one hand, and to algebraic geometry, representation theory, and number theory on the other (interestingly, the Pontryagin duality plays a vital rôle in number theory as exemplified by André Weil's *Basic Number Theory* [55]). And it was eventually accompanied by the rise of categorical language in the late 20th century, which, in particular, allowed one to identify a universal form of duality (before category theory it was only vaguely understood what exactly different dualities have in common, and no one was able to spell out what presicely duality is). Today there are a great variety of dualities found across quite different kinds of science as in the following figure (and even in engineering such as optimisation, linear programming, control theory, and electrical circuit theory; duality thus goes far beyond pure mathematics, and it may sometimes be of genuine practical use as in those engineering theories):

	Ontic	**Epistemic**	
Complex Geometry	*Complex Surface*	*Function Field*	Riemann
Algebraic Geometry	*Variety/Scheme*	*k-Algebra/Ring*	Hilbert-Grothendieck
Galois Theory	*(Profinite) G-Set*	*Algebra Extension*	Galois
Representation Th.	*Compact Group*	*Representations*	Pontryagin-Tannaka
Anabelian Geometry	*Elliptic Curve*	*Frobenioid*	Mochizuki
Topology	*Topological Space*	*Algebra of Opens*	Isbell-Papert
Convex Geometry	*Convex Space*	*Semantic Domain*	Maruyama
Logic	*Space of Models*	*Algebra of Theories*	Gödel-Stone
Computer Science	*System*	*Observable Properties*	Abramsky-Smyth
System Scicence	*Controllability*	*Observability*	Kalman
Quantum Physics	*State Space*	*Alg. of Observables*	von Neumann
General Relativity	*Spacetime Manifold*	*Vector Field*	Weyl

These dualities are diverse at first sight, and yet tightly intertwined with each other in their conceptual structures. To pursue links between different dualities is indeed one of the principal aims of duality theory in category theory. Having a look at the above picture of dualities, it is particularly notable that the physics duality between states and observables is, in a way, akin to the informational duality between systems and observable properties/behaviours. How could we, then, shed light on structural analogies and disanalogies between diverse dualities? What is the generic structure or architecture of duality in the first place? Such questions propel the investigation of categorical duality theory. The duality-theoretical correspondence between logic and algebraic geometry illustrates what Ulam calls an analogy between analogies in mathematics; note that the Stone duality is concerned with equivalence between syntax and semantics, and it is actually a strengthened version of the Gödel's completeness theorem (to reinforce this point, it is named Gödel-Stone in the picture; technically, the injectivity of an evaluation map in the Stone duality exactly amounts to completeness, whereas the surjectivity, though more involved to prove, is a pure surplus from the completeness point of view). The computer science duality above is, in its mathematical substance, a form of Stone duality, and even the physics duality between states and observables may be formulated in the Stone duality style. Quite some part of the duality picture above, therefore, boils down to Stone duality, the bird's-eye view of which shall be given below. As evident in the above picture of categorical dualities, category theory today has found widespread applications in diverse disciplines of science beyond mathematics; it would now be more like foundations of science in general than foundations of mathematics in particular.[3]

Duality is even crucial for Hilbert's programme, as Coquand et al. [13] assert:

> A partial realisation of Hilbert's programme has recently proved successful in commutative algebra [..] One of the key tools is Joyal's point-free version of the Zariski spectrum as a distributive lattice [..]

In [13] they contrive a constructive version of Grothendieck's schemes by replacing their base spaces with point-free ones through the Stone duality for distributive lattices. From a categorical point of view, we could say that the spectrum functor Spec : $\mathbf{Alg}^{\mathrm{op}} \to \mathbf{Spa}$ from an algebraic category **Alg** to a topological category **Spa** amounts to the introduction of ideal elements in Hilbert's sense, and its adjoint functor the elimination of them. Duality, therefore, has contributed to Hilbert's programme and constructivism. The point-free Tychonoff theorem is constructive; this is classic. Yet the state-of-the-art goes far beyond it, encompassing not just general topology but also some mainstream mathematics such as Grothendieck's scheme theory.

In light of rich duality theories developed so far, we give succinct answers to the three questions on the mechanism of duality (for detail, see my DPhil thesis [39]):

- How does duality emerge? It is when the dual aspects of a single entity are in "harmony" with each other; what I call the harmony condition explicates this harmony. Dual adjunctions emerge when algebraic structures are harmonious with topological structures, according to (the harmony condition of) the duality theory via categorical topology and algebra. In dual adjunctions between algebras and spaces, the harmony condition basically means that the algebraic operations induced on the spectra of algebras are continuous. Dual equivalences are determined by the ratio of existing term functions over all functions, according to (my understanding of) natural duality theory.

[3]This is the thing category theory has aimed at since its early days. Granted that quite some category theorists had more or less foundationalist doctrines, nevertheless, it would be appropriate to think of category theory as local relative foundations rather than global absolute foundations, which is what set theory is about in the nature of the universe or the cumulative hierarchy of sets, just as base change is a fundamental idea of category theory. Set theory can support a multiverse view as shown in recent developments, and yet category theory intrinsically does so, I would say. Note also that the practice of mathematics is concerned with different combinations of set-theoretical and category-theoretical ideas, and the binary opposition between set theory and category theory are not very constructive or fruitful in practice, apart from philosophical issues.

- How does duality mutate? Dual structures get simplified as term functions increase; this is what natural duality theory tells us. As a limiting case, if existing term functions are all functions (i.e., functional completeness in logical terms), then dual spaces are Stone (aka. Boolean) spaces (this is the primal duality theorem; extra structures on space are indispensable in the quasi-primal duality theorem). If continuous functions coincide with term functions, then dual structures are coherent spaces (this could be called continuous functional completeness, which entails Stone duality with respect to coherent spaces).

- How does duality break? It is caused by either an excess of the ontic or an excess of the epistemic, as shall be discussed below. There are some impossibility theorems known in non-commutative algebra [6], which exhibits an excess of the epistemic. This can however be remedied by means of sheaf theory. The idea of non-commutative duality theory via sheaf theory is simple: we take the commutative core of a non-commutative algebra, dualise it, and equip the dual space with a sheaf structure to account for the non-commutative part. The same methods works for a broad variety of non-commutative algebras, including operator algebras, quantales, and substructural logics. In substructural logics, the method is further extended in such a way that in general we take the structural core of a substructural logic, dualise it, and endow a sheaf structure with it to take care of the substructural part. This process may be expressed by means of the general concept of Grothendieck situations. (For detail, see my DPhil thesis [39].)

To elucidate how duality changes in logical contexts in particular, for example, when you weaken/strengthen your logic or extend it with operators, let us also present a bird's-eye view of different logical dualities in a rough and yet intuitive manner. Stone-type dualities basically tell us that the algebras of propositions are dual to the spaces of models in the following fashion:

- Classical logic is dual to zero-dimensional Hausdorff spaces.
  - Propositions are closed opens, for which the law of excluded middle (LEM) holds, since the union of a closed open and its complement, which is closed open again, is equal to the entire space.
- Intuitionistic logic is dual to certain non-Hausdorff spaces, that is, compact sober spaces such that its compact opens form a basis, and the interiors of their boolean combinations are compact.[4]

[4]This definition of Heyting spaces came out of my joint work with K. Sato; Lurie's Higher Topos Theory [33] gives yet another definition.

  - Propositions are compact opens. The topological meaning of LEM is zero-dimensionality. In general it does not hold because the complement of a compact open is not necessarily compact open.

- Modal logic is dual to Vietoris coalgebras over topological spaces.

  - Modal operators amount to Kripke relations or Vietoris hyperspaces. This is what is called Abramsky-Kupke-Kurz-Venema duality in the thesis, relating to powerdomain constructions in domain theory as well.

Note that the existence of unit ensures that duals spaces are compact (all elements of a finitary algebra concerned yield compact subspaces, and so, if there is a unit element, the entire space is compact); otherwise they are only locally compact. The same holds for the Gelfand duality as well. There are, of course, even more logical systems you can think of:

- First-order logic may be dualised by two approaches: topological groupoids (i.e., spaces of models with automorphisms) and indexed/fibrational topological spaces (i.e., duals of Lawvere hyperdoctrines).

  - The latter approach extends to higher-order logic, thus giving duals of triposes or higher-order hyperdoctrines. It just topologically dualise the propositional value category of a hyperdoctrine or tripos.

- Infinitary logic forces us to take not even locally compact spaces into account, just like the duality for frames (aka. locales). And the resulting duality is a dual adjunction in general, rather than a dual equivalence.

  - There may not be enough models or points to separate non-equivalent propositions. There is no need for the axiom of choice thanks to infinitary operations, i.e., no need to reduce infinitaries on the topological side into finitaries on the algebraic. Note that all the other dualities require the axiom of choice to warrant the existence of enough points.

- Many-valued logics are diverse. It depends what sort of dual structure appears. It is, e.g., rational polyhedra for Łukasiewicz logic. For other logics, dual structures often include multi-ary relations on spaces as in natural duality theory.

  - Dualities for many-valued logics are mostly subsumed under the framework of dualities induced by Janusian (aka. schizophrenic) objects Ω, or Chu duality theory on value objects Ω, which may be multiple-valued. (For detail, see [39].)

You can combine some of these, and thereby obtain more complex dualities for more complex systems. Some compatibility conditions between different sorts of structures are usually required, and yet there is no general method to generate them so far. The structure of duality combinations and coherency conditions thus required would be worth further elucidation. Note that this is a rough picture of dualities in logic, and there are some inaccuracies and omissions. Notice also that not all of these dualities are induced by Janusian (aka. schizophrenic) objects, including those for intuitionistic and modal logics, in which implication and modality, respectively, are not pointwise operations on their spectra (for mode detailed accounts see [39]).

4 Disduality

The absence of duality, what is named disduality in the present article, is just as interesting on its own right as the presence of duality. According to the discussion so far, duality is about the relationships between the epistemic and the ontic. What is disduality then? In a nutshell, disduality is about an excess of the epistemic or the ontic; the duality correspondence collapses when either of the ontic and the epistemic is excessive. To articulate what is really meant here, let us focus upon two cases of disduality in the following: one is caused by incompleteness and the other by non-commutativity as in quantum theory. The former shall give a case of the excess of the ontic, and the latter a case of the excess of the epistemic.

As mentioned above, completeness may be seen as a form of duality between theories and models. What Gödel's first incompleteness theorem tells us is that there are not enough formal theories to characterise the truths of intended model(s) concerned, or to put it differently, there are some models which are unable to be axiomatised via formal theories, where theories are, of course, assumed to be finitary (or recursively axiomatisable) and stronger than the Robinson arithemetic (the technical statement of this is that the set of stronger-than-Robinson truths is not recursively enumerable). If you allow for infinitary theories, you can nonetheless obtain a complete characterisation, for example, of arithmetical truths, and yet this is not acceptable from an epistemological point of view, such as Hilbert's finitism. This is a case of disduality due to the excess of the ontic. We now turn to the other kind of disduality.

Let us have a look at a case of the excess of the epistemic. There is some sort of incompleteness in quantum algebra. The Gelfand duality tells us there is a dual equivalence between (possibly nonunital) commutative C^*-algebras and locally compact Hausdorff spaces. There have been different attempts to generalise it so as to include non-commutative algebras, in particular algebras of observables in quantum theory, and yet, as long as the duals of non-commutative algebras are purely topological, this is actually impossible (see [6]). The duality between space and algebra, thus, does not extend to the quantum realm of non-commutativity. This is indeed a case of disduality due to the excess of the epistemic: there are too many non-commutative algebras, compared to the available amount of topological spaces. The disduality may be remedied to extend the notion of space so as to include, for example, sheaves of algebras in addition to topological spaces *per se* (just as Grothendieck indeed did in his scheme-theoretical duality); in such a case, however, both sides of duality get more or less algebraic (the same may be said about the Tannaka duality for noncommutative compact groups, in which case duals are categories of representations, and so fairly algebraic).

There is another thought on the notion of disduality. No canonical agreement exists on what duality means in the first place even among category theorists as well as among philosophers. For example, some say duality is dual equivalence, whilst others say it is dual adjunction in general. A weaker notion of duality could count as a kind of (weaker) disduality. In that case we can see how far dual adjunctions are from, and yet how they (technically always but practically sometimes) transform into, dual equivalences. The different between dual equivalences and dual adjunctions do matter from a philosophical point of view. Think, for example, of physics, which may be seen as pursuing the duality between reality and observation (recall the state-observable duality above). If there is a perfect balance between reality and observation that means there is a dual equivalence between them. Yet if there is more reality than can be reconstructed from observation then it is a dual adjunction which is not a dual equivalence. Likewise if there are more observational or epistemic differences than reality can metaphysically accommodate then it is, again, a dual adjunction which is not a dual equivalence. This is not just about physics, and there are, for example, subtle theories of different balances between states and observation in theoretical computer science. What this sort of story tells us is that there can still be some sort of weaker duality (e.g., adjunction) even in the presence of disduality (e.g., non-equivalence). From this point of view, the difference between duality and disduality may be considered relative and continuous, the transition between them being gradual.

Disduality is not anything uncommon. If you have more models than theories, or if you have more theories than models, you have disduality. If you have more spaces than equations can represent as solution spaces, or if you have more equations than spaces can distinguish, you have disduality. If you have more reality than language can express, or you have more language than reality can differentiate, you have disduality. The entire enterprise of science is, in a way, about elucidating duality or disduality between formulae and solutions (i.e., substantival entities satisfying them), just as philosophy has centred around the dualism between the epistemic and the ontic. The duality/disduality between formulae and solutions is crystal-clear in logic and geometry, as seen in the formal correspondence between logic and algebraic geometry, and it further holds up for analysis and physics as well. Given the Schrödinger equation, for example, you can think of the Hilbert space of solutions, which inform us of the micro structure of the quantum universe. What if the Schrödinger equation is non-linear? You have just got an infinite-dimensional simplectic manifold as the solution space. Given the Einstein equation, you can think of the manifold of solutions, which tells you about the macro structure of the relativistic universe.[5] If there are not enough solutions to realise equations, or if there are not enough equations to formalise solutions, you have disduality (otherwise you have complete duality), and knowing about that is a gain in the enterprise of science, as Gödel incompleteness served as a fruitful theorem for later developments in different fields. Disduality is a general idea of the limit of the epistemic or the ontic. In Gödel incompleteness, what is incomplete is the epistemic, and yet in principle, it can be the other way around. And indeed it is the case in quantum theory that what is incomplete is the ontic as non-commutativity tells us (that is to say, reality is incomplete rather than quantum theory is incomplete). Duality and disduality are not fancy rhetoric, but they do pin down the fundamental meaning and limit of the scientific enterprise. Although this sounds like a bold claim, nonetheless, it is arguably supported, and to some extent justified, by numerous cases of science in which duality and disduality play central rôles.

We can even shed new light on the so-called frame problem in (philosophy of) artificial intelligence from the disduality point of view. It is concerned with the fundamental limitation of the computational theory of mind. My abstract formulation of the frame problem is as follows:

- Dimensions of reality are possibly infinite;
- Need a (finitary) frame to reduce possibly infinite dimensions and to identify the finitary scope of relevant information;
- Need a (finitary) meta-frame to choose a frame because there are possibly infinitely many frames;
- This meta-frame determination process continues *ad infinitum.*

Here every frame is assumed to be finitary, as every formal system is assumed to be finitary (i.e., recursively axiomatisable) in the standard formulation of incompleteness theorem. This argument applies to any sort of finitary entity, and so, if the human is a finitary entity then it applies to the human as well as the machine. What is essential in the frame problem is the finitude of beings. From this point of view, the frame problem is about the

[5]What the structure of a physical theory is has been a central issue in recent developments of structural realism in the analytic philosophy of science. There could be different solutions. For one thing, the structure of a physical theory may be understood as the structure of the solution space of fundamental equation as explained here.

fundamental disduality between the finitude of beings and the infinity of reality; Gödel incompleteness tells us there is no finitary means to characterise (stronger-than-Robinson-arithmetic) truths (more formally saying, the truths are not recursively enumerable). And the finitude of beings is rooted in the temporality of beings (as Heidegger says); if temporality is ignored the intelligence system may compute for an infinite amount of time and hence no need for singling a finitary frame out and for being worried about the frame problem. Or if an infinitary frame is allowed then there is no frame problem any more. The same happens in incompleteness: if an infinitary system or infinitary computation is allowed then there is no incompleteness any more, that is, there are complete systems characterising the truths. The finitarity assumption is crucial in both phenomena. It may thus be said that the frame problem and Gödelian incompleteness are essentially the same phenomena. The notirous Lucas-Penrose argument [40] for the impossibility of artificial intelligence on the ground of incompleteness has already been much criticised by many (e.g., by Feferman). In my opinion, they may be wrong about the impossibility of artificial intelligence; however, their overall point seems to be something similar to the point of my argument above, or at least my (possibly biased) interpretation of their argument is so.

The Lucas-Penrose argument is essentially about a limitation of the computational theory of mind, and it could be adapted so as to be applicable for the computational theory of the universe or the so-called information physics (see, e.g., [12]):

- If anything (or any mechanism) in the universe allows for solving uncomputable problems then the universe cannot be computational, i.e., any computational system cannot simulate the universe (if it can that means the computer can solve uncomputable problems and this is a contradiction);
- The corresponding version for the computational theory of mind (there are many versions of the Lucas-Penrose argument): if the human mind can solve undecidable problems it cannot be computational, i.e., there is no computational system to simulate it (put another way, artificial intelligence is impossible).
- What is crucial here is that if the human mind can solve undecidable problems then there is something in the universe which can compute uncomputable problems and thus the universe cannot be computational. That is, if the human mind is more powerful than the computing system the universe cannot be simulated by the computing system because humans are part of the universe.

The last point may be put this way: if artificial intelligence is impossible, information physics is impossible, too. So it exposes an interesting, unexpected link between the possibility of artificial intelligence and that of information physics. And all this is about the fundamental disduality between finitary beings and infinitary reality. This completes the discussion on the trichotomy of dualism, duality, and disduality.

5 The Logic of Duality in the Kyoto School

This is the final section for concluding remarks; notwithstanding, I would dare to address yet another facet of duality rather than to give boringly superficial wrapping-up conclusions. No one would ever imagine categorical duality relates to the Kyoto school of philosophy [15], and nevertheless it indeed does as I am going to argue in the following. And the argument is really simple, I would say. We have started the discussion with the understanding of modernism as the disunity of worldviews, which is actually shared by the Kyoto school of philosophy. The Kyoto school's ideal of "overcoming modernity" [44] precisely lies in going beyond the disunity of worldviews (via integration of Western and Japanese/Eastern/Oriental thoughts). Indeed, Nishitani, a member of the Kyoto school, says that modernisation has divided our worldview into three conflicting ones: the nature-centred worldview, the human-centred worldview, and the God-centred worldview [44].

It may be said that, after the loss of global order or global meaning, which is the major characteristic of modern disunitism, the cosmos as a holistically united whole has become the universe as a physical entity, and accordingly cosmology has come down to mere materialistic studies on the universe. Although some argue for the value of disunity as the Stanford school indeed does, nonetheless, the Kyoto school regarded the "Construction of a Unified Worldview as the Fundamental Challenge of the Contemporary Era" in Nishitani's terms [44]. This sort of ideal for unity may be traced back to the even older Amane Nishi's programme "Interweaving a Hundred Sciences" [43], and it would be fair to say that it has been a distinctively Japanese ideal to integrate the Western system of knowledge and the Japanese/Eastern/Oriental one. Yet at the same time the Kyoto school philosophy of overcoming modernity is naturally interlaced with the problematics of modernity in the Western world in many respects and in astonishingly substantial manners. For instance, Nishida [47, 43], the central founder of the Kyoto school, attempts to overcome the division of subjects and objects on the basis of his theory of pure experience, and his problematisation of the subject-object dichotomy is exactly the same as Berman's in his enterprise of re-enchantment. The philosophy of the Kyoto school has centred around the resolution of dichotomies; this would already suggest it may somehow relate to the idea of duality. Yet if we have a look at the Kyoto school's studies on logic (which is never meant to be mathematical logic, apart from the early philosophy of science by Tanabe, who paid a lot of attention to developments of mathematical sciences) the compellingly tight relationships between duality and the Kyoto school would have become crystal-clear.

Major members of the Kyoto school have endorsed the common logic of (absolute) nothingness, including Nishida and Nishitani.[6] It is also called the logic of topos in Nishida's philosophy. The logic of nothingness is inspired by both Japanese/Eastern/Oriental philosophy and Western one such as Hegelian dialectics, and there are actually diverse accounts of it, some of which might not be equivalent. Here I follow Nishida's account in his treatise entitled "Absolutely Contradictory Self-Identity" or "Absolutely Paradoxical Self-Identity" [45] (the title is hard to translate into English; the former is adopted more frequently than the latter). He there addresses the absolutely contradictory self-identity between unity and plurality, arguing that that is the right way to conceive of the world. The details of the argument does not matter here; what does matter, rather, is the pattern of his logic. He then relates it to the absolutely contradictory self-identity between appearance and reality, to that between subjects and environments, and to that between the mechanistic view and the teleological view. And he further repeats, again and again, this absolutely contradictory self-identity logic for different dichotomies in the same article and elsewhere as well. The same pattern can be found in other Kyoto school thinkers' writings. Given a dichotomy between two things, they resolve the conflict or division between them by means of equating them. This is the simple structure of the logic of (absolute) nothingness. You would, of course, wonder why it is justified to equate them. There are some complex arguments for it, yet from a contemporary point of view, we can appeal to duality to equate them. Duality is precisely the means of equating two things whilst keeping them different. And this is the fundamental reason why the logic of duality shares the same structure as the logic of (absolute) nothingness.

In face of a conflict/division/dichotomy between two things/views/terms, there are different strategies to treat it, possibly including doing nothing for it. One is Hegelian dialectics, that is, to find yet another, global thing/view/term as encompassing both of them. Another would be Derrida's deconstruction, which attacks the condition of possibility of the conflict/division/dichotomy, thereby destructing it in its very foundation. And yet another strategy is duality or the logic of (absolute) nothingness, which does not rely upon any grand narrative such as Hegelian synthesis, and which elucidates absolutely non-contradictory structural identity between them. In particular, philosophy tends to differentiate and oppose positions, and there has not been much attention to looking at

[6] Tanabe's logic of species is a little different from their logic of nothingness, concerned with the logic of the relationships between individuals and societies, or those between points and spaces. As recent research [25] shows, Tanabe indeed read Brouwer's theory of continuums, which is an origin of what is now called point-free topology, and applied it in social contexts. The priority of societies/spaces over individuals/points had let to some sort of totalitarianism, which may be called point-free sociology, and he came to be criticised by that after the WWII. It made post-war researchers away from his philosophy, thus having delayed the dissemination of his work after the war. And yet there is some revival now.

structural similarities between different philosophies such as realism and antirealism. The realist conception of space and the antirealist one are provably equivalent [39]. The realist conception of meaning and the antirealist one are arguably equivalent (as model-theoretic and proof-theoretic semantics are categorically equivalent [36, 38]). These are exactly the same sort of enterprises as the Kyoto school undertook for different opposing positions as addressed above. Duality and the Kyoto school are united at this cardinal point.[7]

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[7] This completes the concluding remarks; I hope they were more informative than merely summing-up conclusions. The present article ought to be seen as a prolegomenon to the entire story of the philosophy of duality, and there are indeed numerous other faces of duality to be explored from philosophical perspectives. Even so, I believe it has shed new light on several fundamental issues in the philosophy of duality, and has thereby paved the way for further investigation in this emerging field. Discussions omitted or only briefly touched on here shall be elaborated somewhere else in the future. Last, but not least, I would express my sincere gratitude to those colleagues in Oxford and Kyoto who kindly discussed some of the ideas spelled out here together with me, and for the reader's patience as well. Some materials here are taken from my DPhil thesis in Oxford [39]. The work was partially supported by my Hakubi and JSPS grants.

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Graduate School of Letters
The Hakubi Centre for Advanced Research
Kyoto University
Kyoto 606-8502
JAPAN
E-mail address: maruyama@i.h.kyoto-u.ac.jp

Department of Computer Science
Mathematical, Physical, and Life Sciences Division
University of Oxford
Oxford OX1 3QD
UNITED KINGDOM
E-mail address: maruyama@cs.ox.ac.uk