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Calculations of exponent semigroups aided by a computer

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1 Introduction

Let S be a semigroup and let P denotes the multiplicative semigroup of positive integers. The subset of P defined by

$$E(S) = \{n \in P \mid (xy)^n = x^n y^n \text{ for all } x, y \in S\}$$

forms a subsemigroup of P and is called the *exponent semigroup* of S. This notion was introduced by Tamura [7]. Even for a general semigroup S, the structure of $E(S)$ is rather restrictive. We call S an *E-m semigroup* if $m \in E(S)$. If S is an E-2 semigroup, then $E(S)$ is equal to either P or $P \setminus \{3\}$ by Clarke, Piefer and Tamura [6].

In this article after recalling some basic known results on E-m semigroups, we study the exponential semigroups of E-3 semigroups. In our study we need calculations aided by a computer.

2 E-m semigroups and free E-m semigroups

For $m \geq 2$ if $m \in E(S)$, S is called an *E-m semigroup*. More generally, for $m_1, m_2, \ldots, m_k \geq 2$, if $m_1, m_2, \ldots, m_k \in E(S)$, S is called *E-$(m_1, m_2, \ldots, m_k)$ semigroup.* We write

$$m_1, m_2, \ldots, m_k \Rightarrow n$$

if $m_1, m_2, \ldots, m_k \in E(S)$ implies $n \in E(S)$ for any semigroup S, that is, $n \in E(S)$ for any E-$(m_1, m_2, \ldots, m_k)$ semigroup S. Define

$$E(m_1, m_2, \ldots, m_k) = \{n \in P \,|\, m_1, m_2, \ldots, m_k \Rightarrow n\}.$$

*This is a digest version of Kobayashi [5].

Specifically, set

$$\begin{aligned} E(m) = E(m,1) &= \{n \in P \mid m \Rightarrow n\} \\ &= \{n \in E(S) \mid S : \text{an arbitrary } E\text{-}m \text{ semigroup}\}. \end{aligned}$$

Consider two symbols a and b, and let F the free semigroup generated by $\{a,b\}$. Let $\equiv_m$ be the congruence generated by $\{x^m y^m = (xy)^m \mid x, y \in F\}$, and let $F(m) = F/\equiv_m$ the quotient semigroup of F modulo $\equiv_m$. We call $F(m)$ the *free E-m semigroup of rank* 2. More generally, for $m_1, m_2, \ldots, m_k \geq 2$ $\equiv_{m_1,\ldots,m_k}$ is the congruence generated by

$$\{x^{m_i} y^{m_i} = (xy)^{m_i} \mid i = 1, 2, \ldots k,\ x, y \in F\},$$

and $F(m_1, \ldots, m_k) = F/\equiv_{m_1,\ldots,m_k}$ is the *free E-*$(m_1, m_2, \ldots, m_k)$ *semigroup of rank* 2. We have

$$E(m) = E(F(m)) \quad \text{and} \quad E(m_1, \ldots, m_k) = E(F(m_1, \ldots, m_k)).$$

3 Modular exponent semigroups

The following results are known (Kobayashi [4], Tamura [8] and Cherubini-Varisco [1]).

Theorem 3.1. *Let $m \geq 2$ and $k \geq 1$. We have*
(i) $m, k \Rightarrow am(m-1)+k$ for any $a \geq 2$.
(ii) $m, k \Rightarrow m(m-1)+k$ if $k \geq m$.
(iii) $m(m-1)+1 \notin E(m)$.

Corollary 3.2. *Let $m \geq 2$.*
(i) $m \Rightarrow am(m-1)+m$ for any $a \geq 1$.
(ii) $m \Rightarrow am(m-1)+1$ for any $a \geq 2$.

Corollary 3.3. *For an E-2 semigroup S, $E(S) = P$ or $P \setminus \{3\}$.*

We are interesting in the parameters missing in Theorem 3.1 and pose the following problem.

$\mathbf{P}(m,k)$: $m, k \Rightarrow m(m-1)+k$ for $m > k > 1$?

$\mathbf{P}(m,k)$ has an affirmative answer if $m \leq 7$ or $k \leq 4$. We wonder if $\mathbf{P}(8,6)$ has an affirmative answer;

$$8, 6 \Rightarrow 8 \cdot 7 + 6 = 62\,?$$

Viewing the above theorem we define the *modular exponent semigroup* $\overline{E}_m(S)$ of an E-m semigroup S by

$$\overline{E}_m(S) = \{\overline{n} \in \mathbb{Z}_{m(m-1)} \mid n \in E(S)\}.$$

This is a subsemigroup of the multiplicative semigroup $\mathbb{Z}_{m(m-1)}$. Due to Theorem 3.1, if $\alpha \in \overline{E}_m(S)$ then $n \in E(S)$ for all sufficient large $n \in P$ such that $\overline{n} = \alpha$.

Note that (modular) exponent semigroups are closed under intersection. In fact, for E-m semigroups S and T we have

$$E(S) \cap E(T) = E(S \times T) \text{ and } \overline{E}_m(S) \cap \overline{E}_m(T) = \overline{E}_m(S \times T).$$

For $n > 0$ set

$$M(n) = \{kn+1, kn+n \mid k = 0, 2, \dots\}, \ N(n) = \{kn+1 \mid k = 0, 1, \dots\},$$

and

$$\overline{M}_m(n) = \overline{M(n)}, \ \overline{N}_m(n) = \overline{N(n)} \pmod{m(m-1)}.$$

Theorem 3.4. (Kobayashi [4]) *For any $n > 0$ and $n_1, \dots n_s > 0$, there is an E-m semigroup S with*

$$\overline{E}_m(S) = \bigcap_{i=1}^{s} \overline{M}_m(n_i) \bigcap \overline{N}_m(n). \tag{1}$$

For many types of semigroups S, the modular exponent semigroups are expressed as (1) (for example it is true for all finite semigroups), but we do not know if any modular exponent semigroup is so expressed (see Kobayashi [3]).

4 E-3 semigroups

In this section S is an E-3 semigroup, that is, $x^3y^3 = (xy)^3$ for all $x, y \in S$. For E-3 semigroups we write

$$m_1, \dots, m_k \Rightarrow n$$

if $3, m_1, \dots, m_k \Rightarrow n$.

Lemma 4.1. *For all $x, y \in S$ and $n \geq 0$ we have*

$$(xy)^3 x^{2n+1} = x^{2n}(xy)^3 x \tag{2}$$

Proof.

$$\begin{aligned}(xy)^3 x^{2n+1} &= x^3(yx)^3 x^{2(n-1)} = x^2(xy)^3 x^{2(n-1)+1} = \cdots \\ &= x^{2n}(xy)^3 x.\end{aligned}$$

□

Using (2) we can show

Lemma 4.2. *For $n \geq 1$ we have*

$$n, n+2, n+3 \Rightarrow n+5.$$

$$n, n+3, n+4 \Rightarrow n+7.$$

Corollary 4.3. *For $n \geq 0$ we have*

$$6n+4 \;\Rightarrow\; 6(n+1),$$

$$6(n+1) \;\Rightarrow\; 6(n+1)+4,$$

$$6n+5,\; 6(n+1)+2 \;\Rightarrow\; 6(n+1)+4,$$

$$6n+5,\; 6(n+1) \;\Rightarrow\; 6(n+1)+2.$$

By Theorem 3.1 and Corollary 3.2 we see

$$\Rightarrow\; 6n+1 \; (n \geq 2),\; 6n+3 \; (n \geq 0).$$

$$\not\Rightarrow\; 7.$$

By Kobayashi [2] we have

$$6n+a \;\Rightarrow\; 6(n+1)+a \quad (n \geq 0, a \geq 2).$$

Moreover, [2] determines the modular exponent semigroups of E-3 semigroups as follows.

Theorem 4.4. *For any E-3 semigroup S, $\overline{E}_3(S)$ is equal to one of the subsemigroups:*

(i) $\{1,3\}$, (ii) $\{1,3,5\}$, (iii) $\{0,1,3,4\}$, (iv) $\{0,1,2,3,4,5\}$.

5 Determination of exponents with 3

Let S be an E-3 semigroup. For $r \geq 0$ set

$$\begin{array}{rcl} P_0 & = & \{6a+1, 6a+3 \,|\, a = 0,1,2,\dots\}, \;\; P_0' = P_1 \setminus \{7\}, \\ P_r & = & \{6a+1, 6a+3, 6(r+a)+5 \,|\, a = 0,1,\dots\}, \;\; P_r' = P_{1,r} \setminus \{7\}. \\ Q_r & = & \{6a+1, 6a+3, 6(r+a)+6, 6(r+a)+10 \,|\, a = 0,1,2,\dots\}, \\ R_r & = & \{6a+1, 6a+3, 6(r+a)+6, 6(r+a)+4 \,|\, a = 0,1,2,\dots\}, \\ Q_r' & = & Q_{1,r} \setminus \{7\}, \;\; R_r' = R_r \setminus \{7\}. \end{array}$$

Proposition 5.1. *(1) If $\overline{E}_3(S) = \{1,3\}$, then $E(S) = P_0$ or P_0'.*
(2) If $\overline{E}_3(S) = \{1,3,5\}$, then $E(S) = P_r$ or P_r' for some $r \geq 0$.
(3) If $\overline{E}_3(S) = \{0,1,3,4\}$, then $E(S) = Q_r$, Q_r', R_r or R_r' for some $r \geq 0$.

Finally suppose $\overline{E}_m(S) = \mathbb{Z}_6$, and let

$$k = \min\{n \in E(S) \,|\, n \equiv 2 \bmod 6\}.$$

Set $X(S) = P \setminus E(S)$, which is a finite set.

Proposition 5.2. *(1) If $k = 2$, then $E(S) = P$, that is, $X(S) = \emptyset$.*
(2) If $k = 8$, then $X(S) \setminus \{2\}$ is one of 13 sets:

$$\{\},\; \{4\},\; \{5\},\; \{4,5\},\; \{4,6\},\; \{4,5,6\},$$

$$\{4,7\},\ \{5,7\},\ \{4,5,7\},\ \{4,6,7\},\ \{4,5,6,7\},$$
$$\{4,5,6,10\},\ \{4,5,6,7,10\}.$$

(3) If $k = 14$, $X(S) \setminus \{2,8\}$ is one of 28 sets:

$$\{5,(7),(11)\},\ \{4,5,(7),(11)\},\ \{4,6,(7),(10)\},$$
$$\{4,5,6,(7),(10),(11)\},\ \{4,(5),6,(7),10,12\},$$
$$\{4,5,6,(7),10,11,12,(16)\}.$$

6 Depth-first search

For words f, g over the alphabet $\{a, b\}$, $f = g$ holds in the free E-m semigroup $F(m) = \{a,b\}^+/\equiv_m$, if there exists a sequence $f = f_0, f_1, \ldots, f_n = g$ such that

$$f_{i-1} = u(xy)^m v, f_i = ux^m y^m v \text{ or } f_{i-1} = ux^m y^m v, f_i = u(xy)^m v$$

for some $x, y, u, v \in F = \{a,b\}^+$. This can be calculated utilizing the depth first search. For example, in $F(3)$ we see

$$\begin{array}{rcl} a^{10}b^{10} & \to & a^4(ab)^6b^4 \\ & \to & a(a^2b)^3(ab)^3b^4 \\ & \to & a(a^2b)^3a^3b^7 = a^3(ba^2)^3ab^7 \\ & \to & (aba^2)^3ab^7 = a(ba^3)^3b^7 \\ & \to & a(ba^3b^2)^3b = ab(a^3b^3)^3 \\ & \to & (ab)^{10}. \end{array}$$

This shows $6 \Rightarrow 10$ for E-3 semigroups. We find $3, 14 \Rightarrow 18$ by a computer calculation. Our search program gives a sequence from $a^{18}b^{18}$ to $(ab)^{18}$ of length 12037856 in $F(3,14)$.

Suggested by the computer calculations we conjecture that for any $n \geq 1$

$$6n+4 \notin E(3, 6n+2) \text{ and } 6n+5, 6n+6 \in E(3, 6n+2). \qquad (3)$$

If (3) is true, we would be able to completely describe the exponent semigroups of E-3 semigroups.

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