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Note on derivations of lattices

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Abstract

In this paper we consider some properties on derivations of lattices and show that (i) for a derivation d of a lattice L with the maximum element 1, it is monotone if and only if $d(x) \leq d(1)$ for all $x \in L$ (ii) a monotone derivation d is characterized by $d(x) = x \wedge d(1)$ and (iii) simple characterization theorems of modular lattices and of distributive lattices are given by derivations.

1 Introduction

A notion of derivations of algebras with two operations $+$ and $\cdot$ has introduced as an analogy of derivations of analysis and then some properties of derivations are considered. For an algebra $A = (A, +, \cdot)$, a map $f : A \to A$ is called a derivation if it satisfies the conditions: For all $x, y \in A$,

$$f(x + y) = f(x) + f(y)$$
$$f(x \cdot y) = f(x) \cdot y + x \cdot f(y)$$

The notion of derivation is important in the theory of rings ([5]). After that, it is applied to lattices ([4]), where operation $+$ and $\cdot$ are interpreted as lattice operations $\vee$ and $\wedge$, respectively. Following the naive interpretation, the derivation d of a lattice L may be defined by

$$(a) \quad d(x \vee y) = d(x) \vee d(y)$$
$$(b) \quad d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)).$$

As proved in [4, 6], the condition (a) says d to be monotone and then the condition (b) is equivalent to the condition $d(x \wedge y) = d(x) \wedge y$. Hence, as proved later, a monotone derivation $f : L \to L$ is characterized by $f(x \wedge y) = f(x) \wedge y$ for all $x, y \in L$. It follows from the result that a monotone derivation d has the form of $d(x) = x \wedge d(1)$ if L has the maximum element 1 and thus every monotone derivation is determined completely by the value $d(1)$.

In order to obtain more interesting properties of derivations of lattices, we adopt another definition of derivations according to [1, 2, 3, 7] and prove some fundamental properties of them, from which we get new results about derivations of lattices and provide accurate statements described in [1, 2, 3, 6, 7]. Moreover, we consider properties of generalized derivation ([1, 2]).

Concretely, we prove that

(i). For a derivation d of a lattice L with a maximum element 1, it is monotone if and only if $d(x) \leq d(1)$ for all $x \in L$.

(ii). A monotone derivation d is just the form of $d(x) = x \wedge d(1)$.

(iii). For any lattice L and a derivation d, the condition

$$d \text{ is monotone} \Leftrightarrow d(d(x) \vee y) = d(x) \vee d(y) \quad (\forall x, y \in L)$$

is equivalent to that L is a modular lattice.

(iv). For any lattice L and a derivation d, the condition

$$d \text{ is monotone} \Leftrightarrow d(x \vee y) = d(x) \vee d(y) \ (\forall x, y \in L),$$

is equivalent to that L is a distributive lattice.

2 Derivations

According to [6, 7], we give a definition of derivation of a lattice. Let $L = (L, \vee, \wedge)$ be a lattice. A map $d : L \to L$ is called a *derivation* of L if it satisfies the condition

$$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)) \quad (\forall x, y \in L)$$

Moreover, a derivation d is called *monotone* if

$$x \leq y \Rightarrow d(x) \leq d(y) \quad (\forall x, y \in L).$$

We note that the notion of monotone is called isotone in [1, 2, 3, 7])

Example 1. Let L be a lattice and $a \in L$. If we define a map $d_a : L \to L$ by $d_a(x) = x \wedge a$, then d_a is a monotone derivation. Indeed, for all $x, y \in L$, we have $d_a(x \wedge y) = (x \wedge y) \wedge a = ((x \wedge a) \wedge y) \vee (x \wedge (y \wedge a)) = (d_a(x) \wedge y) \vee (x \wedge d_a(y))$.

Example 2. ([3]) Let $L = \{0, a, b, 1\}$, $(0 < a < b < c < 1)$. We define $d : L \to L$ by

$$d(x) = \begin{cases} 0 & (x = 0) \\ a & (x = a, b) \\ c & (x = c, 1) \end{cases}$$

It is clear that $d : L \to L$ is the derivation of L.

We have fundamental results about derivations of lattices.

Proposition 1. *Let L be a lattice and d be a derivation of L. For all $x, y \in L$,*
(1) $d(x) \leq x$
(2) $d(d(x)) = d(x)$
(3) *If* $1 \in L$, *then* $d(x) = d(x) \vee (x \wedge d(1))$
(4) *If* $1 \in L$, *then* $d(1) = 1 \Leftrightarrow d = id_L$
(5) $d(x) \wedge d(y) \leq d(x \wedge y) \leq d(x) \vee d(y)$
(6) $d(d(x) \wedge d(y)) = d(x) \wedge d(y)$
(7) *If d is monotone, then* $d(d(x) \vee d(y)) = d(x) \vee d(y)$
(8) *If* $d(d(x) \vee y) = d(x) \vee d(y)$, *then d is monotone.*

We note that the derivation $d_a(x) = x \wedge a$ in Example 1 is monotone. Moreover, any monotone derivation d has just the form of $d(x) = x \wedge a$ for some $a \in L$. In order to prove this fact, we deeply think about properties of monotone derivations.

Theorem 1. *For any derivation d, the following conditions are equivalent to each other.*
(1) *d is monotone ;*
(2) $d(x \wedge y) = d(x) \wedge d(y) \quad (\forall x, y \in L)$;
(3) $d(x) \vee d(y) \leq d(x \vee y) \quad (\forall x, y \in L)$.

Proof. We only show the cases (1) $\Rightarrow$ (2). The other cases can be proved easily.

Since $x \wedge y \leq x, y$, we have $d(x \wedge y) \leq d(x), d(y)$. On the other hand, since $d(x \wedge y) \leq d(x) \wedge d(y) \leq x \wedge y$, we get $d(x \wedge y) = d(d(x \wedge y)) \leq d(d(x) \wedge d(y)) \leq d(x \wedge y)$. Thus $d(x \wedge y) = d(d(x) \wedge d(y))$. It follows that

$$\begin{aligned} d(x \wedge y) &= d(d(x) \wedge d(y)) \\ &= \{d(d(x)) \wedge d(y)\} \vee \{d(x) \wedge d(d(y))\} \\ &= (d(x) \wedge d(y)) \vee (d(x) \wedge d(y)) \\ &= d(x) \wedge d(y). \end{aligned}$$

□

From the result above, a monotone derivation can be characterized as follows.

Theorem 2. *Let L be a lattice and $f : L \to L$ be a map. Then*
(1) $f(x \wedge y) = f(x) \wedge y \quad (\forall x, y \in L) \Rightarrow f$ *is a monotone derivation.*
(2) *f is a monotone derivation* $\Rightarrow f(x \wedge y) = f(x) \wedge y \quad (\forall x, y \in L)$
(3) $f(x \wedge y) = f(x) \wedge y \quad (\forall x, y \in L) \Leftrightarrow \; f(x) = x \wedge f(1) \quad (\forall x \in L)$

Proof. We only show the cases (1) and (2).

(1) Since $f(x \wedge y) = f(y \wedge x) = f(y) \wedge x$, we get $f(x \wedge y) = f(x) \wedge y = x \wedge f(y)$ and $f(x \wedge y) = (f(x) \wedge y) \vee (x \wedge f(y))$, that is, f is a derivation. Moreover, if $x \leq y$ then $f(x) = f(y \wedge x) = f(y) \wedge x \leq f(y)$ and f is monotone.

(2) Let f be a monotone derivation. Since $x \wedge y \leq x, y$, we get $f(x \wedge y) \leq f(x), f(y)$ and $f(x \wedge y) \leq f(x) \wedge y, x \wedge f(y)$ by $f(x \wedge y) \leq x \wedge y \leq x, y$. On the other hand, since f is the derivation, we have $f(x \wedge y) = (f(x) \wedge y) \vee (x \wedge f(y)) \geq f(x) \wedge y, x \wedge f(y)$. This means that $f(x \wedge y) = f(x) \wedge y = x \wedge f(y)$. □

Corollary 1. *If L has a maximum element* 1 *then the following conditions are equivalent.*
(1) *d is a monotone derivation.*
(2) *$d(x) = x \wedge d(1)$ for all $x \in L$.*
(3) *$d(x) \leq d(1)$ for all $x \in L$.*

Corollary 2. *If d is a monotone derivation of L, then $d(d(x) \vee d(y)) = d(x) \vee d(y)$ for all $x, y \in L$.*

Unfortunately, the converse of the result above does not hold, namely, d may not be monotone even if $d(d(x) \vee d(y)) = d(x) \vee d(y)$ holds. We have a counterexample. Let $L = \{0, a, b, 1\}$ with $0 < a < b < 1$. If we define $d : L \to L$ by $d(0) = d(1) = 0, d(a) = d(b) = b$, then it is easy to show that d is a derivation and $d(d(x) \vee d(y)) = d(x) \vee d(y)$, but d is not monotone.

Remark 1. A map $f : L \to L$ for a lattice L is called an *interior operator* if

(io1) $x \leq y \Rightarrow f(x) \leq f(y)$
(io2) $f(x) \leq x$
(io3) $f(f(x)) = f(x)$

It follows from our result above that a monotone derivation is an interior operator.

Remark 2. A similar results to our theorem 1 are already proved in [7] as Theorem 3.19 and Theorem 3.21.

Theorem 3.19. Let L be a modular lattice and d be a derivation of L. Then the following conditions are equivalent:

(1) d is a monotone;
(2) $d(x \wedge y) = d(x) \wedge d(y)$;
(3) If $d(x) = x$, then $d(x \vee y) = d(x) \vee d(y)$,

where a lattice L is called *modular* if

$$x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z \quad (\text{for all } x, y, z \in L).$$

Theorem 3.21. Let L be a distributive lattice and d be a derivation of it. Then the following conditions are equivalent:

(1) d is a monotone;
(2) $d(x \wedge y) = d(x) \wedge d(y)$;
(3) $d(x \vee y) = d(x) \vee d(y)$.

Our results are stronger than those of above, because our results say that monotone is equivalent to the condition (2) $d(x \wedge y) = d(x) \wedge d(y)$ for all lattice L, namely, we do not assume modularity nor distributivity to get such results.

Moreover, we obtain a following identity condition instead of (3) If $d(x) = x$, then $d(x \vee y) = d(x) \vee d(y)$ in Theorem 3.19 in [7].

Theorem 3. *Let L be a modular lattice and d be a derivation. Then we have d is a monotone $\Leftrightarrow d(d(x) \vee y) = d(x) \vee d(y) \quad (\forall x, y \in L)$*

Moreover we prove the converse.

Theorem 4. *For any lattice L and derivation d of it, if the condition holds*

$$d \text{ is monotone} \Leftrightarrow d(d(x) \vee y) = d(x) \vee d(y) \quad (\forall x, y \in L),$$

then L is a modular lattice.

Proof. For every $z \in L$, if we consider a map $d_z(x) = x \wedge z$ then it is a monotone derivation. By assumption, the map d_z satisfies

$$d_z(d_z(x) \vee y) = d_z(x) \vee d_z(y) \quad (\forall x, y \in L)$$

and hence $((x \wedge z) \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$. This implies that if $x \leq z$ then $(x \vee y) \wedge z = x \vee (y \wedge z)$. Therefore L is the modular lattice. □

We also have a similar result about distributive lattices.

Theorem 5. *Let L be a distributive lattice and d be a derivation. Then we have*

$$d \text{ is monotone} \Leftrightarrow d(x \vee y) = d(x) \vee d(y) \ (\forall x, y \in L)$$

Conversely,

Theorem 6. *For any lattice L and derivation d of it, if the condition holds*

$$d \text{ is monotone} \Leftrightarrow d(x \vee y) = d(x) \vee d(y) \ (\forall x, y \in L),$$

then L is a distributive lattice.

The above results provide characterization theorems of modular lattices and of distributive lattices in terms of derivations.

Remark 3. If d is a monotone derivation then a subset

$$\mathrm{Fix}_d(L) = \{x \in L \mid d(x) = x\}$$

of L is an *ideal* of L, that is, $\mathrm{Fix}_d(L)$ satisfies the conditions

(I1) $0 \in \mathrm{Fix}_d(L)$
(I2) $x \in \mathrm{Fix}_d(L), y \leq x \ \Rightarrow \ y \in \mathrm{Fix}_d(L)$
(I3) $x, y \in \mathrm{Fix}_d(L) \ \Rightarrow \ x \vee y \in \mathrm{Fix}_d(L)$.

3 Generalized derivations

Some types of derivations, such as *generalized derivation*, *generalized (f,g)-derivation* and *f-derivation*, are defined and properties of them are considered in [1, 2, 3]. We only treat gfeneralized derivations according to [1]. A map $D : L \to L$ is called a *generalized derivation* if it satisfies the condition: For a derivation d,

$$D(x \wedge y) = (D(x) \wedge y) \vee (x \wedge d(y))$$

We get basic results about a generalized derivation D without difficulty.

Proposition 2 (cf. Proposition 3.4, 3.9 [1]). *Let d be a derivation and D be a generalized derivation. Then we have*
(1) $d(x) \leq D(x) \leq x$
(2) $D(D(x)) = D(x)$
(3) $D(x) \wedge D(y) \leq D(x \wedge y)$
(4) $D(x) \wedge D(y) = D(D(x) \wedge D(y))$
(5) $D(x) = d(x) \vee (x \wedge D(1))$

We also have a new result about a generalized derivation D.

Proposition 3. *Let d be a derivation and D be a generalized derivation. Then we have $D \circ d = d \leq d \circ D$*

It follows from our result that a characterization theorem about monotone generalized derivations can be proved similarly.

Proposition 4. *(Proposition 3.12 [1]) For a generalized derivation D, the following conditions are equivalent to each other:*
(1) *D is monotone;*
(2) $D(x \wedge y) = D(x) \wedge D(y)$;
(3) $D(x) \vee D(y) \leq D(x \vee y)$;
(4) $D(x) = x \wedge D(1)$ *if L has a maximum element* 1.

Proposition 5. *If L has a maximum element 1, then any generalized derivation D has a following form*

$$D(x) = (D(1) \wedge x) \vee d(x)$$

Corollary 3. $D(1) = 1 \Leftrightarrow D = id_L$

Lemma 1. *If L has a maximum element 1 and $d(x) \leq D(1)$ for all $x \in L$, then*

$$D(x) = x \wedge D(1)$$

In this case, the generalized derivation D is monotone. Conversely, if D is monotone then $d(x) \leq D(1)$ for all $x \in L$. Therefore, we have another characterization of monotone generalized derivations.

Theorem 7. *For any generalized derivation D,*

$$D \text{ is monotone} \Leftrightarrow d(x) \leq D(1) \quad (\forall x \in L)$$

Corollary 4. *If d is monotone, then so D is.*

We may ask whether the converse holds, that is, if a generalized derivation D is monotone then so d is ?

Unfortunately, this does not hold by the following example.

Example 3 Let $L = \{0, a, b, 1\}$, $(0 < a < b < 1)$ and $d, D : L \to L$ be maps defined by

$$d(x) = \begin{cases} 0 & (x = 0, 1) \\ a & (x = a, b) \end{cases}$$

$$D(x) = \begin{cases} x & (x = 0, a, b) \\ b & (x = 1) \end{cases}$$

It is easy to show that d is a derivation and D is a generalized derivation. Moreover D is monotone. However, it is obvious that d is not monotone.

In the previous section, we provide characterization theorems of modular lattices and of distributive lattices in terms of derivations. We also have similar results about generalized derivations.

Theorem 8. *For any lattice L and generalized derivation D of it, if the condition holds*

$$D \text{ is monotone} \Leftrightarrow D(D(x) \vee y) = D(x) \vee D(y) \quad (\forall x, y \in L),$$

then L is a modular lattice.

Proof. For every $z \in L$, if we define maps d_z and D_z by $d_z(x) = x \wedge z = D_z(x)$ for all $x \in L$. It is clear that d_z is a derivation and D_z is also a generalized derivation. Since D_z is monotone, it follows from assumption that $D_z(D_z(x) \vee y) = D_z(x) \vee D_z(y)$ and thus $((x \wedge z) \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$. This implies that if $x \leq z$ then $(x \vee y) \wedge z = x \vee (y \wedge z)$. Therefore L is the modular lattice. □

Theorem 9. *(Theorem 3.14 [1]) Let L be a distributive lattice and D be a generalized derivation. Then we have*

$$D \text{ is monotone} \Leftrightarrow D(x \vee y) = D(x) \vee D(y) \ (\forall x, y \in L)$$

Conversely,

Theorem 10. *For any lattice L and generalized derivation D of it, if the condition holds*

$$D \text{ is monotone} \Leftrightarrow D(x \vee y) = D(x) \vee D(y) \ (\forall x, y \in L),$$

then L is a distributive lattice.

The above results provide characterization theorems of modular lattices and of distributive lattices in terms of generalized derivations.

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