数理解析研究所講究録
第2051巻 2017年 35-42

Exploring Meta-Symmetry for Configurations in Closure Spaces

Marcin J. Schroeder
Akita International University, Akita, Japan
mjs@aiu.ac.jp

1. Introduction

This paper continues research presented earlier in which the study of symmetry was generalized to arbitrary closure spaces. [1] The present paper is exploring one of the consequences of this generalization. All possible closure spaces on a given set S (i.e. all possible closure operators on this common set S) can be associated with corresponding Moore families $\mathcal{M}$ of closed subsets. Then each of these Moore families of closed subsets $\mathcal{M}$ can be considered a closed subfamily of the power set $\wp = 2^S$ of S. For arbitrary family $\mathcal{B}$ of subsets of S we can find the least Moore family $\mathcal{M}$ of subsets including this family ($\mathcal{B} \subseteq \mathcal{M}$). Equivalently, for every family $\mathcal{B}$ of subsets of S we can consider a larger family $\mathcal{M}$ with all intersections of subsets belonging to the original family $\mathcal{B}$. Obviously this will be a Moore family $\mathcal{M}$ considered before, i.e. the least Moore family including the original one. We assume that the intersection of every empty family is entire power set of S.

Thus, the set of all closure spaces defined on S, or all closure operators defined on S defines one specific closure operator f on the power set $\wp$. Now, in the earlier paper a theory of symmetries for configurations of closed subsets of an arbitrary closure space is developed. In this paper we apply this generalization of symmetry to the particular case of the closure space defined on the power set $\wp$ of the set S by the closure operator which extends any family of subsets to the least Moore family, i.e. our closure operator f on $\wp$. The closure space defined by this closure operator on the power set of S is called a meta-closure space. The closed subsets of a meta-closure space are directly and bijectively corresponding to closure operators on S. The symmetry of configurations of closure operators (or spaces) defined this way is called here "meta-symmetry".

2. From Erlangen Program to General Concept of Symmetry

Spectacular success of the concept of symmetry in mathematics, where it became understood as invariance with respect to a class of transformations (in the consequence of the Erlangen Program of Felix Klein published in 1872 [2]) and following this success rise to the fundamental role of symmetry and symmetry breaking in physics and physical sciences generated interest in this concept among representatives of other disciplines as far from physics as those in the humanities. The immense popularity of the book "Symmetry" published by Hermann Weyl in 1952 greatly contributed to this wide spread of interest. [3] Weyl demonstrated the universal character of symmetry as a tool for the study of structures not only in mathematics and natural sciences, but also in art.

A half century earlier before symmetry and invariance with respect to transformations became the focus of intellectual intercourse across the wide range of non-scientific disciplines, structural (synchronic) studies started to be considered as an alternative to those with historical (diachronic) methodology. The opposition of synchronic and diachronic perspectives was introduced by Ferdinand de Saussure in the context of linguistics (more specifically in his lectures 1907-1911 posthumously published by his disciples in 1916 [4]). However the need for this distinction was recognized even earlier in the works of psychologists.

Structuralism as a philosophical direction can be understood as the response to the calls for a methodology for disciplines in which the use of measures and numbers dominating natural sciences seemed impossible or ineffective. Explosion of enthusiastic interest in structuralism and in its tool of symmetry studies initiated by the Erlangen Program did not last long outside of science. Only quite recently structuralism and structural realism became the subject of revived interest.

My own diagnostic of the disappointment, of the lost popularity and of the decline of interest in structuralism outside of mathematics and physical sciences was that its methodology derived from the Erlangen Program of Klein was heavily dependent on coordinatization, very natural in geometry and physics, but of limited meaning outside these disciplines. [1] Transformations of a geometric space can be easily defined with the use of coordinates of points. Without coordinates the only alternative was to consider finite, discrete, or combinatorial cases in which all possible permutations are considered. This is why structuralism in anthropology, [5] or in developmental psychology, [6] engaged only extremely simple forms of symmetry described in terms of the Klein group.

This issue was convoluted with quite frequent misunderstanding of the core idea of symmetry in Klein's Erlangen Program. In some extent this confusion can be "blamed" on Klein, because his original exposition excluded completely the case of topological groups of transformations relegated to the reference to the works of Sophus Lee. Thus, even if the group of all rotations around a point is not discrete, its topological properties were not investigated or even mentioned.

Of course blaming Klein would be an anachronism, as general topology did not exist at that time and Lee just referred to continuity for metric spaces. Exclusion of the continuity of groups such as that of rotations from consideration in 1872 was not problematic. After all, Klein clearly required the introductory step of the selection of the group of transformations which defines the general concept of geometry (for Klein it was projective geometry). Subgroups of this group selected by appropriate conditions defined particular forms of geometry, or what Weyl called symmetries of the space. [2] Only after appropriate group of transformations characterizing the choice of geometry was selected, symmetries of particular configurations of geometric objects were studied in terms of subgroups of transformations for which configurations were invariant. There was no reason to involve in this entire "symmetric group", i.e. group of all permutations. For instance, Euclidean geometry was characterized by the subgroup of all Euclidean isometries (transformations that preserve Euclidean distance of points). Only in the next step after establishing this symmetry of the space, specific configurations of points could be considered.

The omission of the fact that for the study of symmetry of given configuration not all transformations for which this configuration is invariant are important and actually many such transformations have to be excluded had grave consequences. Many recent books popularizing the concept of symmetry promote nonsensical views such as symmetry of a configuration is any function which makes this configuration invariant and all these functions form its "group of symmetries". Of course this kills the very idea of Klein's Erlangen Program in which it is necessary to start from the distinction of a specific group of transformations defining the context of study (for instance geometry, topology, or whatever it is) and then to consider a Galois connection between subgroups of this group and configurations of points (or possibly other objects, such as lines) ordered by inclusion. The distinction of the original group of transformations does not preclude the choice of the symmetric group (group of all transformations), but this is only very special case rarely interesting outside of combinatorics.

The problem studied in my earlier work was how to eliminate the need for coordinatization of the set (in geometry entire plane or space) in order to define transformations and to distinguish appropriate groups and their subgroups. This goal was achieved by a formulation of the theory of symmetry in terms of an arbitrary closure space. To make this paper self-contained a brief summary of the earlier paper will be included.

3. Revisiting Concept of Symmetry in General Closure Spaces

The following notation and terminological conventions will be used throughout the text:

Greek letters such as ϕ, φ, θ, etc. indicate functions on the elements of a given set S and with the values belonging to a set T. Small Latin letters such as *f, g, h,* etc. indicate functions defined on the subsets of a given set and with the values which are subsets of this set. The double use of the symbol $\varphi^{-1}(A)$, as the set of values for the inverse function of φ, and as an inverse image of a set A with respect to function φ which may not have inverse, should not cause problems. The composition of functions is written as a juxtaposition of their symbols, unless the fact of the use of a composition of functions is contrasted with constructing function images. The symbol $\cong$ indicates a bijective correspondence or isomorphism. Throughout the paper, partially ordered sets are often called posets.

The purpose of these preliminaries is to specify terminological and notational conventions, not to present the introduction to the subject which can be found elsewhere [7] These preliminaries are followed by the brief presentation of main result of the earlier paper on symmetry in an arbitrary closure space. [1]

Definition 3.1 *Let f be a function from the power set of a set S to itself which satisfies the following two conditions:*

(1) $\forall A \subseteq S: A \subseteq f(A)$,

(2) $\forall A,B \subseteq S: A \subseteq B \Rightarrow f(A) \subseteq f(B)$,

(3) $\forall A \subseteq S: ff(A) = f(f(A)) = f(A)$.

Then f is called a closure operator (or transitive closure operator) on S. The set of all closure operators on the set S is indicated by I(S). A set equipped with a closure operator will be called a closure space $<S, f>$.

The third conditions can be replaced by a condition: which is easier to use in proofs, but which in combination with other two gives exactly the same concept:

(3)* $\forall A,B \subseteq S: A \subseteq f(B) \Rightarrow f(A) \subseteq f(B)$.

The stronger form of this condition $\forall A,B \subseteq S$: $A \subseteq f(B)$ *iff* $f(A) \subseteq f(B)$ *can be used instead of all three conditions to define a transitive operator, but this fact does not have a significant practical importance.*

Definition 3.2 *Let f be a closure operator on a set S. The subsets A of S satisfying the condition f(A)=A, called f-closed sets form a Moore family f-Cl, i.e. it is closed with respect to arbitrary intersections and includes the set S (which can be considered the intersection of the empty subfamily of subsets). Every Moore family* $\mathcal{M}$ *defines a transitive operator* $f(A) = \cap \{M \in \mathcal{M}: A \subseteq M\}$. *Set theoretical inclusion defines a partial order on* ***f-Cl*** *with respect to which it is a complete lattice. To this structure we will refer as the complete lattice* L_f *of f-closed (or just closed) subsets.*

Let f and g be operators on a set S. The relation defined by $f \leq g$ if $\forall A \subseteq S$: $f(A) \subseteq g(A)$ is a partial order on I(S), with respect to which it is a complete lattice. This partial order corresponds to the inverse of the inclusion of the Moore families of closed subsets

Definition 3.3 *Let f be a closure operator on a set S, g a closure operator on set T, and φ be a function from S to T. The function φ is (f,g)-continuous if $\forall A \subseteq S$: $\varphi f(A) \subseteq g\varphi(A)$. We will write continuous, if no confusion is likely.*

Proposition 3.1 *Continuity of the function φ as defined above is equivalent to:*

$\forall B \in g\text{-}Cl$: $\varphi^{-1}(B) \in f\text{-}Cl$.

Definition 3.4 *Let f be a closure operator on a set S, g a closure operator on set T, and φ be a function from S to T. The function φ is (f,g)-isomorphism if it is bijective and $\forall A \subseteq S$: $\varphi f(A) = g\varphi(A)$. We will write isomorphism, if no confusion is likely. If S=T, we will call φ an (f,g)-automorphism, or smply automorphism.*

Proposition 3.2 *The conditions for a function φ to be an isomorphism, as defined above, are equivalent to either one below:*

(1) φ has an inverse φ^{-1}, and both are continuous,

(2) There exists a function ψ from T to S such that $\varphi\psi=id_T$ and $\psi\varphi=id_S$ and both φ and ψ are continuous.

Proposition 3.3 *Let f be a closure operator on a set S, g a closure operator on set T, and φ be a function from S to T. Then, every (f,g)-isomorphism φ generates a lattice isomorphism φ^* between the complete lattices of closed subsets L_f and L_g defined by $\forall A \in L_f$: $\varphi^*(A) = \varphi(A) \in L_g$. Also, if a function $\varphi: S \rightarrow T$ is bijective and is generating a lattice isomorphism φ^* between lattices L_f and L_g., then φ is an (f,g)-isomorphism.*

Proposition 3.4 *Every f-authomorphism φ of <S,f> generates a unique lattice automorphism of L_f. However, more than one f-authomorphism φ of <S,f> can correspond to the same lattice automorphism of L_f.*

Proposition 3.5 *The set of all f-automorphisms of <S,f> forms a group Aut<S,f> under the function composition. This group is isomorphic to $Aut(L_f)$ of lattice automorphisms of L_f.*

We will refer to the concept of an (antisotone) Galois connection between two posets.

Definition 3.5 *Let $<P,\leq>$ and $<Q,\leq>$ be posets and φ and ψ be anti-isotone (order inverting) functions $\varphi: P \rightarrow Q$ and $\psi: Q \rightarrow P$. Then the functions define a Galois connection between the posets if: $\forall x \in P$: $x \leq \psi\varphi(x)$ and $\forall y \in Q$: $y \leq \varphi\psi(y)$.*

Galois connection can be defined in an equivalent way as a pair of functions $\varphi: P \rightarrow Q$ and $\psi: Q \rightarrow P$ such that $\forall x \in P \, \forall y \in Q$: $y \leq \varphi(x)$ iff $x \leq \psi(y)$.

Proposition 3.6 *If a pair of functions $\varphi: P \rightarrow Q$ and $\psi: Q \rightarrow P$ defines a Galois connection, then the functions $\psi\varphi: P \rightarrow P$ and $\varphi\psi: Q \rightarrow Q$ are closure operators, i.e. they satisfy the conditions 1)-3) of Definition 3.1 generalized from the inclusion $\subseteq$ to the partial order $\leq$. Moreover, the functions $\varphi: P \rightarrow Q$ and $\psi: Q \rightarrow P$ define order anti-isomorphism (order reversing functions preserving all infima and suprema) between the complete lattices of closed elements in the posets P and Q.*

Proposition 3.7 *Given an anti-isotone function $\varphi: P \rightarrow Q$. If the function $\varphi: P \rightarrow Q$ defines together with $\psi: Q \rightarrow P$ a Galois connection, then the function ψ is unique. However, there are anti-isotone functions which do not form a Galois connection with any function.*

Proposition 3.8 *If posets* $\langle P, \leq \rangle$ *and* $\langle Q, \leq \rangle$ *are complete lattices, then for every anti-isotone function* $\varphi: P \rightarrow Q$, *there exists (by Prop. 3.6 unique) function* $\psi: Q \rightarrow P$, *such that they form a Galois connection. The function* $\psi: Q \rightarrow P$ *is defined by:* $\forall y \in Q$: $\psi(y) = \vee\{x \in P: y \leq \varphi(x)\}$, *where is the lowest upper bound of the set, which must exist in a complete lattice.*

Remark 3.9 *We were using only the fact that the poset* $\langle P, \leq \rangle$ *is a complete lattice.*

In the abstract formulation of geometry on the plane in the terms of closure spaces the only closed subsets are entire plane, empty subset, points and straight lines. Geometric configurations are collections of points or lines. However, the concepts of closure spaces do not give us any tools for analysis of such configurations beyond the intersections of lines producing points and pairs of points defining lines. Our goal is to provide the tools for the analysis of such configurations not only for abstract geometries, but for arbitrary closure spaces. The approach presented below was informed by the analogy with geometric symmetries in the choice of group theory as a foundation.

Thus, it is a study of symmetry of configurations of closed subsets in a selected, but arbitrary closure space <*S, f*> with the group G = Aut<*S, f*> of its f-automorphisms. A configuration in this space will be an arbitrary, but not empty set $\mathfrak{I}$ of f-closed subsets of S. It is a natural question how the complete lattice of subgroups of the group G is related to symmetries of configurations, i.e. to symmetries of subsets of the complete lattice L_f of closed subsets in <*S, f*>. The main result of the earlier research presented below was that for arbitrary closure space there is a Galois connection between the lattice of subgroups of the group of all its automorphisms and the partially ordered set of its configurations of closed subsets.

To avoid confusion it is important to notice that we are not interested in stabilizers of sets of elements of the closure space <*S, f*>, but of the families of closed subsets. The asterisk in the formulation of the following lemma refers to the lattice isomorphism φ^* between the complete lattices of closed subsets L_f and L_g generated by (f,g)-isomorphism φ between closure spaces <*S,f*> and <*T,g*>, which always exists by Proposition 3.3.

Lemma 3.10 *Let H be a subgroup of the group* $G = Aut(L_f)$. *Define the family* $\eth_H$ *of subsets of* L_f *by* $\forall \mathcal{K} \subseteq L_f$: $\mathcal{K} \in \eth_H$ *iff* $\forall A \in \mathcal{K}$ $\forall \varphi \in H$: $\varphi^*(A) \in \mathcal{K}$. *Then* $\eth_H$ *is a complete lattice with respect to the order of inclusion of sets.*

Lemma 3.11 *Function* $\Phi: H \rightarrow \eth_H$ *defined in Lemm 3.10 is anti-isotone function between two posets, one of them (the lattice of subgroups of a group G) is a complete lattice.*

Now we can define a Galois connection. By Proposition 3.8 and Remark 3.9 we know that there exists a Galois connection between the poset of complete lattices $\eth_H$ and the complete lattice of subgroups of $G = Aut(L_f) \cong Aut$ <*S,f*>.

Theorem 3.12 *The following two functions form a Galois connection:*
$\Phi: H \rightarrow \eth_H$ *defined by* $\forall \mathcal{K} \subseteq L_f$: $\mathcal{K} \in \eth_H$ *iff* $\forall A \in \mathcal{K}$ $\forall \varphi \in H$: $\varphi^*(A) \in \mathcal{K}$ *and*
$\Psi: \eth \rightarrow H$ *defined by* $\vee\{K$ *subgroup of* G: $\eth \subseteq \eth_K\} = \{\varphi \in G: \varphi(\eth) \subseteq \eth\}$. *The last equality is a consequence of the fact that* $\{\varphi \in G: \varphi(\eth) \subseteq \eth\}$ *is a subgroup of G.*

Remark 3,13 Summary of the Concept of Symmetry in Closure Spaces:
$G = Aut(L_f)$ *is the group of automorphisms of the logic* L_f *of a closure space* <*S,f*>
H is a subgroup of the group $G = Aut(L_f)$
$\mathcal{K} \subseteq L_f$ is a configuration of closed subsets (e.g. in the geometry on a plane of points or lines)
We get a mutual correspondence between subgroups H of transformations of <*S,f*> *a*nd invariant families of configurations $\mathcal{K}$ defined by the Galois connection between the lattice of subsets of $G = Aut(L_f)$ and the lattice of families of closed subsets of the closure space <*S,f*> defined by two mappings $\Phi: H \rightarrow \eth_H$ & $\Psi: \eth \rightarrow H$.

This Galois connection defines anti-isomorphism of the lattice of subgroups of G and the lattice of invariant families of closed subsets of *<S,f>*

The existence of this Galois connection and its definition give us tools to analyze symmetry of configurations of closed subsets in an arbitrary closure space. The selection of transformations for the symmetry subgroups is determined by the condition of continuity with respect to the closure operator. Of course, in the case of a closure space describing Euclidean geometry, the continuity of transformations means isometry, i.e. preservation of Euclidean distance. But there is nothing in this formalism which requires any particular form of coordinatization. All we need is the restriction to transformations for which action of closure operator is invariant.

Now, we can observe that the restriction of the symmetric group of all permutations to the specific subgroup corresponding to symmetries of some type (in the case of Euclidean geometry, the restriction to the group of isometries) can be determined by the complete lattice of L_f closed subsets. Since we are concerned here only with the restriction of the group of transformations to the group of f-authomorphisms, we can think in purely lattice theoretic terms. We can call this lattice a "logic for symmetry". Thus, establishing of the Galois connection requires the choice of the logic for symmetry. After Galois connection is established, we can proceed to the study of symmetries of particular configurations.

4. Meta-Closure Space, its Logic &Meta-Symmetry

The most striking feature of the theory of closure spaces is that it is "autological". The description of all possible closure operators on a given set S can be achieved by one specific closure operator on the power set of S. This closure operator extends any family $\mathscr{B}$ of subsets of S to the least Moore family $\mathscr{M}$ including $\mathscr{B}$, but of course many different families can be extended to the same family $\mathscr{M}$.

Definition 4.1 *We can define this closure operator* $\boldsymbol{f}$ *on* $2^S = \wp(S)$ *by:*

$\forall \mathscr{B} \subseteq 2^S: f(\mathscr{B}) = \{B \subseteq S: \exists \mathfrak{C} \subseteq \mathscr{B} : B = \cap \mathfrak{C}\}$. *The power set equipped with this closure operator can be called a meta-closure space.*

The fact that one closure operator on the power set 2^S of S describes all closure operators on S was known from the beginning of the studies of closure spaces, but the properties of this operator started to be explored in more systematic way relatively recently and mainly in the context of combinatorics (finite closure spaces) which in this case significantly limits the generality of results. This has to be carefully considered as there are some results in literature presented without explicit assumption of the finitness of the sets are false in the infinite case. [8]

One of the results of Caspard & Monjardet [9,10] which actually can easily be extended to infinite case is that the lattice of closed subsets L_f is atomistic (i.e. every non-zero element of L_f is a join of the atoms below it). Atoms (i.e. minimal non-zero elements of the lattice) in L_f are defined by very simple Moore families {A, S} for each of proper subsets A of S (if A = S, then the Moore family defines the least element of L_f or $f(\varnothing)$). It is surprising that the following property of the closure operator f with the well-known important consequences for the lattice L_f of closed subsets of f was apparently never recognized.

Theorem 4.1 *Let* $<2^S, f>$ *is defined for an arbitrary set S by:*
$\forall \mathscr{B} \subseteq 2^S: f(\mathscr{B}) = \{B \subseteq S: \exists \mathfrak{C} \subseteq \mathscr{B} : B = \cap \mathfrak{C}\}$. *Then f satisfies the* ***anti-exchange property****:*
(awE) $\forall \mathscr{B} \subseteq 2^S: \forall A,B \subseteq S: A \neq B \;\&\; A \notin f(\mathscr{B}) \;\&\; A \in f(\mathscr{B} \cup \{B\}) \Rightarrow B \notin f(\mathscr{B} \cup \{A\})$.

Proof: Let $\mathcal{M}$ be any Moore family on S and A, B $\subseteq$ S, A $\neq$ B, A, B $\notin\mathcal{M}$ and $\exists\mathcal{B}\subseteq\mathcal{M}$: A = $\cap$ $\mathcal{B}$ $\cap$B and $\exists\mathfrak{C}\subseteq\mathcal{M}$: B = $\cap$ $\mathfrak{C}\cap$A. Then A $\subseteq$ B & B $\subseteq$ A, i.e. A = B, contradiction.

Corollary 4.2. *Meta-closure closure space <2^S, f> is a convex geometry, i.e. the lattice of closed subsets is meet-distributive.*

Definition 4.2 *Let <S, f> be a closure space. Subsets A of S satisfying the condition:*
$\forall x\in A$: $x\notin f(A\setminus\{x\})$ are called f-independent. The family of all independent subsets is represented by the symbol f-Ind.
Subsets A of S satisfying the condition: f(A) = S are called f-generating S. The family of all f-generating subsets is represented by f-Gen.
A subset A of S is called an f-base (or just base) if f $\in$f-Ind $\cap$f-Gen. Obviously every f-base is the same as minimal generating subset.
Not all closure spaces have bases!

Remark 4.3 In the finite case (i.e. for a finite set S), convex geometry <2^S,f> always has a basis and moreover this basis is unique. This makes the study of the group of authomorphisms of <2^S, f> relatively simple, as it can be carried out in terms of the group of permutations of its unique base. However this is not true when S is infinite.

Theorem 4.4 *If <S,f> is a convex geometry and the set S is infinite, then for every infinite and co-finite subset B of S there exists an f-closed family of subsets $\mathcal{B}_B$ of S, such that there is no minimal subfamily $\mathcal{B}$ of $\mathcal{B}_B$ satisfying f($\mathcal{B}$) = f($\mathcal{B}_B$).*

Proof: Consider $\mathcal{B}_B$ the principal filter of B. It can be shown that it does not have a minimal generating subset.

Corollary 4.5 *If S is infinite, then <2^S, f> does not have a base.*

This Corollary is consistent with the fact that for the infinite set S <2^S,f> is "essentially" infinite.

Definition 4.3 *We call a closure space <S, f> of finite character if*
(fC) $\forall A\subseteq S$ $\forall x\in S$: $x\in f(A) \Rightarrow \exists B\in Fin(A)$: $x\in f(B)$.

Of course all closure spaces with finite S are of finite character. Infinite closure spaces of finite character retain many characteristics of finite closure spaces. In absence of finite character, typically most of the results for finite closure spaces cannot be recovered. This is unfortunately the case of meta-closure space.

Proposition 4.6 *If S is infinite, then <2^S, f> is not of finite character (fC).*

Now we can see that while for meta-closure spaces on finite sets there are many results waiting in the literature of finite convex geometries, little is known about more general cases.

Finally, we can observe that there is another example of a closure space on the power set of S of special interest.

Definition 4.4 *A binary relation T on a set S is a weak tolerance if it is symmetric and satisfies the condition: $\forall x\in S$: [$xT^c x \Rightarrow \forall y\in S$: $xT^c y$]. Every weak tolerance which is reflexive ($\forall x\in S$: xTx) is called a tolerance relation. Equivalence relations are transitive tolerance relations.*

Proposition 4.7 *There is a bijective correspondence between weak tolerance relations on set S which generalize equivalence relations extending them to a general concept of similarity and closed subsets of the closure operator on the power set of S, i.e. closure space* $<2^S, f>$ *defined by:*
$\forall \mathcal{B} \subseteq 2^S: f(\mathcal{B}) = \{B \subseteq S: \forall x,y \in B\ \exists A \in \mathcal{B}: \{x,y\} \subseteq A\}$.

Open Problem: This article is concluded with the open problem. We could see that two particular closure spaces on the power set of S define and characterize in one case all closure spaces on S in the other case all binary relations generalizing equivalence. What are the other structures on the set S that are determined and characterized by closure operators defined on the power set of S?

References

[1] Schroeder, M. J. Concept of Symmetry in Closure Spaces as a Tool for Naturalization of Information . In Horiuchi, K. (Ed.) *Algebraic System, Logic, Language and Computer Science. RIMS Kokyuroku, Kyoto: Research Institute for Mathematical Sciences, Kyoto University*, 2016, No. 2008, pp. 29-36. Available at http://www.kurims.kyoto-u.ac.jp/~kyodo/kokyuroku/contents/ 2008.html

[2] Klein, F. C. *A Comparative Review of Recent Researches in Geometry (Vergleichende Betrachtungen über neuere geometrische Forschungen, 1872)*. Haskell, M. W. (Transl.), 2008. arXiv:0807.3161v1

[3] Weyl, H. *Symmetry*. Princeton: Princeton Univ. Press., 1952.

[4] de Saussure, F. *Course in General Linguistics*. Transl. Wade Baskin. New York: Columbia Univ. Press., 2011.

[5] Lévi-Strauss, C. *Structural Anthropology*. Transl. Claire Jacobson and Brooke Grundfest Schoepf. New York: Doubleday Anchor Books, 1967.

[6] Piaget, J. *Le structuralism*. Paris: Presses Universitaires de France, 1968; Engl tr. *Structuralism*. New York : Basic Books, Harper & Row, 1970.

[7] Birkhoff, G. *Lattice Theory, 3rd. ed.* American Mathematical Society Colloquium Publications, Vol XXV, Providence, R. I.: American Mathematical Society, 1967.

[8] Pfaltz, J.L. Closure Lattices. *Discrete Math.,* 154 (1996), 217-236.

[9] Caspard, N. & Monjardet, B. The lattices of closure systems, closure operators, and implicational systems on a finite set: a survey. *Discrete Applied Mathematics*, 127 (2003), 241-269.

[10] Caspard, N. & Monjardet, B. Some lattices of closure systems, on a finite set. *Discrete Mathematics and Theoretical Computer Science*, 6 (2004), 163-190.

[11] Schroeder, M.J & Wright, M.H. Tolerance and weak tolerance relations, *J. Combin. Math. and Combin. Comput.,* 11 (1992), 123-160